How to Solve a Quadratic Equation, Part 2

James F. Blinn

Microsoft Research

Last time, in the November/December 2005 issue, we derived an algorithm to find the real roots of the homogeneous quadratic equation

$$Ax^2 + 2Bxw + Cw^2 = 0$$

Because the equation is homogeneous, a root consists of an $[x, w]$ pair where any nonzero multiple represents the same root. We strove to find an algorithm that didn't blow up no matter what values of A, B, and C we were given, including various combinations of zeroes. At the end of the article I wrote the final algorithm in tabular form. For reference, Figure 1 shows it in a more algorithmic form. The big trick was that we found two possible formulations for each of the two homogeneous roots:

$$\begin{aligned}\begin{bmatrix}x_1 & w_1\end{bmatrix} &= \begin{bmatrix}-B+R & A\end{bmatrix} \text{ or } \begin{bmatrix}C & -B-R\end{bmatrix}\\ \begin{bmatrix}x_2 & w_2\end{bmatrix} &= \begin{bmatrix}-B-R & A\end{bmatrix} \text{ or } \begin{bmatrix}C & -B+R\end{bmatrix}\end{aligned} \tag{1}$$

$$\begin{aligned}
&\text{Discr} = B^2 - AC\\
&\text{if } (\text{Discr} < 0) \text{ exit(“no real roots”)}\\
&R = \sqrt{\text{Discr}}\\
&\text{if } (B > 0)\\
&\qquad \begin{bmatrix}x_1 & w_1\end{bmatrix} = \begin{bmatrix}C & -B-R\end{bmatrix}\\
&\qquad \begin{bmatrix}x_2 & w_2\end{bmatrix} = \begin{bmatrix}-B-R & A\end{bmatrix}\\
&\text{elseif } (B < 0)\\
&\qquad \begin{bmatrix}x_1 & w_1\end{bmatrix} = \begin{bmatrix}-B+R & A\end{bmatrix}\\
&\qquad \begin{bmatrix}x_2 & w_2\end{bmatrix} = \begin{bmatrix}C & -B+R\end{bmatrix}\\
&\text{elseif } (|A| \ge |C|)\\
&\qquad \begin{bmatrix}x_1 & w_1\end{bmatrix} = \begin{bmatrix}+R & A\end{bmatrix}\\
&\qquad \begin{bmatrix}x_2 & w_2\end{bmatrix} = \begin{bmatrix}-R & A\end{bmatrix}\\
&\text{else}\\
&\qquad \begin{bmatrix}x_1 & w_1\end{bmatrix} = \begin{bmatrix}C & -R\end{bmatrix}\\
&\qquad \begin{bmatrix}x_2 & w_2\end{bmatrix} = \begin{bmatrix}C & +R\end{bmatrix}
\end{aligned}$$

1 Homogeneous quadratic root algorithm.

The two formulations for $[x_1, w_1]$ are homogeneously equivalent; one is a scalar multiple of the other. Likewise for the two formulations of $[x_2, w_2]$. Having two formulas was convenient because it allowed us, for each root, to pick the formula that worked best numerically. Even though Figure 1 is pretty much the last word in quadratic solution stability, I'm still not done playing with this problem. My main purpose, though, is to gain some insights and establish some techniques that will help us in future articles where we will ramp up to cubic and quartic homogeneous equations. We'll start by trying to figure out what the existence of two quadratic solution formulations really means.

Derivation of quadratic formula

Let's review the derivation of the quadratic formula by starting with a nonhomogeneous version of the equation:

$$Ax^2 + 2Bx + C = 0$$

The standard trick is to apply a coordinate translation to the parameter by the substitution

$$x = \tilde{x} - \frac{B}{A}$$

This is carefully engineered so that it makes the linear term go away, leaving us with a much simpler quadratic in $\tilde{x}$:

$$A\tilde{x}^2 + \frac{(AC - B^2)}{A} = 0$$

We can easily solve this to get $\tilde{x}$:

$$\tilde{x} = \pm\frac{\sqrt{B^2 - AC}}{A}$$

and then apply the coordinate transformation to get the answer, x, in the original coordinate space:

$$x = \frac{\pm\sqrt{B^2 - AC}}{A} - \frac{B}{A}$$

Homogenizing the problem

Now let's see how this looks when we switch to a homogeneous quadratic and, further, write it in matrix notation. The quadratic equation is

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \tag{2}$$

The parameter-space coordinate transformation is

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B/A & 1 \end{bmatrix} \tag{3}$$

Those of you who are experienced in homogeneous transformation matrices can recognize this as a 1D version of a standard 3D homogeneous translation matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

We can clear fractions from Equation 3 by homogeneously multiplying the whole matrix by A. With some foresight, though, I'm going to do a bit more than that. I am going to set the top left element back to 1 giving

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & A \end{bmatrix} \tag{4}$$

We can see that this still works to get rid of the linear term by plugging the transformation (Equation 4) into the quadratic (Equation 2), giving us

$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & A \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} 1 & -B \\ 0 & A \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} =$$
$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & A\left(AC - B^2\right) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = 0$$

Our foresight has paid off by giving a result that has a common factor of A, which we can homogeneously throw out. (We'll deal with the case $A = 0$ later.) This leaves us with the easily solvable

$$\tilde{x}^2 + \left(AC - B^2\right)\tilde{w}^2 = 0 \tag{5}$$

which has the two solutions

$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{B^2 - AC} & 1 \end{bmatrix}$$

Finally, to go back to original parameter space, we apply the transformation matrix (Equation 4):

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \pm\sqrt{B^2 - AC} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -B & A \end{bmatrix}$$
$$= \begin{bmatrix} -B \pm \sqrt{B^2 - AC} & A \end{bmatrix}$$

This gives us the first of the two root choices in Equation 1.

The other homogeneous solution

But what happens when $A = 0$ (or, nearly as numerically disastrous, when A is quite small compared with B?). The 2×2 transformation is singular (or at least ill conditioned). In that case, we can use the following:

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -C & B \end{bmatrix} \tag{6}$$

This transformation effectively reverses the polynomial, swapping A and C, before applying Equation 4 (while encoding all these transforms in the matrix for later unraveling). Plugging Equation 6 into Equation 2, we get

$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C\left(AC - B^2\right) \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{w} \end{bmatrix} = 0$$

Tossing out the common factor of C gives us exactly the same base quadratic to solve, Equation 5, so the transformed answer is again

$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{B^2 - AC} & 1 \end{bmatrix}$$

Applying Equation 6 to go back to original parameter space gives the other solution from Equation 1:

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \pm\sqrt{B^2 - AC} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ C & -B \end{bmatrix}$$
$$= \begin{bmatrix} C & -B \pm \sqrt{B^2 - AC} \end{bmatrix}$$

A general solution

So what do we do if both A and C are zero (or small)? Are there any other transformations that we can use? Let's see by writing the general parameter-space transformation matrix as

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} \begin{bmatrix} t & u \\ s & v \end{bmatrix} \tag{7}$$

Now transform the coefficient matrix by applying this to Equation 2:

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{bmatrix} = \begin{bmatrix} t & u \\ s & v \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} t & s \\ u & v \end{bmatrix}$$
$$= \begin{bmatrix} t^2A + 2tuB + u^2C & tsA + \left(us + tv\right)B + uvC \\ tsA + \left(us + tv\right)B + uvC & s^2A + 2svB + v^2C \end{bmatrix}$$

So we have

$$\tilde{A} = t^2A + 2tuB + u^2C$$
$$\tilde{B} = stA + \left(su + tv\right)B + uvC$$
$$\tilde{C} = s^2A + 2svB + v^2C \tag{8}$$

We want to pick s, t, u, v to make $\tilde{B} = 0$. As you might expect, a whole bunch of matrices (choices of s, t, u, v) can make this happen. How can we characterize them? First, pick any two values for t and u (this selection will effectively parameterize our class of transformations). Next, rewrite the expression for $\tilde{B}$ from Equation 8 as

$$(tA+uB)s+(tB+uC)v=0$$

An appropriate choice for s and v will be

$$s=-(tB+uC)$$
$$v=(tA+uB)$$

So any transformation of the form

$$\begin{bmatrix} t & u \\ -tB-uC & tA+uB \end{bmatrix}$$

will result in a $\tilde{B}$ coefficient of zero. In fact, we can scale either of the two rows $[t, u]$ and $[s, v]$ by any arbitrary nonzero factor and it will still work. We will only get into trouble if this matrix is singular, which will happen if the determinant is zero:

$$\begin{aligned} 0 &= \det\begin{bmatrix} t & u \\ -tB-uC & tA+uB \end{bmatrix} \\ &= t^2A+2tuB+u^2C \\ &= \tilde{A} \end{aligned}$$

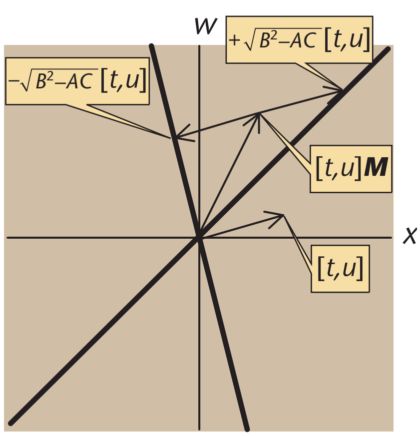
So the only time this won't work is if the (t, u) we pick is already a root of the quadratic.

Now let's look at the value of $\tilde{C}$:

$$\begin{aligned} \tilde{C} &= s^2A+2svB+v^2C \\ &= (tB+uC)^2A-2(tB+uC)(tA+uB)B+(tA+uB)^2C \end{aligned}$$

Expanding this out and doing some obvious factoring we get

$$\begin{aligned} \tilde{C} &= (t^2A+2tuB+u^2C)(AC-B^2) \\ &= \tilde{A}(AC-B^2) \end{aligned}$$



2 Geometric interpretation of Equation 10.

This means that, no matter what (t, u) we pick, the transformed coefficient matrix is

$$\begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{B} & \tilde{C} \end{bmatrix} = \tilde{A}\begin{bmatrix} 1 & 0 \\ 0 & AC-B^2 \end{bmatrix}$$

and the solutions will be

$$\begin{bmatrix} \tilde{x} & \tilde{w} \end{bmatrix} = \begin{bmatrix} \pm\sqrt{B^2-AC} & 1 \end{bmatrix}$$

Transforming back to the original coordinate system gives us

$$\begin{aligned} \begin{bmatrix} x & w \end{bmatrix} &= \begin{bmatrix} \pm\sqrt{B^2-AC} & 1 \end{bmatrix}\begin{bmatrix} t & u \\ -tB-uC & tA+uB \end{bmatrix} \\ &= t\begin{bmatrix} -B\pm\sqrt{B^2-AC} & A \end{bmatrix} \\ &\quad -u\begin{bmatrix} C & -B\mp\sqrt{B^2-AC} \end{bmatrix} \end{aligned} \tag{9}$$

These are just the two solutions from Equation 1, blended by t and $-u$. So okay, we haven't uncovered anything radically new here, but at least we've investigated all possible transformations that make $\tilde{B} = 0$. (When we go to cubic and quartic polynomials, this question will become meatier.) Now let's see what Equation 9 looks like geometrically.

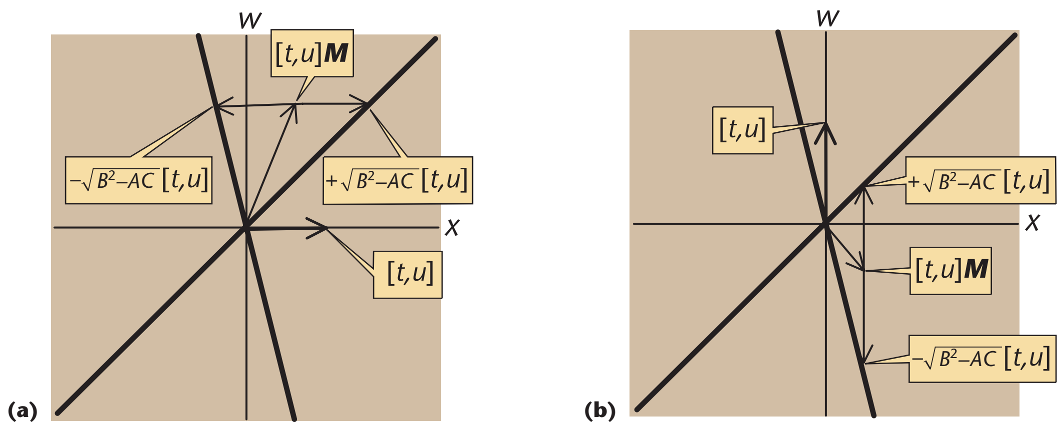
Geometric interpretation

We can tinker with Equation 9 a bit to write $[x, w]$ as

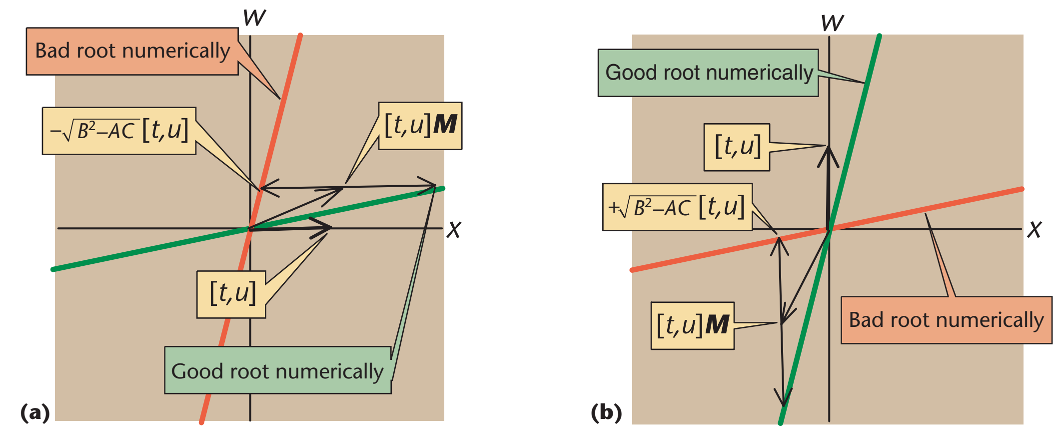
$$\begin{aligned} \begin{bmatrix} x & w \end{bmatrix} &= \begin{bmatrix} -tB-uC & +tA+uB \end{bmatrix} \pm\sqrt{B^2-AC}\begin{bmatrix} t & u \end{bmatrix} \\ &= \begin{bmatrix} t & u \end{bmatrix}\underbrace{\begin{bmatrix} -B & A \\ -C & B \end{bmatrix}}_{\boldsymbol{M}} \pm\sqrt{B^2-AC}\begin{bmatrix} t & u \end{bmatrix} \end{aligned} \tag{10}$$

Remember that $[t, u]$ can be any vector. The only restriction is that it cannot itself be a root of the quadratic. So what's going on here? We take an arbitrary vector $[t, u]$, transform it by the matrix $\boldsymbol{M}$, and then add and subtract the scalar $\sqrt{B^2-AC}$ times the original vector $[t, u]$. Figure 2 illustrates this. It shows a plot of x, w space with the two dark lines representing the two roots of the quadratic (different points on each line are just homogeneous scales of each other).

Note the following geometric relationships. First, the double-ended arrow (representing plus and minus $\sqrt{B^2-AC}$ times $[t, u]$) is parallel to $[t, u]$. Second, the vector $[t, u]\boldsymbol{M}$ points halfway along it. You can enhance your intuition by imagining an interactive program that lets you grab $[t, u]$ and drag it around, having $[t, u]\boldsymbol{M}$ automatically calculated and plotted. You would find that as you rotate $[t, u]$ around the origin, the vector $[t, u]\boldsymbol{M}$ rotates in the opposite direction. Whenever $[t, u]$ crosses a root, the vector $[t, u]\boldsymbol{M}$ also crosses that root, either pointing in the same direction as



3 (a) The case where [*t*, *u*] = [1, 0]. (b) The case where [*t*, *u*] = [0, 1].



4 A more difficult case where (a) [*t*, *u*] = [1, 0] and (b) [*t*, *u*] = [0, 1].

$[t, u]$ or in the opposite direction. In other words, the roots of the quadratic are the same as the eigenvectors of $\boldsymbol{M}$.

The transformation used in Equation 4 is just the special case where $[t, u] = [1, 0]$ and looks like Figure 3a. The transformation used in Equation 6 has $[t, u] = [0, 1]$ and looks like Figure 3b. Now we can see why the small-root–large-root situation gives us headaches. Figure 4 shows a mild form of the problem for visualization purposes. In Figure 4a you can see that the vector $[t, u]\boldsymbol{M}$ and the vector $-\sqrt{B^2 - AC}[t, u]$ are almost antiparallel to each other; their vector sum will be numerical noise. In Figure 4b you can see that the vector $[t, u]\boldsymbol{M}$ and the vector $+\sqrt{B^2 - AC}[t, u]$ are almost antiparallel to each other and their vector sum is noise. Also, note how each diagram has the good and bad numerical roots swapped.

An idea

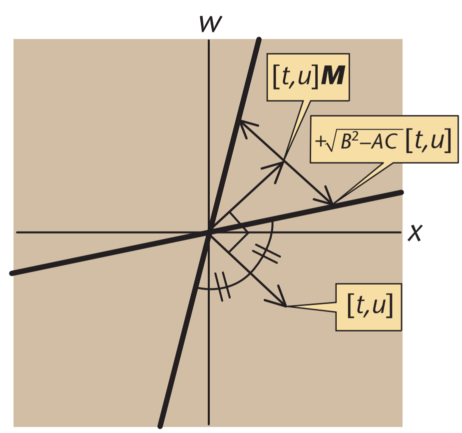
The transformations in Equations 3 and 6 are both shearing transformations. If the values of $|B| \gg |A|$ or $|B| \gg |C|$, then these matrices are horrible numerically. They are nearly singular and it serves us right that we get into trouble using them. Wouldn't it be nice if we could find a single, well-conditioned matrix that works for both roots? To avoid problems caused by adding almost-antiparallel vectors, we would really like to find a $[t, u]$ so that the vector $[t, u]\boldsymbol{M}$ is perpendicular to it. This is easily done by solving for $[t, u]$ in

$$
\begin{aligned}
\begin{bmatrix} t & u \end{bmatrix} \cdot \left\{ \begin{bmatrix} t & u \end{bmatrix} \boldsymbol{M} \right\} &= \\
\begin{bmatrix} t & u \end{bmatrix} \cdot \begin{bmatrix} -tB - uC & +tA + uB \end{bmatrix} &= \\
-t^2 B + tu\left(A - C\right) + u^2 B &= 0
\end{aligned}
$$

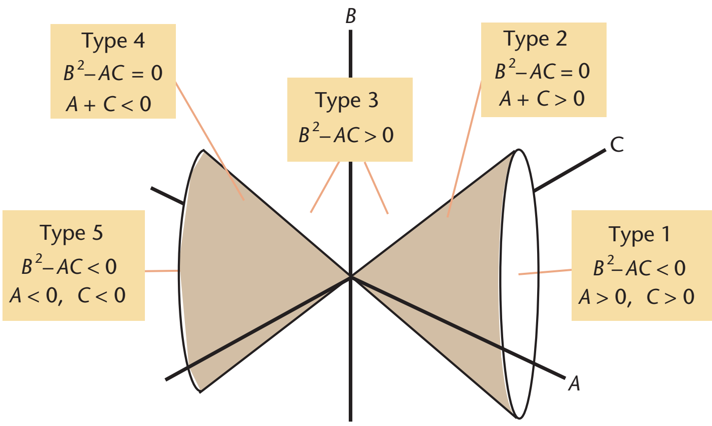
which gives

$$
\begin{bmatrix} t & u \end{bmatrix} = \begin{bmatrix} \left(C - A\right) \pm \sqrt{\left(C - A\right)^2 + 4B^2} & -2B \end{bmatrix}
$$

This gives the situation in Figure 5. Note that $[t, u]$ and $[t, u]\boldsymbol{M}$ both point halfway between the roots. Having these two vectors perpendicular (that is, having the two



5 Picking [*t*, *u*] to be perpendicular to [*t*, *u*]***M***.



6 **The space of possible quadratic coefficients.**

rows of the 2×2 transformation perpendicular) means just one thing: the 2×2 transformation looks a lot like a rotation in $[x, w]$ space. This seems much nicer than a nasty old shear.

Sadly, this does not seem to help us much. Some numerical experiments show that the algorithm in Figure 1 still wins. This is because our new rotation-based algorithm spreads its numerical errors equally to each root. The Figure 1 algorithm, for each choice of $[t, u]$, places all its accuracy into one root while doing badly with the other. But because it uses two transforms it gets the best of both worlds.

But this exercise is not wasted. It raises an important question with which I will close. We are playing around with transformations of parameter space represented by Equation 7. This transformation implies a transformation of the quadratic coefficients $[A, B, C]$ shown in Equation 8, which I will write in matrix form:

$$\begin{bmatrix}\tilde{A}\\ \tilde{B}\\ \tilde{C}\end{bmatrix}=\begin{bmatrix}t^2 & 2tu & u^2\\ ts & tv+us & uv\\ s^2 & 2sv & v^2\end{bmatrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix} \tag{11}$$

So our final goal is to get some intuition about the relationship between the transformation in Equation 7 and the transformation in Equation 11.

Relating spaces

Let's begin by looking at the catalog of all possible quadratics in Figures 6 and 7. Figure 6 contains five regions in the $[A, B, C]$ space, represented by the five examples in Figure 7. The transform in Equation 7 operates on the $[x, w]$ space of Figure 7. It can rotate, scale, or shear the shape of the function, but (as long as it's nonsingular) it cannot change the number of roots or change a positive definite matrix (rightmost) to a negative definite matrix (leftmost). The corresponding transform in Equation 11 operates on points in Figure 6, but it cannot move a point from one of the numbered regions into another one. For example, any point on the cone represents a quadratic with a double root. A coordinate transformation cannot change this property, so a transformation of a point on the cone must stay on the cone. On the other hand, for any two points within a particular numbered region, there will always exist a transformation that connects them.

Now let's see what happens if we transform the quadratic by a rotation in parameter space. We have

$$\begin{bmatrix}t & u\\ s & v\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Plugging these values for s, t, u, v into Equation 11 gives

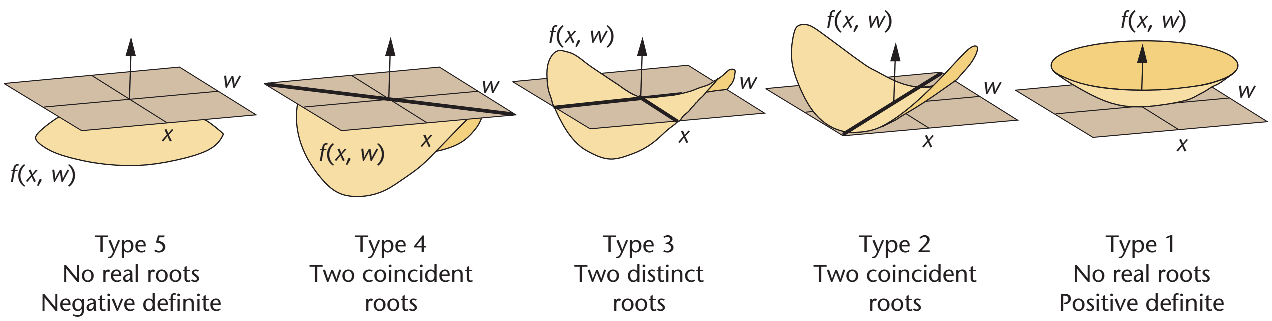
$$\begin{bmatrix}\tilde{A}\\ \tilde{B}\\ \tilde{C}\end{bmatrix}=\begin{bmatrix}\cos^2\theta & 2\cos\theta\sin\theta & \sin^2\theta\\ -\cos\theta\sin\theta & \cos^2\theta-\sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & -2\sin\theta\cos\theta & \cos^2\theta\end{bmatrix}\begin{bmatrix}A\\ B\\ C\end{bmatrix} \tag{12}$$

This 3×3 matrix is itself almost a rotation matrix in 3D. The vector $[A, B, C] = [1, 0, 1]$ remains unchanged upon applying this matrix, so it's like an axis of rotation. The glitch is that the cone in Figure 6 isn't a circular cone; it has an elliptical cross section. So the transformation in Equation 12 is a sort-of squashed rotation that keeps cone points on the cone.

We can make what's going on more obvious by writing the original quadratic in polar coordinates. If we define

$$\begin{bmatrix}x & w\end{bmatrix}=\begin{bmatrix}r\cos\alpha & r\sin\alpha\end{bmatrix}$$

the quadratic becomes



7 **Possible types of quadratic functions.**

$$
\begin{aligned}
&Ax^2+2Bxw+Cw^2\\
&=A\left(r\cos\alpha\right)^2+2B\left(r^2\cos\alpha\sin\alpha\right)+C\left(r\sin\alpha\right)^2\\
&=r^2\left(A\cos^2\alpha+B\left(2\cos\alpha\sin\alpha\right)+C\sin^2\alpha\right)
\end{aligned}
$$

If we now define new values D and E such that

$$
\begin{aligned}
A&=D+E\\
C&=D-E
\end{aligned}
$$

we have

$$
\begin{aligned}
&A\cos^2\alpha+B(2\cos\alpha\ \sin\alpha)+C\sin^2\alpha\\
&=D+E(\cos^2\alpha-\sin^2\alpha)+B(2\cos\alpha\ \sin\alpha)
\end{aligned}
$$

Now applying a trigonometric double angle identity gives us the net result:

$$Ax^2+2Bxw+Cw^2=r^2(D+E\cos(2\alpha)+B\sin(2\alpha))$$

So we've turned the polynomial into a biased sine wave. (Imagine surrounding the plots of Figure 7 with a unit radius cylinder. The sine wave is the intersection of the function and the cylinder.) This new coordinate system $[D, B, E]$ is a more natural way to represent the quadratic's properties. For example, the condition of having real roots is just that the amplitude $\sqrt{E^2+B^2}$ is greater than the magnitude of the bias $|D|$. Equivalently, the discriminant of the quadratic becomes

$$B^2-AC=B^2-D^2+E^2=0$$

This is the equation of a circular cone around the D axis. Now writing the new coordinates as

$$
\begin{bmatrix}1&0&1\\0&1&0\\1&0&-1\end{bmatrix}\begin{bmatrix}D\\B\\E\end{bmatrix}=\begin{bmatrix}A\\B\\C\end{bmatrix}
$$

and plugging into Equation 12 gives us

$$
\begin{bmatrix}\tilde{D}\\\tilde{B}\\\tilde{E}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos^2\theta-\sin^2\theta&-2\cos\theta\sin\theta\\0&2\sin\theta\cos\theta&\cos^2\theta-\sin^2\theta\end{bmatrix}\begin{bmatrix}D\\B\\E\end{bmatrix}
$$

Again, with the double-angle formulas we have

$$
\begin{bmatrix}\tilde{D}\\\tilde{B}\\\tilde{E}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos 2\theta&-\sin 2\theta\\0&\sin 2\theta&\cos 2\theta\end{bmatrix}\begin{bmatrix}D\\B\\E\end{bmatrix}
$$

In other words, rotating the function in 2D $[x, w]$ parameter space by the angle θ rotates the quadratic coefficients in 3D $[D, B, E]$ space by 2θ around the D axis. I think that's pretty neat. But wait 'til you see what happens when you do this analysis with cubic polynomials. ■

Readers may contact Jim Blinn at blinn@microsoft.com.