An introduction to wave mechanics

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The quantum mechanical Dirac equation is a deterministic equation describing the evolution of spin angular momentum density (or spin density). Therefore an understanding of the classical physics description of spin density is a logical prerequisite for understanding quantum mechanics. This paper outlines how a classical wave theory of spin density can be used to describe particle-like waves and their interactions, offering students a conceptual bridge between classical physics and quantum mechanics. We specifically address two common misconceptions by demonstrating that special relativity and spin angular momentum are consequences of classical wave theory. First, a wave equation is derived for infinitesimal shear waves in an elastic solid. Next, a change of variables is used to describe the waves in terms of classical spin density - the field whose curl is equal to twice the classical momentum density. The second-order wave equation is then converted to a first-order Dirac equation. Conceptually, the Dirac equation is much easier to understand than the Schrödinger equation for two reasons: (1) the wave function has a well-defined physical interpretation, and (2) consistency with special relativity is guaranteed by Lorentz-invariance of the wave equation. Bispinors describing transverse plane wave solutions contain a phase factor with half of the phase of the real-valued vector wave functions. Unlike the "parity" operator of relativistic quantum mechanics, the classical operator for spatial inversion exchanges matter and antimatter, consistent with experiments in which some mirror-image processes only occur if matter is replaced by antimatter. The dynamical operators of relativistic quantum mechanics are derived. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials. Classical interpretations of magnetic flux quantization and the Coulomb potential are presented. A classical version of the single-electron Lagrangian for quantum electrodynamics is also presented.

Keywords: classical interpretation, Dirac equation, elastic solid, quantum mechanics pedagogy, spin angular momentum, spin density, teaching quantum mechanics, wave mechanics

1. INTRODUCTION

Students of physics are typically introduced to quantum mechanics via the Schrödinger equation. Although this equation can successfully describe some processes, it suffers from the fact that unlike ordinary wave equations it is not Lorentz-invariant. The Schrödinger equation also does not provide an interpretation of spin angular momentum, which is intrinsic to elementary particles. Isaac Newton reportedly said, "A man may imagine things that are false, but he can only understand things that are true, for if the things be false, the apprehension of them is not understanding." Although the Dirac equation may seem more complicated than the Schrödinger equation, it has the advantage of being a physically realistic, and therefore comprehensible, description of nature. Besides its application to quantum mechanics, the Dirac formalism has been used by various researchers to describe classical wave dynamics. [1–11] Therefore we propose that the Dirac equation, which is both relativistic and describes spin angular momentum, is a better starting point for understanding quantum mechanics.

Although the Dirac equation is a deterministic description of the evolution of physical quantities, it is used to calculate probabilities of various measurement outcomes. Bohmian mechanics, or pilot-wave theory, offers insight into the relationships between deterministic wave processes and quantum statistics. [12–14] Pilot-waves exhibiting quantum statistics have been experimentally demonstrated using silicone droplets bouncing on a vibrating water tank. [15–21] However, we will only analyze the physical dynamics of wave evolution, and not the probabilistic nature of measurements.

In this paper we expand on previous work in deriving a Dirac equation for spin density from the classical model of an ideal elastic solid. [7–9] We then present plane wave solutions, and use these as the basis for explanations of spatial inversion and special relativity. We construct a Lagrangian and derive the dynamical operators of relativistic quantum mechanics. We then describe how wave interactions give rise to the Pauli exclusion principle and electromagnetic potentials. Finally, we relate the classical Lagrangian to that of quantum electrodynamics.

2. METHODS: DERIVING AN EQUATION FOR SPIN DENSITY

2.1. Ideal Elastic Solid

We consider the case of an isotropic, homogeneous solid with a linear relationship between infinitesimal stress and strain. The usual expression for potential energy is:

$$\int U\, d^3\mathbf{r} = \int \left(\frac{1}{2}\lambda(\nabla\cdot\boldsymbol{\xi})^2 + \mu e_{ij}^2\right) d^3\mathbf{r} \tag{1}$$

where $\boldsymbol{\xi}$ represents displacement, $e_{ij} = (\partial_i\xi_j + \partial_j\xi_i)/2$ is the symmetric strain tensor, and λ and μ are the Lame' parameters. This expression has the drawback that it does not cleanly separate compressible and rotational motion. We can remedy this as follows:

Expanding the square of the symmetrical strain tensor yields:

$$\begin{aligned} e_{ij}^2 &= (\partial_x\xi_x)^2 + (\partial_y\xi_y)^2 + (\partial_z\xi_z)^2 \\ &+ \frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2\right). \end{aligned} \tag{2}$$

Add $2(\partial_x\xi_x\partial_y\xi_y + \partial_y\xi_y\partial_z\xi_z + \partial_z\xi_z\partial_x\xi_x)$ to the first term and subtract it from the second term to obtain:

$$\begin{aligned} e_{ij}^2 &= (\nabla\cdot\boldsymbol{\xi})^2 \\ &+ \frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2\right) \\ &- 2(\partial_x\xi_x\partial_y\xi_y + \partial_y\xi_y\partial_z\xi_z + \partial_z\xi_z\partial_x\xi_x). \end{aligned} \tag{3}$$

Integrate the extra terms by parts on each of the two derivatives (assuming no contribution at infinity) to obtain:

$$\begin{aligned} e_{ij}^2 &\to (\nabla\cdot\boldsymbol{\xi})^2 \\ &+\frac{1}{2}\left((\partial_x\xi_y + \partial_y\xi_x)^2 + (\partial_y\xi_z + \partial_z\xi_y)^2 + (\partial_z\xi_x + \partial_x\xi_z)^2\right) \\ &-2(\partial_x\xi_y\partial_y\xi_x + \partial_y\xi_z\partial_z\xi_y + \partial_z\xi_x\partial_x\xi_z). \end{aligned} \tag{4}$$

This is equivalent to:

$$e_{ij}^2 \to (\nabla\cdot\boldsymbol{\xi})^2 + \frac{1}{2}(\nabla\times\boldsymbol{\xi})^2\,. \tag{5}$$

The potential energy density may therefore be expressed as:

$$U = \frac{1}{2}(\lambda+2\mu)(\nabla\cdot\xi)^2 + \frac{1}{2}\mu(\nabla\times\boldsymbol{\xi})^2. \tag{6}$$

This form of the potential energy density separates infinitesimal irrotational and incompressible motion. It is a quadratic function of the first derivatives of displacement. The Lagrangian for infinitesimal incompressible motion is:

$$\mathcal{L} = \int\left(\frac{1}{2}\rho(\partial_t\boldsymbol{\xi})^2 - \frac{1}{2}\mu(\nabla\times\boldsymbol{\xi})^2\right)dV. \tag{7}$$

The Euler-Lagrange equation is the usual equation for infinitesimal shear waves:

$$\partial_t^2\boldsymbol{\xi} = -\frac{\mu}{\rho}\nabla\times\nabla\times\boldsymbol{\xi} \tag{8}$$

for which the wave speed is $c=\sqrt{\mu/\rho}$.

The incompressible potential energy in Eq. 7 is the form of potential energy used by MacCullagh in 1837 to describe light waves [22]. In spite of this well-known result, some have claimed that the rotation vector $(\nabla\times\boldsymbol{\xi})/2$ cannot appear in the energy density because the energy should be independent of rotations (see e.g. Ref. 23). That logic does not apply here for two reasons. First, we assume zero displacement at infinity, so global rotations are not allowed. Second, elastic energy density depends on relative motion at different positions, so integration by parts can alter the spatial distribution of the calculated energy density. There is no need for so-called ”second-gradient” elasticity (inclusion of derivatives of rotation in the energy density), since the energy density depends directly on the infinitesimal rotation vector.

2.2. Spin Angular Momentum

It is well known that elastic waves in solids have two types of momentum: that of the medium $(\rho\partial_t\boldsymbol{\xi})$ and that of the wave: $\rho(\nabla\xi_j)\partial_t\xi_j$ (see e.g. Ref. 24). Clearly there must also be two types of angular momentum in an elastic solid: ”spin” associated with rotation of the medium, and ”orbital” associated with rotation of the wave. However, spin angular momentum has not historically been considered to be a classical physics concept.

The key to understanding classical spin angular momentum is the Helmholtz decomposition of momentum density. The momentum density $\mathbf{p}=\rho\mathbf{v}$ consists of an incompressible (or rotational) part $(\tilde{\mathbf{p}})$, an irrotational (or compressible) part $(\check{\mathbf{p}})$, and a constant part $(\bar{\mathbf{p}})$ determined by the Helmholtz decomposition:

$$\mathbf{p} = \tilde{\mathbf{p}} + \check{\mathbf{p}} + \bar{\mathbf{p}} = \frac{1}{2}\nabla\times\mathbf{s} - \nabla\Phi + \bar{\mathbf{p}} \tag{9}$$

where:

$$\mathbf{s}(\mathbf{r},t) = \frac{1}{2\pi}\nabla\times\int_V \frac{\mathbf{p}(\mathbf{r}',t)-\bar{\mathbf{p}}}{|\mathbf{r}-\mathbf{r}'|}dV' \tag{10a}$$

$$\Phi(\mathbf{r},t) = -\frac{1}{4\pi}\nabla\cdot\int_V \frac{\mathbf{p}(\mathbf{r}',t)-\bar{\mathbf{p}}}{|\mathbf{r}-\mathbf{r}'|}dV'\,. \tag{10b}$$

Previous work has demonstrated that $\mathbf{s}$ represents angular momentum density corresponding to spin in relativistic quantum mechanics. [7–9] Hence we refer to $\mathbf{s}$ as ”spin density”.

Assuming sufficiently rapid fall-off at large distances, the volume integral of spin density is equal to the volume integral of the first moment of momentum $\mathbf{r}\times\tilde{\mathbf{p}}$. The two representations of angular momentum density are related by integration by parts [9]:

$$\begin{aligned}
\int \mathbf{r}\times\frac{1}{2}(\nabla\times\mathbf{s})d^3r &= \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r\\
&= \frac{1}{2}\int\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\right)d^3r\\
&= \int \mathbf{s}\,d^3r\,.
\end{aligned} \tag{11}$$

where the total derivatives are assumed not to contribute to the last line, since they can be converted into surface integrals that are assumed to vanish.

Unlike the "moment of momentum" definition, spin angular momentum density is an intrinsic property defined at each point in space. Coordinate-independent descriptions of rotational dynamics can actually be traced back to the nineteenth century.[25] In 1891 Oliver Heaviside recognized MacCullagh's force density in Eq. 8 as being the curl of a torque density that is proportional to an infinitesimal rotation angle $\mathbf{\Theta} = (1/2)\nabla\times\boldsymbol{\xi}$. [26] However, this idea seems to have been largely forgotten. Students should be encouraged to ponder how physics might have developed differently had a simple interpretation of spin angular momentum been available to the early pioneers of quantum mechanics.

The rotational kinetic energy is: [9]

$$\begin{aligned} K &= \frac{1}{2\rho}\int \tilde{p}^2 d^3r = \frac{1}{2\rho}\int \left[\frac{1}{2}\nabla\times\mathbf{s}\right]^2 dV \\ &= \frac{1}{8\rho}\int \left[\mathbf{s}\cdot[\nabla\times(\nabla\times\mathbf{s})] + \nabla\cdot(\mathbf{s}\times(\nabla\times\mathbf{s}))\right] dV \\ &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, dV, \end{aligned} \tag{12}$$

where $\mathbf{w} = \nabla\times\mathbf{v}/2$ is the angular velocity (sometimes confusingly referred to as "spin" in the literature). In this case the divergence term is assumed not to contribute to the integral.

According to Eq. 12, spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity:

$$\frac{\delta}{\delta w_i}\int \frac{1}{2} w_j s_j\, dV = \frac{1}{2}\int \left(\frac{\delta w_j}{\delta w_i} s_j + w_j \frac{\delta s_j}{\delta w_i}\right) dV = \frac{1}{2}s_i + \frac{1}{2}s_i = s_i\,, \tag{13}$$

where integration by parts was used twice to evaluate the second term in the integral.

As an example, consider a cylinder of radius R aligned with the z-axis and rotating rigidly with angular velocity w_0. The motion is described by these non-zero variables: [9]

$$s_z = \rho w_0[R^2 - r^2] \text{ for } r \le R \text{ and zero for } r > R; \tag{14a}$$

$$v_\phi = -\frac{1}{2\rho}\frac{\partial}{\partial r} s_z = r w_0 \text{ for } r \le R\text{; and zero for } r > R; \tag{14b}$$

$$w_z = \frac{1}{2r}\frac{\partial}{\partial r} r v_\phi = w_0\left[1 - R\delta(r-R)/2\right] \text{ for } r \le R\text{; and zero for } r > R. \tag{14c}$$

The total angular momentum per unit height is

$$\begin{aligned} S_z &= 2\pi\int_0^R s_z r dr = 2\pi\int_0^R \rho w_0[R^2 - r^2] r dr \\ &= \frac{1}{2}\pi\rho R^4 w_0 = \frac{1}{2} M R^2 w_0 \\ &= I w_0\,, \end{aligned} \tag{15}$$

where we have used the mass per unit height $M = \rho\pi R^2$ and moment of inertia per unit height $I = MR^2/2$.

The kinetic energy per unit height is

$$\begin{aligned} K &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, r dr d\phi = \pi\int_0^R w_0\left[1 - R\delta(r-R)/2\right]\rho w_0[R^2 - r^2] r dr \\ &= \pi\rho w_0^2\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{MR^2}{4} w_0^2 = \frac{1}{2} I w_0^2\,. \end{aligned} \tag{16}$$

These are in agreement with standard rotational dynamics. Students should understand that spin angular momentum is well-defined in classical physics.

Defining $\mathbf{\Theta} = (1/2)\nabla\times\boldsymbol{\xi}$, Eq. 8 becomes:

$$\partial_t(\nabla\times\mathbf{s}) + 4\mu\nabla\times\mathbf{\Theta} = 0. \tag{17}$$

Note that $\mathbf{\Theta}$ is a vector and only represents an angle for infinitesimal motion. Assuming $\nabla\cdot\mathbf{s} = 0$, the Helmholtz decomposition yields:

$$\partial_t\mathbf{s} = -4\mu\mathbf{\Theta}. \tag{18}$$

This equation states that the rate of change of angular momentum density is equal to torque density, which is proportional to infinitesimal rotation angle.

The next step is to relate the displacement $\boldsymbol{\xi}$ to the spin density $\mathbf{s}$. For infinitesimal motion, define a vector potential $\mathbf{Q}$ such that $\partial_t \mathbf{Q} = \mathbf{s}$. Since the curl of $\mathbf{s}$ is proportional to velocity, the curl of $\mathbf{Q}$ must be proportional to displacement:

$$\frac{1}{2\rho} \nabla \times \mathbf{Q} = \boldsymbol{\xi} \,. \tag{19}$$

Therefore the equation for $\mathbf{s}$ is equivalent to:

$$\partial_t^2 \mathbf{Q} + c^2 \nabla \times \nabla \times \mathbf{Q} = 0 \,, \tag{20}$$

where $c^2 = \mu/\rho$. The curl of this equation yields Eq. 8.

Thus far we have assumed infinitesimal motion. Previous work attempted to describe finite motion by the equation: [7–9]

$$\partial_t \mathbf{s} + \mathbf{v} \cdot \nabla \mathbf{s} - \mathbf{w} \times \mathbf{s} = -c^2 \nabla \times \nabla \times \mathbf{Q} = \boldsymbol{\tau}. \tag{21}$$

The logic of this equation is that changes due to translation ($\mathbf{v} \cdot \nabla \mathbf{s}$) and rotation ($-\mathbf{w} \times \mathbf{s}$) do not require torque density ($\boldsymbol{\tau}$). Similarly, the momentum density equation may be interpreted as a statement that changes due to translation ($\mathbf{v} \cdot \nabla \mathbf{p}$) do not require force density ($\mathbf{f}$):

$$\partial_t \mathbf{p} + \mathbf{v} \cdot \nabla \mathbf{p} = \mathbf{f}. \tag{22}$$

However, it is not clear if Eq. 21 is correct. In this paper we only consider infinitesimal motion.

2.3. Dirac Equation

Eq. 20 is a second-order vector equation. In order to use variational methods, it is desirable to re-write the wave equation as a first order equation. We will follow Refs. [7] and [9] by starting with one-dimensional waves and then generalizing to three dimensions.

2.3.1. One-Dimensional Waves

Consider a one-component wave propagating in one-dimension with amplitude of $a(z,t)$. If the wave equation is

$$\partial_t^2 a = c^2 \partial_z^2 a \,, \tag{23}$$

then the general solution consists of backward (B) and forward (F) propagating waves:

$$a = a_B(ct + z) + a_F(ct - z) \,. \tag{24}$$

The two directions of wave propagation are clearly independent states, and they are separated in space by a 180° rotation. This property is the fundamental characteristic of spin one-half states. Generalization to three dimensional space should therefore involve spinor wave functions.

The forward and backward waves satisfy the equations:

$$\begin{aligned} \partial_t a_B &= \partial_z a_B \\ \partial_t a_F &= -\partial_z a_F \,. \end{aligned} \tag{25}$$

Defining $\dot{a} = \partial_t a$, we can write the wave equation as a first-order matrix equation:

$$\partial_t \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} c \partial_z \begin{bmatrix} \dot{a}_B \\ \dot{a}_F \end{bmatrix} = 0 \,. \tag{26}$$

We have thus achieved our goal of converting a one-dimensional second-order wave equation into a first-order matrix equation. Generalization to three dimensional vector waves requires additional components. One possibility is to introduce vector components such as (a_{Bi}) and (a_{Fi}) to make a 6-element column vector in the equation above.

Unfortunately, this method does not allow a simple means for changing the direction of the derivative (∂_z). Therefore we follow a different path.

First, note that the procedure above specifies independent components with positive and negative wave velocity, and uses a diagonal matrix to relate spatial and temporal derivatives. We can apply a similar technique to separate positive and negative values of the wave function. Letting a_B and a_F represent the z-components of vectors, separate each component of the wave into positive and negative parts ($\dot{a}_B = \dot{a}_{B+} - \dot{a}_{B-}$ and $\dot{a}_F = \dot{a}_{F+} - \dot{a}_{F-}$) so that each of the four wave components ($\dot{a}_{B+}, \dot{a}_{B-}, \dot{a}_{F+}, \dot{a}_{F-}$) is positive-definite. With these definitions, we have:

$$\dot{a_B} = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = \eta^T \sigma_z \eta \,, \tag{27}$$

and similarly for $\dot{a}_F$. The matrix σ_z is the Pauli spin matrix for the z-component of a vector. Notice that this matrix has the same form (within a minus sign) as the velocity direction matrix in Eq. 26. In one dimension, the significance of simultaneous positive and negative components is unclear. We will see that in three dimensions, simultaneous positive and negative components for one direction indicates polarization in a different direction.

We arrange the components in the following order, corresponding to the chiral representation of the Dirac wave function:

$$\begin{bmatrix} \dot{a}_{B+} \\ \dot{a}_{F-} \\ \dot{a}_{F+} \\ \dot{a}_{B-} \end{bmatrix} \,. \tag{28}$$

We may now write the time derivative $\dot{a}$ as the matrix product:

$$\dot{a} = \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = \psi^T \sigma_z \psi \tag{29}$$

where σ_z now represents the 4×4 Dirac matrix for the z-component of spin density, and the the four-component column vectors are called Dirac bispinors. The spatial derivative is now given by:

$$c\partial_z a = - \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = -\psi^T \gamma^5 \psi \tag{30}$$

where an overall minus sign has been introduced in order to maintain consistency with the chiral representation of the Dirac equation. The matrix γ^5 is the Dirac matrix for chirality. If the amplitude (a) represents rotation angle, then positive and negative chirality ($\partial_z a$) are analogous to left- and right-handed threads on a screw. Wave velocity (v) is obtained by combining the two matrices used above:

$$v\psi = c \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{a}_{B+}^{1/2} \\ \dot{a}_{F-}^{1/2} \\ \dot{a}_{F+}^{1/2} \\ \dot{a}_{B-}^{1/2} \end{bmatrix} = c\gamma^5 \sigma_z \psi \,. \tag{31}$$

The one-dimensional wave equation may be written in the form:

$$\partial_t[\psi^T \sigma_z \psi] + c\partial_z[\psi^T \gamma^5 \psi] = \partial_t^2 a - c^2 \partial_z^2 a = 0 \,. \tag{32}$$

Other matrices may be inserted between the wave functions, resulting in the following corresponding expressions (correcting a mistake in Ref. [9]). Each of these is equal to zero for the wave solutions:

$$\partial_t[\psi^T \psi] + c\partial_z[\psi^T \gamma^5 \sigma_z \psi] = \partial_t|\partial_t a_F| + \partial_t|\partial_t a_B| + c^2\left(\partial_z|\partial_z a_F| - \partial_z|\partial_z a_B|\right) ; \tag{33a}$$

$$\partial_t[\psi^T \gamma^5 \sigma_z \psi] + c\partial_z[\psi^T \psi] = c\left(\partial_t|\partial_z a_F| - \partial_t|\partial_z a_B| + \partial_z|\partial_t a_F| + \partial_z|\partial_t a_B|\right) ; \tag{33b}$$

$$\partial_t[\psi^T \gamma^5 \psi] + c\partial_z[\psi^T \sigma_z \psi] = \partial_t[-c\partial_z a] + c\partial_z[\partial_t a] \,. \tag{33c}$$

2.3.2. Three-Dimensional Vector Waves

Generalization to three dimensions is based on the fact that the matrix σ_z may be regarded as representing one component of a three-dimensional vector. An arbitrary vector $\mathbf{a} = (a_x, a_y, a_z)$ may be written in terms of a 2-component complex spinor η as:

$$\begin{aligned} a_x &= \eta^\dagger \sigma_x \eta = \eta^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eta\,, \\ a_y &= \eta^\dagger \sigma_y \eta = \eta^\dagger \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \eta\,, \\ a_z &= \eta^\dagger \sigma_z \eta = \eta^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \eta\,. \end{aligned} \tag{34}$$

The normalized spinor eigenfunctions for each direction are:

$$\begin{aligned} \sigma_x \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\,; \qquad & \sigma_x \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} &= -\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\,; \\ \sigma_y \begin{bmatrix} 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} \end{bmatrix} &= \begin{bmatrix} 1/\sqrt{2} \\ \mathrm{i}/\sqrt{2} \end{bmatrix}\,; \qquad & \sigma_y \begin{bmatrix} 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} \end{bmatrix} &= -\begin{bmatrix} 1/\sqrt{2} \\ -\mathrm{i}/\sqrt{2} \end{bmatrix}\,; \\ \sigma_z \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,; \qquad & \sigma_z \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= -\begin{bmatrix} 0 \\ 1 \end{bmatrix}\,. \end{aligned} \tag{35}$$

The algebra of the Pauli matrices is called "geometric algebra":

$$\begin{aligned} \sigma_x \sigma_y \sigma_z &= \mathrm{i} I\,, \\ \sigma_i \sigma_j &= \delta_{ij} I + \mathrm{i}\epsilon_{ijk}\sigma_k\,. \end{aligned} \tag{36}$$

where the unit imaginary "i" represents a unit oriented "volume". It is a true scalar if the matrices represent axial vectors, and is a pseudoscalar if the matrices represent polar vectors. The fourth independent matrix in this algebra is the identity matrix (I). For each propagation direction, the direction of the vector $\eta^\dagger \boldsymbol{\sigma} \eta$ can be rotated by operations of the form:

$$\begin{aligned} R_{\phi_j}(\sigma_i) &= \exp\left(-\mathrm{i}\sigma_j \phi_j/2\right)\sigma_i \exp\left(\mathrm{i}\sigma_j \phi_j/2\right) \\ &= \left(\cos\left(\phi_j/2\right) - \mathrm{i}\sigma_j \sin\left(\phi_j/2\right)\right)\sigma_i\left(\cos\left(\phi_j/2\right) + \mathrm{i}\sigma_j \sin\left(\phi_j/2\right)\right)\,. \end{aligned} \tag{37}$$

These rotation matrices may operate either on the wave functions (Schrödinger picture) or on the matrices (Heisenberg picture).

The vector $\mathbf{a}$ may be split into forward and backward wave components by replacing the two-component spinor η with a 4-component bispinor ψ.

Just as there are three Pauli matrices indicating different vector directions, there are also three orthogonal matrices associated with wave velocity. In the chiral notation, these are:

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{I}_2 \\ \mathbf{I}_2 & \mathbf{0} \end{pmatrix}, \quad \gamma^4 = -\begin{pmatrix} \mathbf{0} & -\tilde{\mathrm{i}}\mathbf{I}_2 \\ \tilde{\mathrm{i}}\mathbf{I}_2 & \mathbf{0} \end{pmatrix}, \quad \gamma^5 = -\begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I}_2 \end{pmatrix} \tag{38}$$

where $\mathbf{0}$ and $\mathbf{I}_2$ are the 2×2 null and identity matrices, respectively. The γ-matrices above have the same form as the Pauli spin matrices except that two of the matrices have minus signs, as in a 180-degree rotation about the direction represented by γ^0. The unit imaginary here is denoted as $\tilde{\mathrm{i}}$ because velocity is a polar vector, so this unit imaginary $\tilde{\mathrm{i}} = \gamma^0\gamma^4\gamma^5$ is a pseudoscalar (spatial inversion changes $\tilde{\mathrm{i}}$ to $-\tilde{\mathrm{i}}$). [27] However, we will continue to use the notation "i" for either scalar or pseudoscalar unit imaginary whenever behavior under spatial reflection is not explicit.

The one-dimensional wave equation (Eq. 32) has the bispinor form:

$$\psi^T \left\{\sigma_z \partial_t \psi + c\gamma^5 \partial_z \psi\right\} + \text{Transpose} = 0\,. \tag{39}$$

We can separate a common factor of $\psi^\dagger \sigma_z$:

$$\psi^\dagger \sigma_z \left\{\partial_t \psi + c\gamma^5 \sigma_z \partial_z \psi\right\} + \text{Transpose} = 0\,. \tag{40}$$

For arbitrary vector components and derivatives, the bispinors are complex and the wave equation is:

$$\psi^\dagger\sigma_i\left\{\partial_t\psi + c\gamma^5\sigma_j\partial_j\psi\right\} + \text{adjoint} = 0\,. \tag{41}$$

This is the first-order wave equation for vector waves in three dimensions.

Expanding the spatial derivative term in Eq. (41) yields the 3-D generalization of the wave equation (32):

$$\begin{aligned} 0 &= \partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] + c\nabla\left[\psi^\dagger\gamma^5\psi\right] - \mathrm{i}c\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\} \\ &= \partial_t^2\mathbf{a} - c^2\nabla(\nabla\cdot\mathbf{a}) + c^2\nabla\times(\nabla\times\mathbf{a})\,, \end{aligned} \tag{42}$$

where corresponding terms are in the same order in each line of the equation. This is the result we have been seeking. We have rewritten the second-order vector wave equation as a first order equation involving Dirac bispinors. The validity of this correspondence, which we will confirm with examples, demonstrates that the Dirac equation of relativistic quantum mechanics is simply a special case of an ordinary vector wave equation.

Replacing the vector $\mathbf{a}$ by $2\mathbf{Q}$ yields the following physical correspondences:

$$\mathbf{s} = \partial_t\mathbf{Q} \equiv \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\psi\right]; \tag{43a}$$

$$c\nabla\cdot\mathbf{Q} \equiv -\frac{1}{2}\left[\psi^\dagger\gamma^5\psi\right]; \tag{43b}$$

$$c^2\left\{\nabla\times\nabla\times\mathbf{Q}\right\} \equiv -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\}; \tag{43c}$$

$$0 = \frac{\mathrm{i}c}{2}\nabla\cdot\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\}\ . \tag{43d}$$

These identifications provide seven independent constraints on the eight free parameters of the complex Dirac bispinor: three for the first, one for the second, two for the third (since a curl has only two independent components), and one for the fourth. There is also an arbitrary overall phase factor. The last identification simply states that the divergence of a curl is zero. This condition is necessary for consistency.

The first-order wave equation (Eq. 41) can be reduced to:

$$\partial_t\psi + c\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi + \mathrm{i}\chi\psi = 0\,, \tag{44}$$

where χ is any operator with the property

$$\mathrm{Re}\left\{\psi^\dagger\sigma_j\mathrm{i}\chi\psi\right\} = 0\,. \tag{45}$$

Behavior of the unit imaginary (and χ) under spatial inversion is not specified and will be left undetermined from here on. The equation for an electron is obtained by the choosing $\chi = M\gamma^0$. Hence the Dirac equation for an electron may be interpreted as an ordinary wave equation with a clear dynamical interpretation describing the motion of an elastic solid.

According to the above analysis, the first-order Dirac equation is a kind of factorization (or square root) of a second-order wave equation. Others have made similar factorizations using multivariate 4-vectors and/or quaternions. [28–30]

Multiplying Eq. 44 by $\psi^\dagger$ and adding the adjoint yields a conservation law with density $\psi^\dagger\psi$ and current $\psi^\dagger c\gamma^5\boldsymbol{\sigma}\psi$:

$$\partial_t(\psi^\dagger\psi) + \nabla\cdot(\psi^\dagger c\gamma^5\boldsymbol{\sigma}\psi) = 0\,. \tag{46}$$

In quantum mechanics this equation is regarded as a conservation law for probability density, but in both classical and quantum mechanics it is part of the description of the evolution of spin angular momentum density.

3. RESULTS: APPLICATIONS OF CLASSICAL SPIN DENSITY

3.1. Sample Plane Wave Solutions

As a simple mathematical example, the longitudinal wave $(Q_x, Q_y, Q_z) = (0, 0, Q_0\sin(\omega t - kz))$ propagating along the z-axis may be expressed in the bispinor form:

$$\psi = \sqrt{2\omega Q_0}\,\exp\left[-\mathrm{i}\left(\omega t - kz\right)/2\right]\begin{bmatrix} 0 \\ \sin\left(\left[\omega t - kz\right]/2\right) \\ \cos\left(\left[\omega t - kz\right]/2\right) \\ 0 \end{bmatrix}\,. \tag{47}$$

The phase factor in front is introduced for later consistency with transverse waves. For $\omega = ck$, this wave function yields:

$$\mathbf{s} = \partial_t \mathbf{Q} = \frac{1}{2}\left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = (0, 0, \omega Q_0 \cos(\omega t - kz))\,; \tag{48a}$$

$$c\nabla \cdot \mathbf{Q} = -\frac{1}{2}\left[\psi^\dagger \gamma^5 \psi\right] = -\omega Q_0 \cos(\omega t - kz)\,; \tag{48b}$$

$$c^2(\{\nabla \times \nabla \times \mathbf{Q}\}) = -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right] \times \gamma^5 \boldsymbol{\sigma}\psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla\psi\right\} = (0, 0, 0)\,. \tag{48c}$$

In this case $\nabla \times \mathbf{s} = 0$, so this wave solution is not relevant for describing shear waves in an elastic solid.

In addition to the wave variables described above, there are other "observables" that may be computed from the wave function. These include the vectors:

$$\mathbf{b}_0 = \psi^\dagger \gamma^0 \boldsymbol{\sigma} \psi = \hat{\mathbf{y}} \times \omega \mathbf{Q} = (\omega Q_0 \sin(\omega t - kz), 0, 0)\,; \tag{49a}$$

$$\mathbf{b}_4 = \psi^\dagger \gamma^4 \boldsymbol{\sigma} \psi = \hat{\mathbf{x}} \times \omega \mathbf{Q} = (0, -\omega Q_0 \sin(\omega t - kz), 0)\,; \tag{49b}$$

$$\mathbf{b}_5 = \psi^\dagger \gamma^5 \boldsymbol{\sigma} \psi = \omega\, Q_0 \hat{\mathbf{z}} = (0, 0, \omega Q_0)\,. \tag{49c}$$

In this case the current $\mathbf{b}_5$ is proportional to wave velocity. The other two $\mathbf{b}$-vectors are orthogonal to $\mathbf{b}_5$ and to each other. Hence the matrices $(\gamma^0, \gamma^4, \gamma^5)$ may be interpreted as defining directions relative to the wave velocity direction. Specifically, multiplication of the wave function by $\exp[-\mathrm{i}\gamma^0 \phi_0/2]$ rotates the velocity about the y-axis by angle ϕ_0, provided that the variables (x, y, z) are also rotated accordingly ($kz \to k(z\cos\phi_0 + x\sin\phi_0)$). Likewise, multiplication of the original wave function by $\exp[-\mathrm{i}\gamma^4 \phi_4/2]$ rotates the wave function about the x-axis.

The "$\mathbf{b}$" vectors, representing relative velocity directions, are all polar vectors. This differs from the quantum mechanical interpretation, in which $\mathbf{b}_0$ is regarded as an axial vector. Applying the classical properties to quantum mechanical wave functions leads to the conclusion that matter and antimatter are mirror images of each other. [7, 27] This is consistent with the experimental fact that "matter to the right is symmetrical with antimatter to the left." [31] With this interpretation of spatial reflection, the mirror image of beta decay of cobalt-60 is simply beta decay of anti-cobalt-60. The mirror image of a left-handed neutrino is a right-handed anti-neutrino. According to classical physics, "parity violation" is just the mundane observation that matter is more common than anti-matter.

The conventional view of spatial reflection is much more complicated. The standard "parity" operator inverts the vectors $\mathbf{b}_4$ and $\mathbf{b}_5$, but not $\mathbf{b}_0$. With this interpretation of spatial reflection, the mirror image of matter is also matter, but some mirror-image processes do not occur in nature (so-called "maximal" parity violation). Although antimatter appears to behave exactly like the mirror-image of matter, it is not regarded as such.

Students of physics might wonder why such a convoluted theory of spatial reflection is preferred over the simpler theory based on classical physics. Since it is impossible to actually invert space to directly test the theory of spatial reflection, why not apply the principle of "Occam's razor" and choose the simpler theory? This author has no good answer to that question, except to note that the conventional assumption predates the discovery of antimatter.

To obtain a transverse wave solution, we rotate the wave velocity independently of the polarization direction using the γ matrices. For example, velocity rotation by $\pi/2$ about the y-axis is performed by multiplying the wave function by $\exp[-\mathrm{i}\gamma^0(\pi/4)]$ and changing the wave direction $z \to x$, so that the bispinor becomes:

$$\psi = \sqrt{\omega Q_0}\, \exp[-\mathrm{i}\,(\omega t - kx)/2] \begin{bmatrix} -\mathrm{i}\,\cos([\omega t - kx]/2) \\ \sin([\omega t - kx]/2) \\ \cos([\omega t - kx]/2) \\ -\mathrm{i}\,\sin([\omega t - kx]/2) \end{bmatrix}, \tag{50}$$

and the new wave vector potential is $(Q_x, Q_y, Q_z) = (0, 0, Q_0 \sin(\omega t - kx))$. The phase factor in front is now necessary for satisfaction of the Dirac equation. A bispinor that yields correct real-valued vector wave quantities does not necessarily also satisfy the Dirac equation as presented here (perhaps a more generalized version would be satisfied).

Other wave variables are:

$$\mathbf{s} = \partial_t \mathbf{Q} = \frac{1}{2}\left[\psi^\dagger \boldsymbol{\sigma} \psi\right] = (0, 0, \omega Q_0 \cos(\omega t - kx))\,; \tag{51a}$$

$$c\nabla \cdot \mathbf{Q} = -\frac{1}{2}\left[\psi^\dagger \gamma^5 \psi\right] = 0\,; \tag{51b}$$

$$\begin{aligned} c^2(\{\nabla \times \nabla \times \mathbf{Q}\}) &= -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right] \times \gamma^5 \boldsymbol{\sigma}\psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla\psi\right\} \\ &= (0, 0, c^2 k^2 Q_0 \sin(\omega t - kx))\,. \end{aligned} \tag{51c}$$

Arbitrary monochromatic plane waves can be derived from this one by suitable scaling and rotation operations (including appropriate phase changes). Two constants of the motion are:

$$\mathrm{Re}(\psi^\dagger \mathrm{i}\partial_t \psi) = -\mathrm{Re}(\psi^\dagger \mathrm{i}c\gamma^5 \boldsymbol{\sigma}\cdot\nabla\psi) = \omega^2 Q_0 \,. \tag{52}$$

The "**b**" vectors are:

$$\mathbf{b}_0 = \psi^\dagger\gamma^0\boldsymbol{\sigma}\psi = \hat{\mathbf{y}}\times\omega\mathbf{Q} = (\omega Q_0 \sin(\omega t - kx), 0, 0); \tag{53a}$$
$$\mathbf{b}_4 = \psi^\dagger\gamma^4\boldsymbol{\sigma}\psi = -\omega Q_0\hat{\mathbf{z}} = (0, 0, -\omega Q_0); \tag{53b}$$
$$\mathbf{b}_5 = \psi^\dagger\gamma^5\boldsymbol{\sigma}\psi = \hat{\mathbf{x}}\times\omega\mathbf{Q} = (0, -\omega Q_0 \sin(\omega t - kx), 0). \tag{53c}$$

For this transverse plane wave, the current $\mathbf{b}_5$ is not aligned with the wave velocity direction. Instead it is proportional to a cross product of the wave vector with the vector potential $\mathbf{Q}$.

Notice that the phase factor in front of the bispinor in Eq. 50 is half of the phase of the real vector potential $\mathbf{Q}$. This suggests a classical physics analogue of fermions and bosons: the real-valued vector wave functions have angular quantum numbers (ℓ) that are odd for classical fermions and even for classical bosons. Since the parity of the spherical harmonics is $(-1)^\ell$, particles with odd values of ℓ have distinct mirror-images corresponding to antiparticles. Particles with even values are their own mirror images. In the Standard Model, almost all elementary fermions likewise have distinct antiparticles (including neutrinos if we use the classical definition of spatial inversion). Most bosons are likewise their own antiparticle, notable exceptions being the W^+ and W^- bosons. Classical physics would evidently regard these particles as fermions. Further study is needed to relate the classical picture to the Standard Model.

3.2. Angular Eigenfunctions

Let $\Phi^{(+)}_{j,m_z}(\theta,\phi)$ and $\Phi^{(-)}_{j,m_z}(\theta,\phi)$ be the two-component spinor eigenfunctions of the angular momentum operators J^2, J_z, L^2, S^2, and:

$$\begin{aligned}\boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(+)}_{j,m_z}(\theta,\phi) &= (j-1/2)\Phi^{(+)}_{j,m_z}(\theta,\phi)\\ \boldsymbol{\sigma}\cdot\mathbf{L}\Phi^{(-)}_{j,m_z}(\theta,\phi) &= -(j+3/2)\Phi^{(-)}_{j,m_z}(\theta,\phi)\end{aligned} \tag{54}$$

where j and m_z are half-integer angular momentum quantum numbers. [32] These functions are related by $\sigma_r\Phi^{(+)}_{j,m_z} = \Phi^{(-)}_{j,m_z}$ and yield opposite eigenvalues under coordinate inversion ($\mathbf{r}\to-\mathbf{r}$). Only the true scalar imaginary can appear within these functions. In terms of spherical harmonics $Y_{l,m}(\theta,\phi)$ they are:

$$\begin{aligned}\Phi^{(+)}_{j,m_z}(\theta,\phi) &= \begin{bmatrix}\sqrt{\frac{j+m_z}{2j}}Y^{m_z-1/2}_{j-1/2}(\theta,\phi)\\ \sqrt{\frac{j-m_z}{2j}}Y^{m_z+1/2}_{j-1/2}(\theta,\phi)\end{bmatrix}\\ \Phi^{(-)}_{j,m_z}(\theta,\phi) &= \begin{bmatrix}\sqrt{\frac{j+1-m_z}{2(j+1)}}Y^{m_z-1/2}_{j+1/2}(\theta,\phi)\\ -\sqrt{\frac{j+1+m_z}{2(j+1)}}Y^{m_z+1/2}_{j+1/2}(\theta,\phi)\end{bmatrix}\end{aligned} \tag{55}$$

Consider a bispinor wave function with given j and m_z values of the form:

$$\psi(r,\theta,\phi) = \frac{1}{r}\begin{bmatrix}\tilde{i}g(r)\Phi^{(+)}_{j,m_z}\\ f(r)\Phi^{(-)}_{j,m_z}\end{bmatrix} \tag{56}$$

This functional form is commonly used in the Dirac representation (see e.g. Ref.[32]), but here we are still using the chiral representation. Both representations yield the same spin density. The ambiguity stems from the fact that there is no oscillation, and therefore no distinction between wave propagation directions.

The spin density defined by such functions may be computed directly, and the curl of spin density is proportional to the classical velocity of the wave-carrying medium. For these wave functions with any values of j and m_z, the velocity field is purely azimuthal with the form $(0,0,v_\phi(r,\theta))$ in spherical coordinates. For the simple case of $j=1/2, m_z=1/2$, the spin density is:

$$\mathbf{s} = (s_r, s_\theta, s_\phi) \propto ((1/r^2)(|f(r)|^2+|g(r)|^2)\cos\theta, (1/r^2)(|f(r)|^2-|g(r)|^2)\sin\theta)\,. \tag{57}$$

The contributions from $\Phi^{(+)}_{1/2,1/2}$ and $\Phi^{(-)}_{1/2,1/2}$ can be determined by setting either $g(r)$ or $f(r)$ to zero, respectively. The The $j = 1/2, m_z = 1/2$ wave function with $f(r) = 0$ is:

$$\psi^{(+)}(r, \theta, \phi) = \frac{\mathrm{i}\, g(r)}{\sqrt{4\pi}\, r} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{58}$$

The $j = 1/2, m_z = 1/2$ wave function with $g(r) = 0$ is:

$$\psi^{(-)}(r, \theta, \phi) = \frac{f(r)}{\sqrt{4\pi}\, r} \begin{bmatrix} 0 \\ 0 \\ \cos\theta \\ e^{\mathrm{i}\phi} \sin\theta \end{bmatrix} \tag{59}$$

The angular dependence of spin density for $(j, m_z) = (1/2, 1/2)$ is illustrated in Fig. 1. The orbital angular momentum numbers are $l = 0$ for $\Phi^{(+)}_{1/2,1/2}$ and $l = 1$ for $\Phi^{(-)}_{1/2,1/2}$.

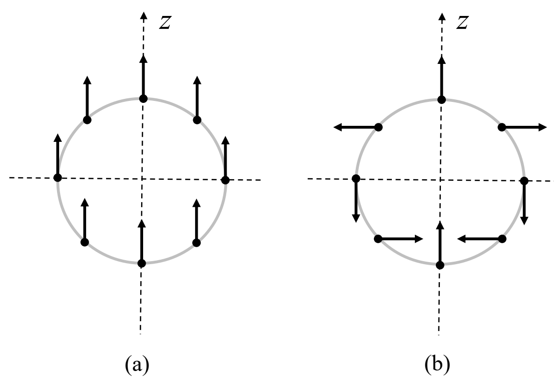

FIG. 1. Contribution to θ-dependence of spin density from (a) $\Phi^{(+)}_{1/2,1/2}$ and (b) $\Phi^{(-)}_{1/2,1/2}$

The velocity field computed from $\Phi^{(+)}_{1/2,1/2}$ by setting $f(r) = 0$ is $(v_r, v_\theta, v_\phi) \propto (0, 0, ((d/dr)|g|^2/r^2)\sin\theta)$. The velocity computed from $\Phi^{(-)}_{1/2,1/2}$ by setting $g(r) = 0$ is $(v_r, v_\theta, v_\phi) \propto (0, 0, ((2|f|^2 - r(d/dr)|f|^2)/r^3)\sin\theta)$.

While these computed velocity fields are consistent with spin densities similar to those found in quantum mechanics, they are not consistent with wave-like motion in an elastic solid. Such circulating motion cannot continue indefinitely. Time-dependence could be introduced explicitly (e.g. $f(r,t)$ and $g(r,t)$) while maintaining constant values of j and m_z, but such oscillations are not assumed in quantum mechanics. However, a constant component of azimuthal velocity is possible as part of a more general oscillatory motion. For example, if you could grab a point in the solid and move it in a circle around a central point, displacements would decrease with distance and the average velocity at all nearby points in space would be in the azimuthal direction. The "walking droplet" analogue of quantum mechanics also interprets the quantum wave function as an average or low-frequency part of a more general oscillation. [21]

3.3. Special Relativity

The mass term in quantum mechanics involves multiplication of the wave function by $\mathrm{i}\gamma^0$, which we have shown above to be a generator of rotation of wave velocity. This fact suggests that particles with mass should be interpreted as waves whose velocity direction continuously rotates, or as standing waves consisting of a superposition of such waves. This behavior is similar to that of de Broglie waves in a central potential, whose rays follow circular paths between two bounding radii.[33] In the quantum mechanical interpretation of the Dirac equation, the fluctuation of position known as "zitterbewegung" is attributable to the particle undergoing circular motion with diameter equal to the Compton wavelength: $\lambda_0 = h/(m_0 c)$.[34]

The model of particles as circulating waves offers a simple means for understanding special relativity (SR). [35–38] Model a particle at rest as in Fig. 2(a) with wave crests propagating in circles at the speed of light around the z-axis

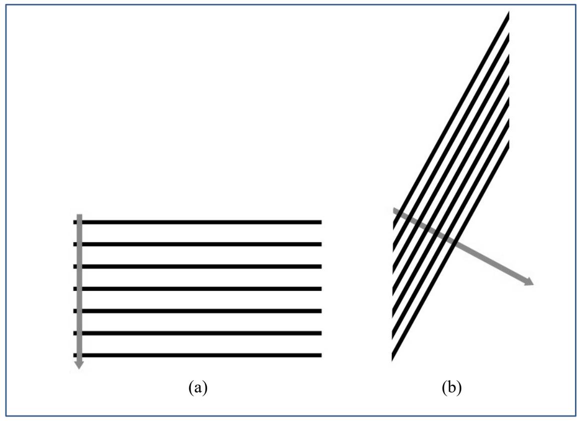

FIG. 2. (a) Model of circular wave propagation with the vertical axis representing the azimuthal direction. (b) Model of helical wave propagation with speed $v = 0.866c$ and $\gamma = 2$. These patterns are designed to be printed on a transparency sheet and rolled into a cylindrical tube (if printed on ordinary paper, shine a light into the tube in order to illuminate the wave pattern).

(horizontal direction) with wave frequency $f_0 = m_0c^2/h$ and wavelength $\lambda_0 = h/m_0c$. The gray arrow represents the distance light travels in one unit of time, as measured by a stationary observer. The internal clock ticks once each time the wave traverses the circle. Rotating the wave crests as in Fig. 2(b) results in helical wave propagation with average velocity v and relativistic factor $\gamma = c/\sqrt{c^2 - v^2}$, and a new wavelength of λ_0/γ and relativistic frequency γf_0. The width of the moving wave packet is reduced by a factor of $1/\gamma$ (this length contraction was proposed by Fitzgerald and made quantitative by Lorentz in order to explain the null result of the Michelson-Morely experiment [39–41]). Propagation in the azimuthal direction, which measures time, is also reduced by a factor of $1/\gamma$ (time dilation). The distance between wave crests along the z-axis is $(\lambda_0/\gamma)(c/v) = h/(\gamma m_0 v) = h/p$, which is the de Broglie wavelength of a moving "particle". Hence the de Broglie wavelength results from a Lorentz boost of a stationary oscillation.

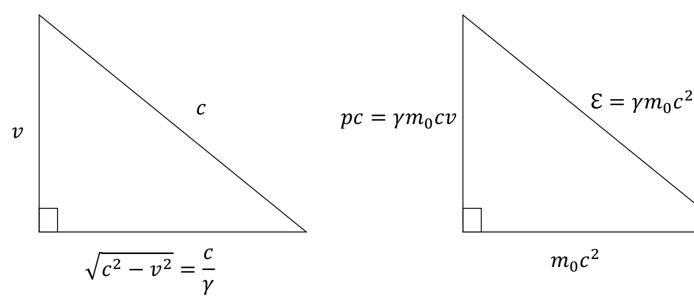

FIG. 3. Left: Velocity triangle with the lower side representing azimuthal propagation. Right: Energy-momentum triangle obtained from multiplication by $\gamma m_0 c$.

Consider the velocity triangle in Fig. 3 with hypotenuse c, one side representing average motion v, and a third side $\sqrt{c^2 - v^2}$ representing circulating motion perpendicular to the average motion. The Pythagorean theorem states that:

$$c^2 = v^2 + \left(\sqrt{c^2 - v^2}\right)^2 . \tag{60}$$

Simply multiply each side by $\gamma m_0 c$, with rest mass m_0, to obtain the energy-momentum-mass triangle. The Pythagorean theorem now yields:

$$(\gamma m_0 c^2)^2 = (\gamma m_0 cv)^2 + (m_0 c^2)^2 \tag{61}$$

which is equivalent to:

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{62}$$

This relationship is valid, averaging over the cyclical motion, even if the average motion is in the plane of circulation. [36]

Since the wave equation is Lorentz-invariant and also arises for many different types of waves, SR should be understood as a general property of waves rather than a property of spacetime [35, 36]. A unifying principle of SR, applicable to all waves, is this:

> Measurements made by differently moving observers using a particular type of wave are related by Lorentz transformations based on the characteristic wave speed.

An explanation of this principle using animations is available online. [42]

Hence the model of the vacuum as an ideal elastic solid existing in a Galilean physical spacetime (with wave measurements comprising Minkowski spacetime) is entirely consistent with the laws of SR. The reader may recall that Maxwell also derived the equations of electromagnetism with the assumption of Galilean spacetime. Curiously, the success of Maxwell's model is sometimes regarded as evidence that his assumptions were wrong! In the words of Robert Laughlin, "Relativity actually says nothing about the existence or nonexistence of matter pervading the universe, only that any such matter must have relativistic symmetry. It turns out that such matter exists." [43] Einstein's postulate of the constancy of the speed of light may be understood as a recognition that all of our measurements are made using waves (including particle-like or standing waves) whose characteristic propagation speed is the speed of light. The current definition of the "meter" guarantees a constant measured speed of light (even though we know that the actual speed of light varies in a gravitational field). [44]

Many researchers have proposed that stationary elementary particles consist of standing waves or "solitons" rather than point-like singularities. [45–51] The model described above is a simplification of these more realistic models.

Interpretation of SR as a property of matter rather than spacetime clarifies the analysis of relative motion. Although it is impossible to measure absolute velocity, it is possible to measure absolute acceleration. If an inertial observer detects relativistic changes to accelerated clocks and rulers, it is certain that those changes are real, and they are consistent with the wave nature of matter. Acceleration changes matter, not the spacetime in which the matter moves. Likewise, an accelerated observer should realize that changes seen in external inertial clocks and rulers are not real, but are due to changes in the co-accelerated clocks and rulers used for comparison. Poincaré's statement that "we have no means of knowing whether it is the magnitude or the instrument that has changed" [52] does not apply to accelerated reference frames.

3.4. Lagrangian and Hamiltonian

Now we construct a Lagrange density $\mathcal{L}$. Lagrange's equation of motion for a field variable ψ is

$$\partial_t \frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi\right]} + \sum_j \partial_j \frac{\partial \mathcal{L}}{\partial\left[\partial_j \psi\right]} - \frac{\partial \mathcal{L}}{\partial \psi} = 0\,. \tag{63}$$

Multiplying Eq. 44 by $\mathrm{i}\psi^\dagger$ yields:

$$\psi^\dagger \mathrm{i}\partial_t \psi + c\psi^\dagger \gamma^5 \boldsymbol{\sigma} \cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi \psi = 0, \tag{64}$$

Derivatives of $\psi^\dagger$ do not appear in this equation. Therefore we can construct a Lagrangian whose Euler-Lagrange equation has the simple form $\partial\mathcal{L}/\partial\psi^\dagger = 0$:

$$\mathcal{L} = \mathrm{i}\psi^\dagger \partial_t \psi + \psi^\dagger c\gamma^5 \boldsymbol{\sigma} \cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi \psi\,. \tag{65}$$

The imaginary part of the Lagrangian has no physical significance, so we may discard it:[53]

$$\mathcal{L} = \mathrm{Re}\left\{\mathrm{i}\psi^\dagger \partial_t \psi + \psi^\dagger c\gamma^5 \boldsymbol{\sigma} \cdot \mathrm{i}\nabla\psi - \psi^\dagger \chi \psi\right\}\,. \tag{66}$$

From here on the representation of physical quantities as the real part of complex quantities will be implicit. The associated Hamiltonian is:

$$\mathcal{H} = p_\psi \partial_t \psi - \mathcal{L} = -\psi^\dagger c\gamma^5 \boldsymbol{\sigma} \cdot \mathrm{i}\nabla\psi + \psi^\dagger \chi \psi\,. \tag{67}$$

If we had kept nonlinear terms in Eq. 21, then the hamiltonian would contain addition terms such as $(1/2)\mathbf{w}\cdot\psi^\dagger(\boldsymbol{\sigma}/2)\psi$ whose volume integral equals kinetic energy.

The Hamiltonian operator defined by $\mathrm{i}\partial_t\psi = H\psi$ is: [7]

$$H\psi = -c\gamma^5\boldsymbol{\sigma}\cdot\mathrm{i}\nabla\psi + \chi\psi\,. \tag{68}$$

In quantum mechanics, the hamiltonian represents energy density. We saw that for infinitesimal elastic plane waves, the quantities $\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi)$ and $-\mathrm{Re}(\psi^\dagger\mathrm{i}c\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi)$ are equal constants of the motion.

The Hamiltonian is a special case ($T^0_{\ 0}$) of the energy-momentum tensor:

$$T^\mu_{\ \nu} = \frac{\partial\mathcal{L}}{\partial\left[\partial_\mu\psi\right]}\partial_\nu\psi - \mathcal{L}\delta^\mu_{\ \nu}\,. \tag{69}$$

The conjugate momenta computed from the Lagrangian have the opposite sign of physical quantities. The dynamical (or wave) momentum density P_i is

$$P_i = -T^0_{\ i} = -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_i\psi = -\psi^\dagger\mathrm{i}\partial_i\psi\,. \tag{70}$$

The wave angular momentum density is likewise

$$\begin{aligned}\mathbf{L} &= -\frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]}\partial_{\boldsymbol{\varphi}}\psi = -\mathrm{i}\psi^\dagger\partial_{\boldsymbol{\varphi}}\psi = -\mathrm{i}\psi^\dagger\frac{\partial r_i}{\partial\boldsymbol{\varphi}}\partial_i\psi \\ &= -\mathbf{r}\times\psi^\dagger\mathrm{i}\nabla\psi = \mathbf{r}\times\mathbf{P}\,. \end{aligned}\tag{71}$$

These dynamical variables are consistent with those of quantum mechanics. For total momentum ($\mathbf{P}+\mathbf{p}$) and angular momentum ($\mathbf{L}+\mathbf{s}$), we must combine the wave and medium contributions:

$$\mathbf{P}+\mathbf{p} = -\psi^\dagger\mathrm{i}\nabla\psi + \frac{1}{2}\nabla\times\psi^\dagger\frac{\boldsymbol{\sigma}}{2}\psi; \tag{72}$$

$$\mathbf{L}+\mathbf{s} = -\mathbf{r}\times\psi^\dagger\mathrm{i}\nabla\psi + \psi^\dagger\frac{\boldsymbol{\sigma}}{2}\psi\,. \tag{73}$$

The angular momentum operator is equivalent to that of quantum mechanics. The addition of intrinsic momentum to the wave momentum makes the energy-momentum tensor symmetric, as required for general relativity [54–56].

If the wave function is an eigenfunction of the spin component s_z with total spin $\hbar/2$, then the wave function should be normalized to $\int_V \psi^\dagger\psi dV = \hbar$. However, it is customary to normalize the wave function to unity, so all operators should be modified to include a factor of $\hbar$:

$$H\psi = -c\gamma^5\boldsymbol{\sigma}\cdot\mathrm{i}\hbar\nabla\psi + \hbar\chi\psi \tag{74a}$$

$$\mathbf{P}+\mathbf{p} = -\psi^\dagger\mathrm{i}\hbar\nabla\psi + \frac{1}{2}\nabla\times\psi^\dagger\hbar\frac{\boldsymbol{\sigma}}{2}\psi \tag{74b}$$

$$\mathbf{L}+\mathbf{s} = -\mathbf{r}\times\psi^\dagger\mathrm{i}\hbar\nabla\psi + \psi^\dagger\hbar\frac{\boldsymbol{\sigma}}{2}\psi\,. \tag{74c}$$

The normalization procedure amounts to regarding quantum mechanical dynamics as a special case of classical physics dynamics. There is no difference in the interpretation of dynamical quantities, so we conclude that spin angular momentum in quantum mechanics has the same interpretation as it has in classical mechanics: it is the angular momentum of the medium in which the waves propagate. Experimental confirmation of spin angular momentum is therefore evidence for the existence of an aether. Students should be encouraged to consider whether or not this evidence is convincing.

3.5. Wave Interactions

Suppose we have two Dirac wave functions ψ_A and ψ_B, representing particle-like waves A and B. Adding the wave functions yields a total wave function ψ_T satisfying:

$$\begin{aligned}\psi_T^\dagger\boldsymbol{\sigma}\psi_T &= (\psi_A+\psi_B)^\dagger\boldsymbol{\sigma}(\psi_A+\psi_B) \\ &= \psi_A^\dagger\boldsymbol{\sigma}\psi_A + \psi_B^\dagger\boldsymbol{\sigma}\psi_B + \psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A\,. \end{aligned}\tag{75}$$

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [7]. This cancelation imposes a vector constraint on the wave functions:

$$\psi_A^\dagger \boldsymbol{\sigma} \psi_B + \psi_B^\dagger \boldsymbol{\sigma} \psi_A = 0\,. \tag{76}$$

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

$$\psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0\,. \tag{77}$$

For waves representing identical particles, this is the Pauli exclusion principle. Hence we can conclude that standing waves described by spin eigenfunctions are fermions.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing a phase shift at each point between the two wave functions. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Of course, this procedure is only valid if the particles interact weakly enough to remain distinguishable during the interaction. This limitation does not invalidate the basic premise that physical quantities are fully determined by the spin density field.

The phase shift (δ) is determined from the constraint:

$$\mathrm{Re}(\psi_A^\dagger \exp{(\mathrm{i}\delta)} \psi_{B'}) = 0, \tag{78}$$

or

$$\mathrm{Re}(\psi_A^\dagger \psi_{B'}) \cos\delta - \mathrm{Im}(\psi_A^\dagger \psi_{B'}) \sin\delta = 0\,. \tag{79}$$

This yields:

$$\tan\delta = \frac{\mathrm{Re}(\psi_A^\dagger \psi_{B'})}{\mathrm{Im}(\psi_A^\dagger \psi_{B'})}\,. \tag{80}$$

If we let $\psi_A^\dagger \psi_{B'} = g \exp \mathrm{i}\beta$, then:

$$\tan\delta = \cot\beta\,. \tag{81}$$

Therefore the phase angles are related by:

$$\delta = \frac{\pi}{2} - \beta \pm n\pi \tag{82}$$

where n is an integer. Note that δ is only unique within an arbitrary multiple of π. The $\pi/2$ phase shift is a constant, so we will ignore it while analyzing phase shifts of individual particles.

Suppose we start with two wave functions ψ_A and ψ_B, initially non-overlapping and normalized to one. We will assume that each particle is shifted by a phase that depends only on the other particle: $\psi_A = \psi_A' \exp{(\mathrm{i}\varphi_B)}$ and $\psi_B = \psi_B' \exp{(\mathrm{i}\varphi_A)}$, where the primed variables have zero interference. As they approach each other, the total wave function is $\psi_T = \psi_A' + \psi_B' = \exp{[-\mathrm{i}\varphi_B]}\psi_A + \exp{[-\mathrm{i}\varphi_A]}\psi_B$.

The phase-shifted wave functions are independent, so they satisfy the free-particle wave equation, e.g.

$$\hbar\partial_t(\exp{[-\mathrm{i}\varphi_B]}\psi_A) = -\mathrm{i}H_0 \exp{[-\mathrm{i}\varphi_B]}\psi_A. \tag{83}$$

We take ψ_A to be an electron wave function with free particle hamiltonian $H_0\psi_A = (-c\gamma^5\boldsymbol{\sigma}\cdot \mathrm{i}\hbar\nabla + m_e c^2\gamma^0)\psi_A$.

Expanding the Dirac equation for ψ_A yields:

$$(-\mathrm{i}\hbar[\partial_t\varphi_B] + \hbar\partial_t)\psi_A = \left(-c\gamma^5\boldsymbol{\sigma}\cdot(-\mathrm{i}\hbar[\nabla\varphi_B] + \hbar\nabla) - \mathrm{i}m_e c^2\gamma^0\right)\psi_A \tag{84}$$

or

$$\left(\hbar(\mathrm{i}\partial_t + [\partial_t\varphi_B]) + \hbar c\gamma^5\boldsymbol{\sigma}\cdot(\mathrm{i}\nabla + [\nabla\varphi_B]) - m_e c^2\gamma^0\right)\psi_A = 0\,, \tag{85}$$

where square brackets indicate that the derivatives inside only apply to the immediately following variables.

The modified Hamiltonian is:

$$H'\psi = \mathrm{i}\hbar\partial_t\psi = -\hbar[\partial_t\varphi_B]\psi + c\gamma^5\boldsymbol{\sigma}\cdot\hbar(-\mathrm{i}\nabla - [\nabla\varphi_B])\psi + m_e c^2\gamma^0\psi\,. \tag{86}$$

This looks similar to the "minimal substitution" method for introducing electromagnetic potentials with $q\Phi = -\hbar\partial_t\varphi_B$ and $q\mathbf{A} = \hbar\nabla\varphi_{\mathbf{B}}$. However, with those definitions (and the usual momentum operator $\mathbf{P} = -\mathrm{i}\nabla - \mathbf{qA}$) the electric field would be determined from $q\mathbf{E} = \hbar(\nabla\partial_t\varphi_B - \partial_t\nabla\varphi_B)$. If the space and time derivatives commute then we have no electric field in the linear theory. And if we keep the nonlinear terms that have hitherto been neglected, then the electric field would be a pure gradient since the time derivative of $\mathbf{A}$ is cancelled out. However, "minimal substitution" is not a physical principle.

A more logical approach is to note that the wave momentum density operator in Eq. 70 is unchanged. We want to compute its total (convective) time derivative using the operator $d_t = \partial_t + c\gamma^5\boldsymbol{\sigma}\cdot\nabla$:

$$d_t\mathbf{P} = \mathrm{Re}\left\{\hbar\psi_A^\dagger(\partial_t + c\gamma^5\boldsymbol{\sigma}\cdot\nabla)(-\mathrm{i}\nabla)\psi_A\right\}\,. \tag{87}$$

The independent phase-shifted wave function is unaffected:

$$\hbar\left\{\psi_A^\dagger\exp[\mathrm{i}\varphi_B](\partial_t + c\gamma^5\boldsymbol{\sigma}\cdot\nabla)(-\mathrm{i}\nabla)\exp[-\mathrm{i}\varphi_B]\psi_A\right\} + c.c. = 0\,. \tag{88}$$

The temporal derivatives yield:

$$\hbar\left\{\psi_A^\dagger\partial_t(-\mathrm{i}\nabla)\psi_A - \psi_A^\dagger[\partial_t\varphi_B]\nabla\psi_A + \psi_A^\dagger[\partial_t(-\nabla\varphi_B)]\psi_A + \psi_A^\dagger[\nabla\varphi_B\mathrm{i}\partial_t\varphi_B]\psi_A - \psi_A^\dagger[\nabla\varphi_B]\partial_t\psi_A\right\} + c.c. \tag{89}$$

To simplify analysis, we define the vector potential by $q\mathbf{A} \equiv -\hbar\nabla\varphi_B$, the charge density by $\rho_A \equiv q\psi_A^\dagger\psi_A$, and the current by $\mathbf{J}_A \equiv \psi_A^\dagger qc\gamma^5\boldsymbol{\sigma}\psi_A$. The first term above represents the partial derivative $\partial_t\mathbf{P}$. The fourth term is purely imaginary and cancelled by the complex conjugate. The other terms are:

$$-\hbar\partial_t\varphi_B\nabla(\rho_A/q) + 2\rho_A\partial_t\mathbf{A} + \mathbf{A}\partial_t\rho_A\,. \tag{90}$$

This is equivalent to:

$$-\hbar\nabla(\rho_A\partial_t\varphi_B) + \rho_A\partial_t\mathbf{A} + \mathbf{A}\partial_t\rho_A\,. \tag{91}$$

The spatial derivatives yield:

$$\begin{aligned}&\hbar\{\psi_A^\dagger c\gamma^5\boldsymbol{\sigma}\cdot\nabla(-\mathrm{i}\nabla)\psi_A - \psi_A^\dagger[c\gamma^5\boldsymbol{\sigma}\cdot\nabla\varphi_B]\nabla\psi_A + \psi_A^\dagger[c\gamma^5\boldsymbol{\sigma}\cdot\nabla(-\nabla\varphi_B)]\psi_A\\ &+\psi_A^\dagger[\nabla\varphi_B\mathrm{i}c\gamma^5\boldsymbol{\sigma}\cdot\nabla\varphi_B]\psi_A - \psi_A^\dagger[\nabla\varphi_B]c\gamma^5\boldsymbol{\sigma}\cdot\nabla\psi_A\} + c.c.\end{aligned} \tag{92}$$

The first term is the convective part of $d_t\mathbf{P}$. The fourth term is purely imaginary and cancelled by the complex conjugate. In terms of components, the three remaining terms represent:

$$(\partial_i J_{Aj})A_j + 2J_{Aj}\partial_j A_i + A_i\partial_j J_{Aj}\,. \tag{93}$$

This is equivalent to:

$$\partial_i(J_{Aj}A_j) + J_{Aj}(\partial_j A_i - \partial_i A_j) + \partial_j(A_i J_{Aj})\,. \tag{94}$$

Combining terms and solving for $d_t\mathbf{P}$ yields:

$$d_t\mathbf{P} = \hbar\nabla(\rho\partial_t\varphi_B) - \rho_A\partial_t\mathbf{A} - \mathbf{A}\partial_t\rho_A - \partial_i(J_{Aj}A_j) - J_{Aj}(\partial_j A_i - \partial_i A_j) - \partial_j(A_i J_{Aj})\,. \tag{95}$$

When integrated over space, the total derivatives will yield zero, assuming that the wave fields fall to zero sufficiently rapidly at infinity (otherwise we need to consider a radiated photon). We also assume that phase varies much more rapidly than charge density: $|\rho_A\partial_t\mathbf{A}| >> |\mathbf{A}\partial_t\rho_A|$. This is closely related to the assumption of incident plane waves in quantum mechanical calculations. This leaves:

$$d_t\mathbf{P} = -\rho_A\partial_t\mathbf{A} + \mathbf{J}_A\times(\nabla\times\mathbf{A})\,. \tag{96}$$

This is equivalent to the Lorentz force if:

$$\mathbf{E} = -\partial_t\mathbf{A}\,, \tag{97a}$$

$$\mathbf{B} = \nabla\times\mathbf{A}\,. \tag{97b}$$

This is the Weyl (or temporal) gauge. We can change the electric field to the usual form $\mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}'$ by performing a Helmholtz decomposition. Changing $\mathbf{A}$ to $\mathbf{A}'$ does not affect the magnetic field calculation.

Others have similarly identified the vector potential $\mathbf{A}$ as the gradient of a multivalued field.[57–59] The curl of the such gradients need not be identically zero. This interpretation is also consistent with Synge's "primitive quantization" in which Planck's constant h represents the action for a single wave cycle. [60]

Suppose the field ψ_B produces a phase shift on ψ_A that varies by some multiple of π along a closed path:

$$\oint \nabla\varphi_B \cdot d\boldsymbol{\ell} = -n\pi \tag{98}$$

for some integer n. Stoke's law yields quantization of magnetic flux:

$$\iint \mathbf{B}\cdot\hat{\mathbf{n}}\,dS = \oint \mathbf{A}\cdot d\ell = n\pi\frac{\hbar}{q} = n\frac{h}{2q}\,. \tag{99}$$

This classical quantization of magnetic flux is consistent with de Broglie's observation in a 1963 interview that "... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion." [61]

3.5.1. Electron Interactions

Alternatively, suppose that the interaction phase shift has the form:

$$\varphi_B = \frac{e^2}{4\pi\epsilon_0\hbar\omega}(m_\phi\phi - \omega t)\left(\int \frac{|\psi_B(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right) \tag{100}$$

with $\hbar\omega = 2m_e c^2$ (corresponding to the frequency of the real-valued vector field, assuming that the spinor field has frequency $m_e c^2/\hbar$). The azimuthal dependence $(m_\phi\phi - \omega t)$ is what one might expect for a spherical harmonic wave. The radial dependence is purely speculative, since we don't know the actual wave functions of isolated elementary particles.

The vector potential is then:

$$\mathbf{A} = -(\hbar/e)\nabla\varphi_B = \frac{e}{4\pi\epsilon_0\omega}\left(\frac{m_\phi}{r\sin\theta}\hat{\boldsymbol{\phi}}\int\frac{|\psi_B(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r' + (m_\phi\phi-\omega t)\int\frac{|\psi_B(\mathbf{r}',t)|^2(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\right) \tag{101}$$

The electric field is that of a negative charge distribution:

$$\mathbf{E} = -\partial_t\mathbf{A} = -\frac{e}{4\pi\epsilon_0}\int\frac{|\psi_B(\mathbf{r}',t)|^2(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r' \tag{102a}$$

$$\Phi = -\frac{e}{4\pi\epsilon_0}\int\frac{|\psi_B(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r'\,. \tag{102b}$$

The magnetic flux inside a circle of radius r centered at $\mathbf{r}' = 0$ in the plane $z = 0$ is:

$$\oint \mathbf{A}\cdot d\ell = \frac{e}{4\pi\epsilon_0\omega}\left(2\pi\frac{m_\phi}{\sin\theta}\int\frac{|\psi_B(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right)\,. \tag{103}$$

For large r we use the fact that $|\psi_B|^2$ is normalized to one and make the approximation:

$$\int\frac{|\psi_B(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r' \approx 1/r\,. \tag{104}$$

This amounts to treating the electron as a point-like particle. The magnetic flux is then:

$$\oint \mathbf{A}\cdot d\ell = \frac{m_\phi e}{2\epsilon_0\omega r} = \frac{m_\phi\mu_0\hbar e}{4m_e r}. \tag{105}$$

For $m_\phi = 1$, this is $\mu_0\hbar e/(4m_e r)$. For comparison, the electron spin dipole moment is $M \approx e\hbar/(2m_e)$ to first order with magnetic field:

$$\mathbf{B} = \frac{\mu_0 M}{4\pi r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \tag{106}$$

In the $z = 0$ plane, the field is entirely in the z-direction:

$$B_z(z=0) = -\frac{\mu_0 M}{4\pi r^3} \tag{107}$$

and its magnetic flux in the plane $z = 0$ is:

$$\iint \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \int -2\pi \frac{\mu_0 M}{4\pi r^3} r dr = \frac{\mu_0 M}{2r} = \mu_0 \hbar e/(4 m_e r), \tag{108}$$

in agreement with our calculation. Hence this choice of phase shift, consistent with interpretation of the real-valued wave function as a vector spherical harmonic, yields the correct relationship between the electron's electric field and magnetic flux.

3.5.2. Maxwell's Equations

The electromagnetic fields defined above are also subject to Maxwell's equations. The definitions of $\mathbf{E}$ and $\mathbf{B}$ imply Faraday's Law and Gauss' magnetic law:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}\,, \tag{109a}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{109b}$$

Gauss' electric law and Ampere's law define the charge and current densities (ρ_e and $\mathbf{J}$, respectively):

$$\nabla \cdot \mathbf{E} = -(\nabla \cdot \partial_t \mathbf{A} + \nabla^2 \Phi) \equiv \frac{\rho_e}{\epsilon_0}\,, \tag{110a}$$

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} &= \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}(\partial_t^2 \mathbf{A} + \partial_t \nabla \Phi) \\ &\equiv \mu_0 \mathbf{J}\,. \end{aligned} \tag{110b}$$

These definitions of charge and current densities are consistent with the continuity equation:

$$\partial_t \rho_e + \nabla \cdot \mathbf{J} = 0\,. \tag{111}$$

Hence particle-like waves in an elastic solid can behave like fermions, with electromagnetic potentials derived from phase shifts that result from wave interference.

3.5.3. Quantum Electrodynamics

It is customary in quantum mechanics textbooks to define $\bar{\psi} \equiv \psi^\dagger \gamma^0$, replace $\psi^\dagger$ with $\bar{\psi}\gamma^0$, and define the "4-vector" of matrices $\gamma^\mu \equiv (\gamma^0, \gamma^0\gamma^5\boldsymbol{\sigma})$. The 4-potential is $A_\mu = (\Phi, -\mathbf{A})$ and the 4-current $(\rho, \mathbf{J})$ is $J^\mu = q\bar{\psi}\gamma^\mu\psi$. These changes of variables are intended to make the theory look more "relativistic". It is also common to use "natural" units with $\mu_0 = \epsilon_0 = c = 1$. Using this notation with $\partial_\mu = (\partial_t, \nabla)$, the Lagrangian density for two interacting electrons is:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_B]\psi_B\,. \tag{112}$$

Separating the interaction of particle B yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu) - m_B]\psi_B - J^\mu A_\mu\,. \tag{113}$$

Since the Dirac equation is satisfied for each particle, this is equivalent to:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + J^\mu A_\mu - J^\mu A_\mu\,. \tag{114}$$

Relationships between potentials and sources are given in Eqs. 110. Assuming time-indepence with zero divergence of the vector potential and zero curl of the electric field, the sources become:

$$\frac{\rho_e}{\epsilon_0} = -\nabla^2 \Phi\,, \tag{115a}$$

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A})\,. \tag{115b}$$

Therefore:

$$J^\mu A_\mu = -\Phi\nabla^2\Phi - \mathbf{A}\cdot(\nabla\times\nabla\times\mathbf{A})\,. \tag{116}$$

According to Green's first identity:

$$-\int_V \Phi\nabla^2\Phi dV = \int (\nabla\Phi)^2 dV - \int_{\partial V} \Phi\mathbf{n}\cdot\nabla\Phi dS\,. \tag{117}$$

Similarly:

$$-\int_V \mathbf{A}\cdot(\nabla\times\nabla\times\mathbf{A})dV = -\int_V (\nabla\times\mathbf{A})^2 dV + \int_{\partial V} \mathbf{A}\times(\nabla\times\mathbf{A})dS\,. \tag{118}$$

Using the definitions of $\mathbf{E}$ and $\mathbf{B}$ while neglecting boundary integrals yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + (E^2 - B^2) - J^\mu A_\mu \tag{119}$$

This differs from the (non-quantized) Lagrangian density of quantum electrodynamics (QED) by a factor of 1/2 in front of $(E^2 - B^2)$. This difference is resolved by the fact that when varying the potentials A_μ, the source densities J^μ should be regarded as functions of A_μ in a manner similar to Eq. 13. However, it is conventional to vary the potentials independently of the source densities, yielding only half of the correct value. When computing variations of E^2 and B^2, both factors in E^2 (and B^2) are varied. To eliminate this double-counting and be consistent with independent variation of the potentials, a factor of 1/2 must be introduced:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \frac{1}{2}(E^2 - B^2) - J^\mu A_\mu \tag{120}$$

This is the Lagrangian density of non-quantized QED, in which a single charged fermion interacts with an electromagnetic field. Generalization to multiple particles requires a quantization procedure.

4. DISCUSSION

We have outlined a plausible path from the classical model of an elastic solid to quantum electrodynamics (QED). This interpretation of QED, and by extension the Standard Model, is that it represents a decomposition of the classical spin density field into interacting elementary particles. Others have also associated quantum mechanical behavior with waves in an elastic solid. [10, 62–65]

The Dirac equation was derived from a description of infinitesimal wave motion in an elastic solid. Unlike the non-relativistic Schödinger equation, the Dirac equation is fully relativistic and physically realistic. Each of the variables has a clear physical interpretation. In particular, spin angular momentum of elementary particles may be regarded as the angular momentum of the vacuum (or "aether" if you prefer that term). While this interpretation might be contested, it is nonsensical to say that the aether is undetectable.

With finite motion, nonlinear terms would be added to the linear wave equation. Nonlinearity is a possible reason for quantized amplitudes. Many researchers have attempted to quantize the Dirac equation by adding nonlinear terms. [45–49, 66–69] Particle-like nonlinear wave solutions are sometimes called "breathers" or "solitons."

Thomas Jefferson famously wrote that "Ignorance is preferable to error; and he is less remote from the truth who believes nothing, than he who believes what is wrong" [70]. The non-relativistic Schrödinger equation is obviously wrong, and is therefore a poor choice for introducing students to the wave nature of matter. Students should first be taught the physical basis for the Dirac equation, after which the Schrödinger equation may be derived as an approximation in order to simplify the mathematics (see e.g. [71]).

The model of the vacuum as an elastic solid also offers a good introduction to general relativity. Gravity, at least when weak, may be interpreted as ordinary refraction of waves toward regions whose wave speed is decreased by the presence of energy. [72–74] Wave speed in an elastic solid may likewise be decreased by stress-induced compression (increased inertial density). For example, twisting a rubber band induces a tension that tends to shorten it.

We showed how a model of stationary matter as standing waves gives rise to the de Broglie wavelength for moving particles, and gave a plausible explanation of magnetic flux quantization. Recent research has revealed that classical physical systems can reproduce other quantum phenomena as well. In particular, silicone droplets bouncing on a vibrating water tank can exhibit single-particle diffraction and interference, wave-like probability distributions, tunneling, quantized orbits, and orbital level splitting.[15–21] Students (and their teachers) should be aware that many quantum behaviors have analogues in classical physics.

5. CONCLUSIONS

This paper offers a new approach for introducing students to the wave nature of matter, based on a classical wave description of incompressible motion in an elastic solid. Unlike the Schrödinger equation, this approach to wave mechanics is fully relativistic and includes spin angular momentum. Spin density is the field whose curl is equal to twice the incompressible momentum density. The second-order wave equation is transformed into a first-order Dirac equation, and sample plane wave solutions are given. The classical spatial inversion operator differs from that of the Standard Model, but provides a simple interpretation of so-called parity-violating experiments. Standing wave solutions are naturally classified into two types, comparable to fermions and bosons, based on their angular quantum numbers. A model of stationary matter as circulating waves yields the relativistic energy-momentum equation for relativistic particles. A Lagrangian and Hamiltonian are constructed, from which the dynamical operators of relativistic quantum mechanics are derived. Wave interference gives rise to the Pauli exclusion principle and electromagnetic potentials, with suggested interpretations of magnetic flux quantization and the Coulomb potential. The Lagrangian density of single-fermion quantum electrodynamics is also given a classical physics interpretation. Hence classical wave theory can be a powerful educational tool for modeling of the wave properties of matter.

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