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$$D=(2\pi)^{-1}\int_0^\infty d^2w\, w^{-3-p^2/2-R_a-R_b}$$

one finds that $R_a - R_b$ is conserved in $\lambda \to \lambda' + n$ scalars, but is not necessarily zero. Then all w's in Eq. (3) become $|w|$, the angular integration is trivial, and we wind up integrating $\int_0^1 dw$. Thus in this case we have electrostatics on the torus. One can use the Jacobi transformation to show that the $w=1$ singularity is just duplicating the $w=0$ one if $D=24$, $E=0$. Note that the line (w real) = (τ pure imaginary) only passes through two copies of the fundamental region.

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Dual N-Point Functions in PGL(N-2,C)-Invariant Formalism

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We derive a new expression for the dual N-point function integrand which is invariant under the action of the projective general linear group PGL(N−2, C). The (N−1)(N−3) free complex parameters of the group are used to make the integrand independent of the values of (N−1) points of complex dimension (N−3) which appear in the integrand. These points uniquely specify the location of all $\frac{1}{2}(N-1)(N-2)$ hyperplanes which appear as branch singularities of the integrand when it is viewed as a function on (N−3)-dimensional complex projective space. In contrast to the Koba-Nielsen formalism, the PGL(N−2, C)-invariant form of the N-point integrand allows transformations which mix the (N−3) integration variables and permits greater freedom in the placement of the branch singularities while preserving a simple hyperplane structure for the singularities.

I. INTRODUCTION

A large portion of the literature dealing with the dual N-point functions[1] has made use of the appealing Koba-Nielsen description[2] of the N-point function integrands. The purpose of this paper is to introduce a generalization of the Koba-Nielsen formalism in which the N-point integrands become invariant under the projective general linear group PGL($N-2, C$). When the dual N-point integrands are written in PGL($N-2, C$)-invariant form, we may move the branch singularities of the integrand wherever we please in ($N-3$)-dimensional complex projective space. We therefore view PGL($N-2, C$) as a natural singularity-structure group of the dual N-point integrands which is no less significant than the Koba-Nielsen group PGL($2, C$).

Let us first review the properties of the dual N-point functions and of the Koba-Nielsen formalism so that we may see the characteristics which suggest the introduction of a group such as PGL($N-2, C$). We define the dual N-point function as[1]

$$B_N(p_1, p_2, \ldots, p_N) = \int_0^1 \cdots \int_0^1 \prod_{k=2}^{N-2} du_{1k} \left(\prod_{k=2}^{N-3} (1 - u_{1k}u_{1,k+1}) \right)^{-1} \prod_{\substack{i=1 \\ (i,j)\neq(1,N-1)}}^{N-2} \prod_{j=i+1}^{N-1} u_{ij}{}^{-\alpha_{ij}-1}, \tag{1.1}$$

where

$$\alpha_{ij} = \alpha_{ij}(0) + \alpha'(p_i + p_{i+1} + \cdots + p_j)^2 = \alpha_{j+1,i-1}, \tag{1.2}$$

and the u_{ij}'s may be expressed uniquely in terms of the u_{1k}'s by using the relations

$$\begin{aligned} u_{ij} &= 1 - \prod_{m=i+1}^{j} \prod_{n=j+1}^{i-2} u_{mn}, \\ u_{ij} &= u_{j+1,i-1} = u_{i+N,j}. \end{aligned} \tag{1.3}$$

By changing variables in (1.1) from the u_{1k}'s to other appropriate sets of the ($N-3$)-independent u_{ij}'s, one may exhibit the cyclic and anticyclic symmetry of B_N in its arguments:

$$\begin{aligned} B_N(1,2,\ldots,N) &= B_N(2,3,\ldots,N,1) = B_N(3,4,\ldots,N,1,2) = \cdots \\ &= B_N(N,N-1,\ldots,2,1) = B_N(N-1,N-2,\ldots,2,1,N) = \cdots. \end{aligned} \tag{1.4}$$

This set of variable changes, the cross-ratio substitutions,[3] suggested to Koba and Nielsen that the u_{ij}'s be identified with the cross ratios

$$u_{ij} = \frac{(z_i - z_j)(z_{i-1} - z_{j+1})}{(z_{i-1} - z_j)(z_i - z_{j+1})} \tag{1.5}$$

of N points lying on a single complex circle; the u_{ij}'s so defined automatically obey Eq. (1.3). Fixing any three of the N variables z_i permitted Eq. (1.1) to be recast in the form[2]

$$B_N(p_1, \ldots, p_N) = \underset{\text{cyclic-ordered circle}}{\int \cdots \int} dz_1 \cdots [dz_a dz_b dz_c] \cdots dz_N (z_a - z_b)(z_b - z_c)(z_c - z_a) \prod_{j>i} (z_j - z_i)^{\theta_{ij}-1}, \tag{1.6}$$

where

$$\begin{aligned} \theta_{i,i+1} &= -\alpha_{i,i+1}, \\ \theta_{i,i+2} &= 1 - \alpha_{i,i+2} + \alpha_{i,i+1} + \alpha_{i+1,i+2}, \\ \theta_{i,j>(i+2)} &= 1 - \alpha_{ij} + \alpha_{i,j-1} + \alpha_{i+1,j} - \alpha_{i+1,j-1}. \end{aligned} \tag{1.7}$$

Using Eq. (1.2) and $\theta_{ji} = \theta_{ij}$, one may show

$$\sum_{\substack{j=1 \\ (j\neq i)}}^{N} \theta_{ij} = N - 3 \tag{1.8}$$

and

$$\theta_{ij} - 1 = \beta_{ij} - 2\alpha' p_i \cdot p_j, \tag{1.9}$$

where $\beta_{ij} = 0$ if $\alpha_{ij}(0) = -\alpha' m_k{}^2 = 1$. Equations (1.6) and (1.9) suggest that we identify each particle momentum p_i with one of the integration variables z_i.

The fact that the cross ratios (1.5) are unchanged when the three-parameter projective transformation

$$z_i \to z_i' = \frac{\alpha z_i + \beta}{\gamma z_i + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{1.10}$$

is applied simultaneously to each z_i automatically requires that Eq. (1.6) be invariant when all the z_i are transformed. We may, in fact, use Eq. (1.10) to transform the points (z_a, z_b, z_c) to the values $(1, \infty, 0)$ in order to derive alternative expressions for B_N such as

$$B_N(p_1, \ldots, p_N) = \int_0^1 dx_1 \int_{x_1}^1 dx_2 \cdots \int_{x_{N-4}}^1 dx_{N-3} \prod_{k=1}^{N-3} [x_k^{\theta_{2,k+2}-1}(1-x_k)^{\theta_{N,k+2}-1}] \prod_{i=1}^{N-4} \prod_{j=i+1}^{N-3} (x_j - x_i)^{\theta_{i+2,j+2}-1}. \quad (1.11)$$

We observe that the Koba-Nielsen notation (1.6) serves to abolish the special role played by the numbers $(0, 1, \infty)$ in Eq. (1.11) and to replace $(0, 1, \infty)$ by three arbitrary numbers (z_a, z_b, z_c); the only way the integrand could be independent of the values of (z_a, z_b, z_c) is to possess a symmetry, namely, invariance under the three-parameter group $\mathrm{PGL}(2, C)$ whose action is given by Eq. (1.10).

For $N=4$, the Koba-Nielsen method removes the branch points of (1.11) from special consideration. For $N=5$, however, we argue that when the integrand of (1.11) is viewed as a function on two-dimensional complex projective space, it gives special consideration not to three numbers $(0, 1, \infty)$ but to *six branch lines*. Since two points determine a line, the minimum amount of information needed to describe the six branch lines is provided by the *four two-dimensional numbers* $(x_1, x_2) = (0, 0), (1, 1), (\infty, 0), (0, \infty)$. For general N, Eq. (1.11) gives special privileges to $(N-1)$ different $(N-3)$-dimensional numbers, $(0, 0, \ldots, 0)$, $(1, 1, \ldots, 1)$, (∞_1), $(\infty_2), \ldots, (\infty_{N-3})$. These $(N-1)$ points determine the location of the $\frac{1}{2}(N-1)(N-2)$ singularities which occur when the B_N integrand of Eq. (1.11) is considered as a function of $\vec{x}$ on an $(N-3)$-dimensional complex projective space. The $(N-1)(N-3)$-parameter projective general linear group $\mathrm{PGL}(N-2, C)$ is then exactly the right group to introduce in order to generalize the concept behind the Koba-Nielsen notation and to write the B_N integrand in a form which is independent of the values of all the privileged numbers determining the branch singularities of Eq. (1.11).

In Sec. II, we derive the general expression for the B_N integrands as invariants under the projective general linear group $\mathrm{PGL}(N-2, C)$. In the Appendix, we work out in detail our prescription for treating the four- and five-point functions.

II. REFORMULATION OF THE DUAL N-POINT INTEGRAND AS A $\mathrm{PGL}(N-2, C)$ INVARIANT

In order to understand the nature of the integrands of the dual N-point functions, it is instructive to look at them as functions of $(N-3)$ complex variables on an $(N-3)$-dimensional complex projective space. Let us therefore briefly review the necessary elements of classical projective geometry in p dimensions.[4] We first define the $(p+1)$ *homogeneous coordinates* $x^{(\alpha)}$. The usual p *inhomogeneous coordinates* $z^{(\alpha)}$ are defined by

$$z^{(\alpha)} = \frac{x^{(\alpha)}}{x^{(p+1)}} . \quad (2.1)$$

Note that the inhomogeneous coordinates are just equal to the first p homogeneous coordinates when $x^{(p+1)} = 1$.

The purpose of introducing the extra homogeneous coordinate is to eliminate all possible ambiguities in the treatment of infinity. It allows us to satisfy this basic postulate of projective geometry: *All parallel lines meet in exactly one point, at infinity.* Groups of parallel lines not parallel to each other do not meet at infinity, thus leading to the concept that infinity is $(p-1)$-dimensional in projective p space. The other basic characteristic of projective space that we shall need is the fact that the space is *coordinatized* once we specify $(p+2)$ nondegenerate points. All other points in the space may be uniquely described with respect to the coordinatizing points. The standard choice for the coordinate system in homogeneous coordinates is

$$\begin{aligned} &\vec{x}_2 = (1, 1, \ldots, 1, 1), \quad \vec{x}_3 = (1, 0, 0, \ldots, 0, 0), \\ &\vec{x}_4 = (0, 1, 0, \ldots, 0, 0), \ldots, \\ &\vec{x}_{p+2} = (0, 0, \ldots, 0, 1, 0), \quad \vec{x}_{p+3} = (0, 0, \ldots, 0, 1). \end{aligned} \quad (2.2)$$

If we now try to use Eq. (2.1) to write (2.2) in inhomogeneous coordinates, we observe that the undefined quotient $0/0$ occurs for $(p-1)$ of the variables when the pth is $(1/0) = \infty$. This happens because at ∞, parallel lines drawn through any values of the undefined points must meet. Thus the inhomogeneous coordinates corresponding to (2.2) are

$$\begin{aligned} &\vec{z}_2 = (1, 1, \ldots, 1, 1), \quad \vec{z}_3 = (\infty, \text{ anything}, \ldots), \\ &\vec{z}_4 = (\text{anything}, \infty, \text{ anything}, \ldots), \ldots, \\ &\vec{z}_{p+2} = (\text{anything}, \ldots, \infty), \quad \vec{z}_{p+3} = (0, 0, \ldots, 0, 0). \end{aligned} \quad (2.3)$$

Finally, let us introduce the *projective transformation* which preserves all the characteristics of a p-dimensional projective space. In terms of the homogeneous coordinates, the projective transformation is the linear substitution

$$x^{(\alpha)} \to x'^{(\alpha)} = \sum_{\beta=1}^{p+1} a^{\alpha\beta} x^{(\beta)} \quad (2.4)$$

given by the $(p+1) \times (p+1)$ matrices $A = \| a^{\alpha\beta} \|$. The group acting on the homogeneous coordinates is then just the general linear group $\mathrm{GL}(p+1, C)$. Using (2.1) and (2.4) we find the following transformation law for the inhomogeneous coordinates:

$$z^{(\alpha)} \to z'^{(\alpha)} = \frac{\sum_{\beta=1}^{p} a^{\alpha\beta} z^{(\beta)} + a^{\alpha,p+1}}{\sum_{\beta=1}^{p} a^{p+1,\beta} z^{(\beta)} + a^{p+1,p+1}} . \tag{2.5}$$

This is, of course, the generalization of the one-dimensional linear fractional transformation (1.10). We observe first that Det$A$ must be nonzero for the transformation to be nondegenerate. Secondly, we notice that the transformation (2.5) is unaltered if we make the replacement

$$A \to cA, \quad c \in C^*, \tag{2.6}$$

where C^* is the field of nonzero complex numbers times the identity matrix. The group acting on the inhomogeneous coordinates is then GL$(p+1, C)/C^*$, which we now define as the *projective general linear group* PGL$(p+1, C)$. This is not isomorphic to the group SL$(p+1, C)$, defined by the matrices of determinant one, for the following reason: If $\underline{I}$ is the identity matrix, $\underline{I}$ and $\underline{I}\exp[2n\pi i/(p+1)]$ are the same element of PGL$(p+1, C)$ but are distinct elements of SL$(p+1, C)$.

The number of free parameters in PGL$(p+1, C)$ is just one less than the number in GL$(p+1, C)$, namely, $(p+1)^2 - 1 = p(p+2)$. Thus we come to understand the meaning of the $p(p+2)$ numbers appearing in the $(p+2)$ vectors of dimension p listed in Eq. (2.3). Under the action of PGL$(p+1, C)$, these vectors can be transformed to any $(p+2)$ arbitrary vectors in projective p space. Furthermore, specifying these points exhausts the free parameters of PGL$(p+1, C)$; the transformed coordinates of any other vector are completely specified in terms of the coordinates of the basis vectors. Now suppose we consider an arbitrary vector

$$\vec{z}_1 = (z_1^{(1)}, z_1^{(2)}, z_1^{(3)}, \ldots, z_1^{(p)})$$

in addition to the coordinatizing vectors $\vec{z}_2, \ldots, \vec{z}_{p+3}$. If we find the form of the PGL$(p+1, C)$ transformation which moves the coordinatizing vectors into $(p+2)$ *arbitrary* vectors $\vec{z}_2', \ldots, \vec{z}_{p+3}'$, we may solve for the old vector $\vec{z}_1$ in terms of all the new ones, $\vec{z}_1', \vec{z}_2', \ldots, \vec{z}_{p+3}'$. Suppose

$$\vec{z}_1 = \vec{u}(\vec{z}_1', \vec{z}_2', \ldots, \vec{z}_{p+3}') .$$

Then, since the projective transformation taking $\vec{z}_i$ into $\vec{z}_i'$ was completely arbitrary and because the projective transformations form a group, $\vec{u}$ is invariant under further projective transformations of the coordinates $\vec{z}_i'$.

Let us now find the explicit form of $\vec{u}$ for $N = p+3$. We first define the homogeneous coordinates $\vec{x}_1, \ldots, \vec{x}_N$ of dimension $(N-2)$ and consider the determinants

$$(1, 2, \ldots, N-2) \equiv \mathrm{Det}\begin{bmatrix} x_1^{(1)} & x_2^{(1)} & \cdots & x_{N-2}^{(1)} \\ x_1^{(2)} & x_2^{(2)} & \cdots & x_{N-2}^{(2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_1^{(N-2)} & x_2^{(N-2)} & \cdots & x_{N-2}^{(N-2)} \end{bmatrix} .$$

Since two of the column variables $\vec{x}_i$ must be omitted in order to form a square matrix, we may label the determinants by the *columns omitted*. We therefore define

$$[i, j] \equiv [ij] = -[ji] = \epsilon\begin{pmatrix} 1\,2 \cdots i \cdots j \cdots N \\ 1\,2 \cdots ij \cdots\cdots N \end{pmatrix}(1, 2, \ldots, [i], \ldots, [j], \ldots, N-1, N) , \tag{2.7}$$

where ϵ is the permutation symbol which here assumes the value ± 1 if $(j-i)$ is odd or even, respectively. When the transformation (2.4) is performed simultaneously on all the $x_i^{(\alpha)}$'s, we may easily show

$$[ij] \to \mathrm{Det}A[ij] . \tag{2.8}$$

We must also notice that, in terms of inhomogeneous coordinates,

$$[ij]_{\mathrm{homog}} = [ij]_{\mathrm{inhom}} \prod_{k \neq i,j} x_k^{(N-2)} . \tag{2.9}$$

Since, in the standard coordinate system,

$$\begin{aligned} &[ji] = x_1^{(j-2)} - x_1^{(i-2)} \quad (i = 3 \to N-1, j = i+1 \to N) , \\ &[2, i] = x_1^{(i-2)} \quad (i = 3 \to N) , \end{aligned} \tag{2.10}$$

we may use (2.1) to make the rough guess

$$z_1^{(\alpha)} \overset{?}{=} \frac{[2, \alpha+2]}{[2, N]} , \quad \alpha = 1 \to (N-3) . \tag{2.11}$$

According to Eq. (2.8), this is projectively invariant. However, (2.9) indicates that (2.11) is not independent of the homogeneous coordinates $x_{\alpha+2}^{(N-2)}$ and $x_N^{(N-2)}$, and therefore cannot be expressed exclusively in inhomogeneous coordinates. Since the determinants $[ij]$ in the standard basis contain contributions from $\vec{x}_1$ which will make the right-hand side of (2.11) quadratic in $x_1^{(\alpha)}$ unless $\vec{x}_1$ is one of the columns omitted, we may use only $[1, \alpha+2]$ and $[1, N]$ to cancel the dependence of (2.11) on $x_{\alpha+2}^{(N-2)}$ and $x_N^{(N-2)}$. We are thus led uniquely to the following PGL$(N-2, C)$-invariant expression for $\vec{z}_1$:

$$z_1^{(\alpha)} = u_\alpha(\vec{z}_1, \ldots, \vec{z}_N) = \frac{[2, \alpha+2][1, N]}{[2, N][1, \alpha+2]} . \tag{2.12}$$

Equation (2.9) shows that (2.12) is equally valid for determinants expressed in homogeneous or inhomogeneous coordinates. The obvious analog of the cross ratios occurring in the Koba-Nielsen notation is

$$u_{ij}=\frac{[i,j][i-1,j+1]}{[i-1,j][i,j+1]}\,. \tag{2.13}$$

Thus Eq. (2.12) may be written

$$u_\alpha=u_{2,\alpha+2}u_{2,\alpha+3}\cdots u_{2,N-1}\,. \tag{2.14}$$

The generalization of the cross ratio (1.5) to many dimensions is thus a cross ratio of $(N-2)$-dimensional determinants.[5] Since each determinant is proportional to the volume of the $(N-3)$-dimensional solid the coordinates of whose vertices are given by the columns of the determinant,[6] the u_{ij}'s are cross ratios of $(N-3)$-dimensional volumes. We notice that a determinant vanishes whenever one of the points lies in the same $(N-4)$-dimensional hyperplane as the others, i.e., when the $(N-3)$-dimensional volume vanishes. These are the places at which the cross ratios vanish or become infinite. Just as the ordinary cross ratio (1.5) is an invariant description of length on the projective line, the generalized cross ratio (2.13) is a projective-invariant description of length in $(N-3)$-dimensional projective space.

Using the following identity for products of determinants,

$$[ik][jl]-[jk][il]=[ij][kl]\,, \tag{2.15}$$

one can in fact show that

$$u_{ij}=1-\prod_{m=i+1}^{j}\;\prod_{n=j+1}^{i-2}u_{mn}\,. \tag{2.16}$$

Since Eq. (2.16) is exactly the "duality condition" which is obeyed by the Koba-Nielsen cross ratios, the cross-ratio expression (1.1) for the dual N-point function is equally correct when the u_{ij}'s are defined by Eq. (2.13).

Our main point is that when B_N is written in the form (1.11), the $\frac{1}{2}(N-1)(N-2)$ singularities of the integrand analogous to branch points in one complex dimension coincide *exactly* with the $\frac{1}{2}(N-1)(N-2)$ hyperplanes of complex dimension $(N-4)$ determined by the possible combinations of the $(N-1)$ coordinate-basis points (2.3) taken $(N-3)$ at a time. Since these particular hypersurfaces may be moved essentially anywhere in complex projective $(N-3)$ space by the $\mathrm{PGL}(N-2,C)$ transformation, rewriting Eq. (1.11) as a $\mathrm{PGL}(N-2,C)$ invariant will make the integrand completely independent of the absolute location of the singularities. We remark that Eq. (1.11) is the most condensed possible way to write the integral since the integration volume is in each case a *simplex*, the simplest nontrivial $(N-3)$-dimensional solid, e.g., the line, the triangle, the tetrahedron, etc.

In order to express Eq. (1.11) in terms of the variables $\vec{z}_i$, we may, of course, simply set $x_k=z_1^{(k)}$, use Eq. (2.12), and proceed by brute force. An easier way is to notice that Eq. (2.12) is projective-invariant and homogeneous of degree zero in the $z_i^{(\alpha)}$'s, or alternatively in the $x_i^{(N-2)}$'s. If we can find any projective-invariant integrand which reduces to (1.11) in the standard basis and is homogeneous of degree zero, that is the answer. Making use of Eq. (2.10), we arrive almost immediately at the correct result:

$$B_N(p_1,\ldots,p_N)=-\int d\vec{z}_1[21][31]\cdots[N1]\prod_{j>i}[ji]^{\theta_{ij}-1}\,. \tag{2.17}$$

This is the $\mathrm{PGL}(N-2,C)$-invariant form of the dual N-point function. Recalling Eq. (1.9), we see that Eq. (2.17) requires the momentum p_i to appear in the exponents of all determinants which *lack* the column vector $\vec{z}_i$. We note that the integrand of Eq. (2.17) may be separated into an invariant volume element

$$dV=d\vec{z}_1[21][31]\cdots[N1]\prod_{k=1}^{N}[k+2,k]^{-1} \tag{2.18}$$

and a $\mathrm{PGL}(N-2,C)$-invariant function

$$f(\vec{z}_1,\ldots,\vec{z}_N)=\prod_{k=1}^{N}[k+2,k]\prod_{j>i}[ji]^{\theta_{ij}-1}=\prod_{i,j}u_{ij}{}^{-\alpha_{ij}-1}\,. \tag{2.19}$$

The invariance of dV and $f(\vec{z}_1,\ldots,\vec{z}_N)$ under the transformation (2.5) is proven using

$$d\vec{z}_1\to d\vec{z}_1'=\Big(\sum_\beta a^{N-2,\beta}z_1^{(\beta)}+a^{N-2,N-2}\Big)^{2-N}\,\mathrm{Det}A\;d\vec{z}_1 \tag{2.20}$$

and

$$[ij](z)\to[ij](z')=\prod_{k\neq i,j}\Big(\sum_\beta a^{N-2,\beta}z_k^{(\beta)}+a^{N-2,N-2}\Big)^{-1}\mathrm{Det}A\;[ij](z) \tag{2.21}$$

along with the relation (1.8). Here $[ij](z)$ means the determinant (2.7) of matrices whose columns are $(z_k^{(1)},\ldots,z_k^{(N-3)},1)$. The derivation of Eq. (2.20) is aided by enlarging the dimension of the Jacobian matrix resulting from the variable change.

Utilizing the symmetry (1.4) of B_N in its arguments due to the cross-ratio variable changes, we conclude that any $(N-1)$ of the $\vec{z}_i$ may be held fixed while the Nth is integrated over. The integration region is fixed by requiring the u_α's to lie in the regions determined by the defining integral (1.11). Translated into restrictions on the determinants, these conditions become the analog of the cyclic-ordered circle in the Koba-Nielsen notation. We find that the boundary of the integration in $d\vec{z}_1$ is given by the equations

$$
\begin{aligned}
&[23]=0\,,\\
&[34]=0\,,\\
&\cdots\\
&[N-2, N-1]=0\,,\\
&[N-1, N]=0\,.
\end{aligned}
\tag{2.22}
$$

Finally, we note that the $(N-3)$-dimensional surfaces containing the real parts of the branch singularities of the PGL($N-2, C$)-invariant integrand (2.17) and of the Koba-Nielsen integrand (1.6) are completely different. The former has the topology of real projective $(N-3)$-space. The latter is an $(N-3)$-dimensional torus,[7] which is equivalent to a direct product of $(N-3)$ one-dimensional projective spaces and quite unlike real projective $(N-3)$-space. The difference is essentially due to the nature of the variable changes leading from Eq. (1.1) to each of the two invariant integrands. This phenomenon is illustrated in the Appendix for the case $N=5$.

III. CONCLUSION

In conclusion, we emphasize that the relevance of PGL($N-2, C$) to the dual N-point functions lies in the fact that the group has precisely the number of parameters needed to describe *all* the $(N-4)$-dimensional complex hyperplanes which appear as singularities in the integrand of Eq. (1.11) when viewed as a function on $(N-3)$-dimensional complex projective space. The Koba-Nielsen formalism, in contrast, effectively treats the integrand of (1.11) as a function on a direct product of $(N-3)$ one-dimensional complex projective spaces. Making the B_N integrand invariant under PGL($N-2, C$) permits greater freedom in the placement of the branch singularities, allows transformations which mix the $(N-3)$ integration variables, and yet maintains a simple hyperplane structure for the singularities.

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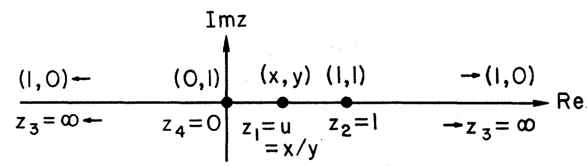

FIG. 1. The real projective line determined by the homogeneous coordinate system $(x,y)=(0,1)$, $(1,1)$, $(1,0)$ corresponding to the points $z=0,1,\infty$. Under a projective transformation, these three points may be moved to any three distinct points in the complex z plane without changing the value of the cross ratio $u=u(z_1,z_2,z_3,z_4)$.

APPENDIX: EXAMPLES B_4 AND B_5

We now work out some familiar examples to illustrate the concepts we have introduced. First, we analyze the integrand of the Euler beta function in one-dimensional projective space. We define

$$B_4(a,b)=\int_0^1 du\, u^{a-1}(1-u)^{b-1}\,. \tag{A1}$$

In the complex u plane, this has singularities at $u=(0,1,\infty)$. Letting $u=x/y$ be a general point, we see that (2.2) specifies the following homogeneous coordinate system for $N=4$:

$$\vec{x}_1=(x,y),\quad \vec{x}_2=(1,1),\quad \vec{x}_3=(1,0),\quad \vec{x}_4=(0,1), \tag{A2}$$

or inhomogeneous coordinates

$$z_1=u,\quad z_2=1,\quad z_3=\infty,\quad z_4=0\,. \tag{A3}$$

This means that the usual definition (A1) of B_4 places the singularities at precisely the conventional coordinatizing points of projective one-space, as shown in Fig. 1. What is significant is that there are *no other* singularities. Rewriting (A1) in PGL($2, C$)-invariant format then lets us put the singularities anywhere we please. We now observe that the projective transformation

$$z\to z'=\frac{z_3(z_2-z_4)z+z_4(z_3-z_2)}{(z_2-z_4)z+(z_3-z_2)} \tag{A4}$$

maps $z=(1,\infty,0)$ into the arbitrary points $z'=(z_2,z_3,z_4)$. Letting $z'=z_1$ be the transformed value of u and solving for u, we find the cross ratio

$$u=\frac{(z_1-z_4)(z_2-z_3)}{(z_2-z_4)(z_1-z_3)}=u(z_1,z_2,z_3,z_4)\,. \tag{A5}$$

Recalling that $z_i=x_i^{(1)}/x_i^{(2)}$, we see that Eq. (A5) may be expressed completely in terms of homogeneous coordinates,

$$u=\frac{\operatorname{Det}\begin{vmatrix}x_1^{(1)} & x_4^{(1)}\\ x_1^{(2)} & x_4^{(2)}\end{vmatrix}\operatorname{Det}\begin{vmatrix}x_2^{(1)} & x_3^{(1)}\\ x_2^{(2)} & x_3^{(2)}\end{vmatrix}}{\operatorname{Det}\begin{vmatrix}x_2^{(1)} & x_4^{(1)}\\ x_2^{(2)} & x_4^{(2)}\end{vmatrix}\operatorname{Det}\begin{vmatrix}x_1^{(1)} & x_3^{(1)}\\ x_1^{(2)} & x_3^{(2)}\end{vmatrix}}
=\frac{[23][14]}{(-[13])(-[24])}=u_{23}\,, \tag{A6}$$

in agreement with Eq. (2.12). From Eq. (2.15), we find

$$1-z=\frac{[12][34]}{[13][24]}=u_{12}\,.$$

The differential is

$$du=\frac{dz_1(z_2-z_3)(z_3-z_4)(z_4-z_2)}{(z_2-z_4)^2(z_1-z_3)^2}
=\frac{dz_1}{(z_2-z_4)^2(z_1-z_3)^2}\operatorname{Det}\begin{vmatrix}1 & 1 & 1\\ z_2 & z_3 & z_4\\ (z_2)^2 & (z_3)^2 & (z_4)^2\end{vmatrix},$$

which is the $x_i^{(2)}=1$ version of the homogeneous differential

$$du=\frac{[12][13][14]}{[13]^2[24]^2}\operatorname{Det}\begin{vmatrix}dx_1^{(1)} & dx_1^{(2)}\\ x_1^{(1)} & x_1^{(2)}\end{vmatrix}.$$

The final expression is thus[8]

$$B_4(a,b)=-\int_{[23]=0}^{[34]=0} dz_1[21][31][41][21]^{b-1}[31]^{(1-a-b)-1}\times[41]^{a-1}[32]^{a-1}[42]^{(1-a-b)-1}[43]^{b-1}\,, \tag{A7}$$

where $[23]=0$ when $z_1=z_4$ and $[34]=0$ when $z_1=z_2$. The integration is along the arc of the circle in the complex z_1 plane determined by (z_2, z_3, z_4). Only along this arc is the integrand real and nonsingular.

Now we turn to the five-point function, which exhibits several new properties. Starting from Eq. (1.11), we may write B_5 as

$$B_5(abcde)=\int_0^1 du\int_u^1 dv(v-u)^{a-1}(1-u)^{b-a-c}\times(1-v)^{c-1}u^{d-1}v^{e-a-d}\,. \tag{A8}$$

Let us now examine the integrand as a function on a two-dimensional projective space. If $u=x/z$ and $v=y/z$, we may write the general point $\vec{x}_1$ and the homogeneous coordinate system as

$$\vec{x}_1=(x,y,z),\quad \vec{x}_2=(1,1,1),\quad \vec{x}_3=(1,0,0),\quad \vec{x}_4=(0,1,0),\quad \vec{x}_5=(0,0,1)\,. \tag{A9}$$

The inhomogeneous coordinates are

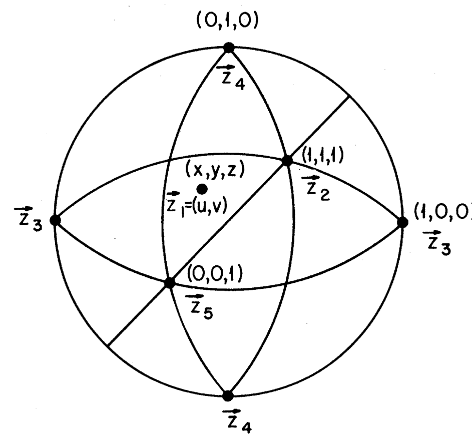

FIG. 2. The real part of the two-dimensional complex projective space determined by the homogeneous coordinate system $(x,y,z)=(1,1,1)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. These four points determine the paths of the six branch lines appearing in the integrand of Eq. (A8). Diametrically opposite points in the figure are understood to be continuously connected so that the surface shown represents the one-sided projective plane.

$$\vec{z}_1=(u,v),\quad \vec{z}_2=(1,1),\quad \vec{z}_3=(\infty_u),\quad \vec{z}_4=(\infty_v),\quad \vec{z}_5=(0,0)\,. \tag{A10}$$

Remembering that infinity is $[(N-4)=1]$-dimensional and that parallel lines through two finite points like $(0,0)$ and $(1,1)$ meet on the line at infinity, we depict in Fig. 2 the points of Eqs. (A9) and (A10). Since any two points determine a line, we also draw the six lines determined by the four coordinatizing points. We observe that the line at infinity which joins the points (∞_u) and (∞_v) is placed on a completely equal footing with the other five lines if we require diametrically opposite points to be continuously connected to one another. The surface shown in Fig. 2 then becomes the topological structure known as the *projective plane*. We caution the reader that Fig. 2 shows only the *real* part of the two-dimensional complex projective space, i.e., the part analogous to the real line of Fig. 1.

One might ask how Fig. 2 compares with the picture implied by the Koba-Nielsen formula for the B_5 integrand:

$$B_5(12345)=\int_{z_1}^{z_3} dz_2\int_{z_3}^{z_5} dz_4(z_1-z_3)(z_3-z_5)(z_5-z_1)\times\prod_{j>i}(z_j-z_i)^{\theta_{ij}-1}\,.$$

Let us place the z_k's on the unit circle, $z_k=\exp(i\phi_k)$. The integrand then returns to its original value when ϕ_2 or ϕ_4 varies from 0 to 2π provided the singularities are properly avoided. For example,

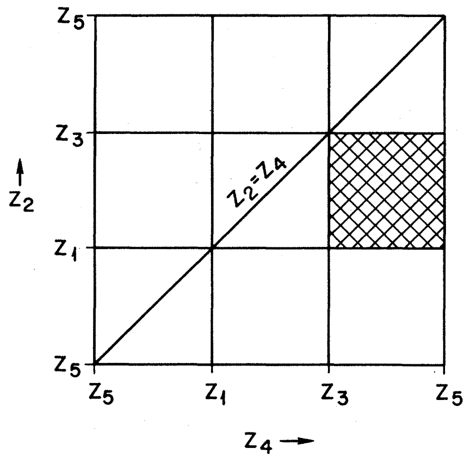

FIG. 3. The surface containing the singularities of the Koba-Nielsen form of the B_5 integrand when (z_1, z_3, z_5) are fixed. The structure is the product of two one-dimensional projective spaces, i.e., a torus. Thus the top horizontal line is to be identified with the bottom line and the left-hand vertical line is identified with the right-hand line. $B_5(12345)$ is given by the integral over the cross-hatched region.

as ϕ_2 varies with $\phi_3 < \phi_4 < \phi_5$, z_2 meets singularities at z_3, z_4, z_5, z_1, and then again z_3. The integrand is thus periodic in each variable separately and suggests the torus shown in Fig. 3. The lines in Fig. 3 represent all the singularities which are encountered as ϕ_2 and ϕ_4 vary. The value of $B_5(12345)$ is given by the two-dimensional integral over the square hatched region bounded by the singularities at $z_2 = (z_1, z_3)$ and $z_4 = (z_3, z_5)$.

We now continue with our derivation of the PGL($3, C$)-invariant form of the B_5 integrand (A8). The explicit form of the PGL($3, C$) matrix $\|a^{\alpha\beta}\|$ which takes the standard coordinate basis (A10) into the arbitrary points $\vec{z}_i' = (x_i, y_i)$ via the transformation (2.5) is easily found to be

$$\|a^{\alpha\beta}\| = \begin{vmatrix} x_3(245) & x_4(253) & x_5(234) \\ y_3(245) & y_4(253) & y_5(234) \\ (245) & (253) & (234) \end{vmatrix}. \tag{A11}$$

Here we have defined the determinants

$$(ijk) = \mathrm{Det}\begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ 1 & 1 & 1 \end{vmatrix},$$

which obey the relation

$$x_k(nij) + x_i(njk) + x_j(nki) = x_n(ijk). \tag{A12}$$

If we now let $\vec{z}_1'$ be the new value of the arbitrary point $\vec{z}_1$ after it has undergone a projective transformation (2.5) with the matrix (A11), we may solve for $\vec{z}_1$ as an explicit function of $\vec{z}_1'$, $\vec{z}_2'$, $\vec{z}_3'$, $\vec{z}_4'$, and $\vec{z}_5'$. Writing $\vec{z}_1 = (u, v)$, we find the result

$$\begin{aligned} u &= \frac{(234)(145)}{(245)(134)} = \frac{[15][23]}{[13][25]} = u_{23}u_{24}, \\ v &= \frac{(135)(234)}{(134)(235)} = \frac{[24][15]}{[25][14]} = u_{24}, \end{aligned} \tag{A13}$$

which agrees of course with Eq. (2.12). Using the identity (2.15), we may show

$$\begin{aligned} 1-u &= \frac{[12][35]}{[25][13]}, \\ 1-v &= \frac{[12][45]}{[25][14]}, \\ v-u &= \frac{[15][12][34]}{[25][13][14]}. \end{aligned} \tag{A14}$$

The Jacobian of the transformation from (u, v) to (x_1, y_1) is

$$J = -\frac{[51]^2[21]^2}{[41][31][52]^3},$$

so we may write Eq. (A8) in the form

$$\begin{aligned} B_5(abcde) &= -\int\int dx_1 dy_1 [21][31][41][51][21]^{b-1}[31]^{c-b-d} \\ &\quad \times [41]^{d-c-e}[51]^{e-1}[32]^{d-1}[42]^{e-d-a} \\ &\quad \times [52]^{a-e-b}[43]^{a-1}[53]^{b-a-c}[54]^{c-1}. \end{aligned}$$

Making the identification

$$a = -\alpha_{34}, \quad b = -\alpha_{12}, \quad c = -\alpha_{13},$$
$$d = -\alpha_{23}, \quad e = -\alpha_{24},$$

we write B_5 in the format of Eq. (2.17):

$$\begin{aligned} B_5(p_1, p_2, p_3, p_4, p_5) &= -\int\int dx_1 dy_1 [21][31][41][51] \prod_{j>i} [ji]^{\theta_{ij}-1}. \end{aligned} \tag{A15}$$

In the standard basis, Eq. (A15) becomes identical to Eq. (A8) and the variables u and v are integrated over the triangle shown in Fig. 2. In general, the determinant (ijk) gives the area of the triangle whose vertices are $\vec{z}_i$, $\vec{z}_j$, and $\vec{z}_k$. The variable u in Eq. (A13) is then a cross ratio of areas, just as (A6) is a cross ratio of lengths. Since u is proportional to (145), u vanishes in any basis when $\vec{z}_1$ becomes collinear with $\vec{z}_4$ and $\vec{z}_5$, i.e., when the area of their triangle vanishes. The boundaries of the triangle of integration in Fig. 2 may then be described in any basis by the conditions

$$(145) = [23] = 0 \rightarrow u = 0\,,$$
$$(215) = [43] = 0 \rightarrow v = u\,, \qquad \text{(A16)}$$
$$(213) = [54] = 0 \rightarrow v = 1\,.$$

Note that in a general basis where $(\vec{z}_2, \vec{z}_3, \vec{z}_4, \vec{z}_5)$ are complex, at least one point of the real projective plane shown in Fig. 2 is mapped to a point on the surface at infinity. Finally, we observe that, just as integrating the beta function integrand along the three intervals $[-\infty, 0]$, $[0,1]$, and $[1,\infty]$ in Fig. 1 gives the three permutations of four external lines, integrating the B_5 integrand over the twelve triangles of Fig. 2 gives the 12 distinct permutations of the five-point function's external lines.[9]

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[6]D. M. Y. Somerville, *An Introduction to the Geometry of N Dimensions* (Dover, New York, 1958), p. 124.

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Infinite-Momentum Helicity States*

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We discuss and generalize to arbitrary spin the kind of single-particle spin states which have appeared naturally in field theories in the infinite-momentum frame. These states transform simply under the Galilean symmetry group which is important in the infinite-momentum frame, rather than under the rotation group. We also find that the spinors $U(P,\lambda)$ representing these states are very simple.

I. INTRODUCTION

The states of a single particle with mass M, spin s are generally represented by a state vector $|P,\lambda\rangle$, where P is the momentum of the particle and λ labels its spin state. Many definitions of spin state are available – the most popular being the Jacob and Wick helicity states.[1]

The presently common kinds of spin states transform simply under rotations. They are thus particularly useful for the description of low-energy phenomena, in which rotational symmetry is important (for instance, two-body scattering in the resonance region). In this paper, we will define and discuss a set of spin states which transform simply under the "Galilean" transformations[2,3] which are useful in the description of particles moving in the $+z$ direction with high energy. These spin states have previously been found to emerge naturally in discussions of field theories in the infinite-momentum frame,[3-7] at least for the cases $s=\frac{1}{2}$ and $s=1$.

We begin with a brief review of the infinite-momentum coordinate system,

$$\tau = 2^{-1/2}(t+z), \quad \mathfrak{z} = 2^{-1/2}(t-z),$$

paying particular attention to the Galilean subgroup of the Poincaré group, which leaves the planes τ = constant invariant. We use this Galilean structure to define a convenient "spin" or "internal angular momentum" operator. It is then a simple matter to construct single-particle eigenstates of this operator. We also show that these "infinite-momentum helicity" states look like ordinary Jacob