Quartic Discriminants and Tensor Invariants

James F. Blinn

Microsoft Research

Hello again. If you haven't noticed, I've been spending most of the last year doing things other than writing this column. I've primarily been writing up a brain dump on tensor diagrams for a course given at Siggraph 2001. A reprise of the course has been accepted for the Siggraph 2002 conference, so if you're interested, stop by. In addition, I have been gathering my latest batch of columns to be published in a new compilation book by Morgan Kaufmann in time for Siggraph 2002. This time, however, I couldn't resist the temptation to do major surgery on most of them, including ideas that I thought of or found out about after their publication in *IEEE Computer Graphics and Applications*. This column presents a few of these.

The quartic discriminant

In my November/December 2000 and January/February 2001 *CG&A* columns, I talked about calculating discriminants of polynomials. The discussion here extends those columns, but I'll try to give a brief recap if you haven't seen them.

The discriminant is a function of the coefficients that indicates if the polynomial has any double roots. In other words, the discriminant being zero tells us that both the function and its derivative are zero at the same parameter value. Quadratic and cubic discriminants are moderately simple, but the discriminant of a quartic

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$$

or more generally, the discriminant of a homogeneous quartic

$$f(x,w) = Ax^4 + 4Bx^3w + 6Cx^2w^2 + 4Dxw^3 + Ew^4 = 0 \qquad (1)$$

is considerably more complicated. (With some foresight, I've built some constant factors into the coefficients and given them new uppercase names.) Several Web references give explicit formulas for the discriminant that, in our notation, look like the incredibly gaudy

$$\begin{aligned}\Delta_4 = {} & A^3E^3 - 12A^2BDE^2 - 18A^2C^2E^2 \\ & + 54A^2CD^2E - 27A^2D^4 + 54AB^2CE^2 \\ & - 6AB^2D^2E - 180ABC^2DE + 108ABCD^3 \\ & - 54AC^3D^2 + 81AC^4E - 27B^4E^2 \\ & + 108B^3CDE - 64B^3D^3 - 54B^2C^3E \\ & + 36B^2C^2D^2\end{aligned}$$

Simpler algebra

You can write this somewhat more simply using resultants. We first define

$$\begin{aligned}\delta_1 &= AC - B^2 & \delta_4 &= BE - CD \\ \delta_2 &= AD - BC & \delta_5 &= CE - D^2 \\ \delta_3 &= BD - C^2 & \delta_0 &= AE - BD\end{aligned}$$

In my January/February 2001 column, I showed that the discriminant equals

$$\Delta_4 = \det\begin{bmatrix} 3\delta_1 & 3\delta_2 & \delta_0 \\ 3\delta_2 & 9\delta_3 + \delta_0 & 3\delta_4 \\ \delta_0 & 3\delta_4 & 3\delta_5 \end{bmatrix}$$

This is pretty but not pretty enough.

Simpler simpler algebra

Another representation of the discriminant of a quartic exists that's even better. It's buried in some 100-year-old lectures by David Hilbert, reprinted recently in Hilbert's *Theory of Algebraic Invariants* (Cambridge University Press, 1993, pp. 72, 74). Hilbert defined two quantities that, translated into our terminology, are

$$\begin{aligned} I_2 &= AE - 4BD + 3C^2 \\ I_3 &= ACE - AD^2 - B^2E + 2BCD - C^3 \end{aligned}$$

Then the quartic discriminant happens to be

$$\Delta_4 = 27(I_3)^2 - (I_2)^3$$

You can verify this for yourself by simple substitution. I won't wait ….

I won't wait because I now have a simpler way to write this using tensor diagram notation.

Tensor diagram review

First, a quick review of tensor diagrams. To make a tensor, we arrange the coefficients of *f* into a $2\times2\times2\times2$ hypercube of coefficients. In writing expressions involving such 4-index quantities, about the best we can do using conventional matrix notation is to use a 2×2 matrix of 2×2 matrices:

$$f(x,w)=\begin{bmatrix}x & w\end{bmatrix}\left\{\begin{bmatrix}x & w\end{bmatrix}\begin{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix} & \begin{bmatrix}B & C\\ C & D\end{bmatrix}\\ \begin{bmatrix}B & C\\ C & D\end{bmatrix} & \begin{bmatrix}C & D\\ D & E\end{bmatrix}\end{bmatrix}\begin{bmatrix}x\\ w\end{bmatrix}\right\}\begin{bmatrix}x\\ w\end{bmatrix}$$

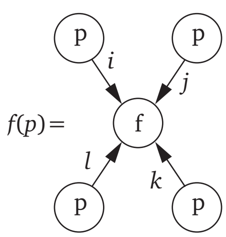
A better way is to use Einstein index notation (EIN). First, we give a new name to the parameter vector and its elements (here superscripts are indices rather than exponents):

$$\mathbf{p}=[x,w]=[p^0,p^1]$$

Then the EIN for the function is

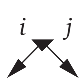
$$f(p)=p^i p^j p^k p^l f_{ijkl}$$

The tensor diagram simply draws each tensor (p and f) as a node, and draws each summed-over index as an arc between the nodes.



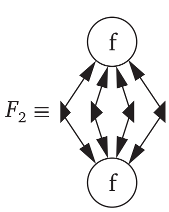
Finally, we define a constant matrix epsilon whose diagram is

$$\varepsilon^{ij}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=$$

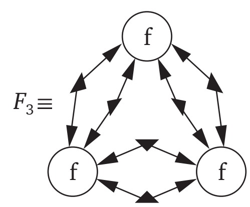


Tensor diagram discriminants

Now let's see how to write the discriminant as a tensor diagram. The two simplest diagrams that you can form from 4-arc *f* nodes and epsilon nodes are



and



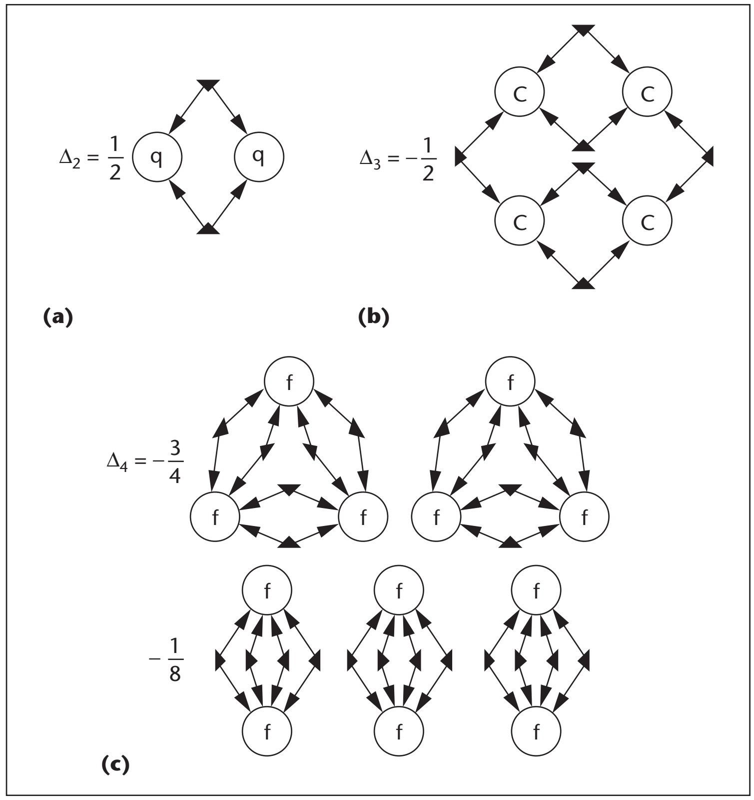
Using techniques similar to those in my March/April 2001 column, I've been able to evaluate these diagrams and verify that

$$\begin{aligned}F_2&=2AE-8BD+6C^2\\&=2I_2\\F_3&=-6ACE+6AD^2+6B^2E-12BCD+6C^3\\&=-6I_3\end{aligned}\qquad(2)$$

Wow … cool! Hilbert's invariants match up with the two simplest possible tensor diagrams! Some fiddling with constants gives us

$$\begin{aligned}\Delta_4&=27\left(\frac{F_3}{-6}\right)^2-\left(\frac{F_2}{2}\right)^3\\&=\tfrac{1}{8}\left(6(F_3)^2-F_2^{\,3}\right)\end{aligned}$$

A family portrait of the tensor diagrams for the discriminants of polynomials of order 2, 3, and 4 appears in Figure 1.

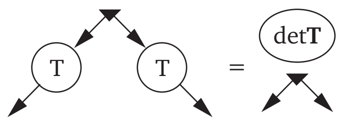


1 Tensor diagrams for polynomial discriminants.

Tensor invariants

The discriminant of a polynomial is an example of an *invariant* quantity. When you calculate such a quantity for a polynomial its sign will remain unchanged if the polynomial is transformed parametrically by a linear transformation matrix ***T***. This makes sense because the number and multiplicity of roots of a polynomial don't change under parameter transformations like translating or scaling.

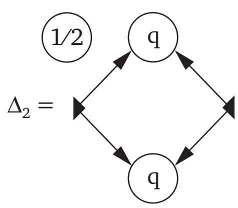
Tensor diagrams are particularly useful to express invariant quantities because of the following identity:



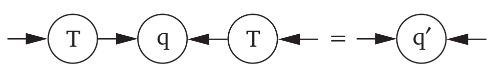
We can easily verify this by explicit calculation:

$$\mathbf{T}\varepsilon\mathbf{T}^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
$$= \begin{bmatrix} 0 & ad-bc \\ bc-ad & 0 \end{bmatrix}$$
$$= (ad-bc)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

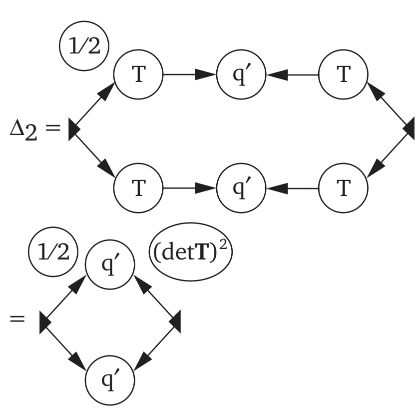
Now, let's apply this to the simplest of our discriminants. The quadratic discriminant is



We now do a parameter transformation on **q**. I showed in the original article in the January/February 2001 issue that the diagram notation of this is



Putting this into our discriminant equation and applying our identity gives



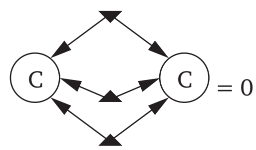
In other words,

$$\text{discr}\,\mathbf{q} = (\det\mathbf{T})^2\,\text{discr}\,\mathbf{q}'$$

As long as we don't do anything silly, like transform by a singular matrix, the sign of the discriminant of a quadratic doesn't change under coordinate transformation. This seems pretty obvious, but there's a bigger idea lurking in it. The big punch line here is that any diagram made up of a collection of polynomial nodes glued together with the appropriate number of epsilon nodes will represent a transformationally invariant quantity.

Actually, the invariant quantity will be multiplied by det **T** raised to a power equal to the number of epsilons in the diagram. If the diagram has an even number of epsilons, we multiply the transformed invariant by a positive number. The sign, or the fact of its being zero, is what remains unmodified. An odd number of epsilons implies multiplying by a nonzero number (could be plus or minus). We simply preserve the zeroness of the invariant.

You can imagine any number of diagrams formed in this way. Each of them represents some invariant property under parameter transformation. However, many of them will be uninteresting. For example, you can show that the following diagram for a cubic polynomial is identically zero:



Hilbert's book is all about some rather complicated algebraic rules for generating invariant quantities. We can do this much more simply with tensor diagrams. For example, we can tell that all the discriminants in Figure 1 are invariants simply because we can write them as tensor diagrams.

2DH diagrams

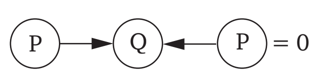
Adding a dimension moves us from the world of homogeneous polynomials—which I think of as 1D-homogeneous (or 1DH) geometry—to 2D-homogeneous (2DH) geometry (curves in the projective plane).

Points on a second order, or quadratic, curve (typically a conic section) satisfy the equation written in various notations as

$$\begin{aligned} &Ax^2 + 2Bxy + Cy^2 \\ &\quad + 2Dxw + 2Eyw \\ &\qquad + Fw^2 = 0 \end{aligned}$$

$$\begin{bmatrix} x & y & w \end{bmatrix}\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

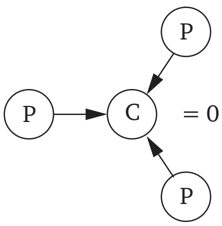
$$\mathbf{P}\cdot\mathbf{Q}\cdot\mathbf{P}^T = 0$$



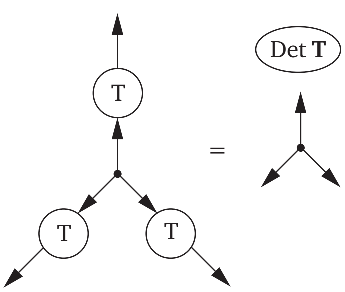
Points on a cubic curve satisfy the equation

$$
\begin{aligned}
&Ax^3+3Bx^2y+3Cxy^2+Dy^3\\
&\quad+3Ex^2w+6Fxyw+3Gy^2w\\
&\qquad+3Hxw^2+3Jyw^2\\
&\qquad\quad+Kw^3=0
\end{aligned}
$$

$$
\begin{bmatrix} x & y & w \end{bmatrix}
\left\{
\begin{bmatrix} x & y & w \end{bmatrix}
\begin{bmatrix}
\begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} \\
\begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} \\
\begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix}
\end{bmatrix}
\right\}
\begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$



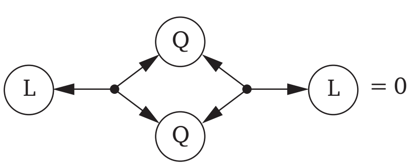
The 2DH epsilon is the same as the 3D Levi-Civita epsilon used in theoretical physics. Its diagram has three arcs leading to it, and it has a similar identity involving transformation matrices that we had in 1DH:



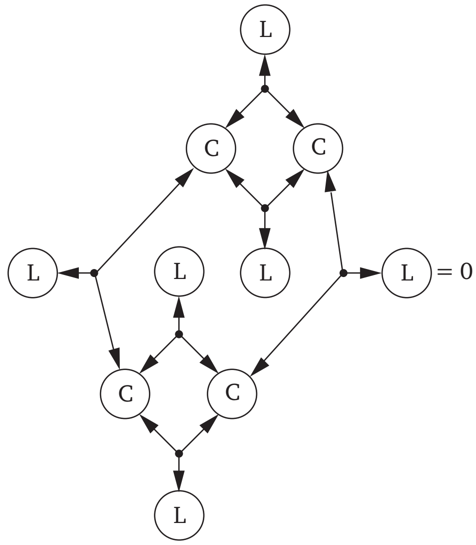
This means that any 2DH tensor diagram consisting of polynomial nodes and epsilons represents a transformational invariant.

Tangency

Given the diagrams for the discriminant of a 1DH polynomial, I showed that you can use them to solve line tangency problems in 2DH geometry. For example, the line L is tangent to a quadratic curve Q if



The line L is tangent to the cubic curve C if



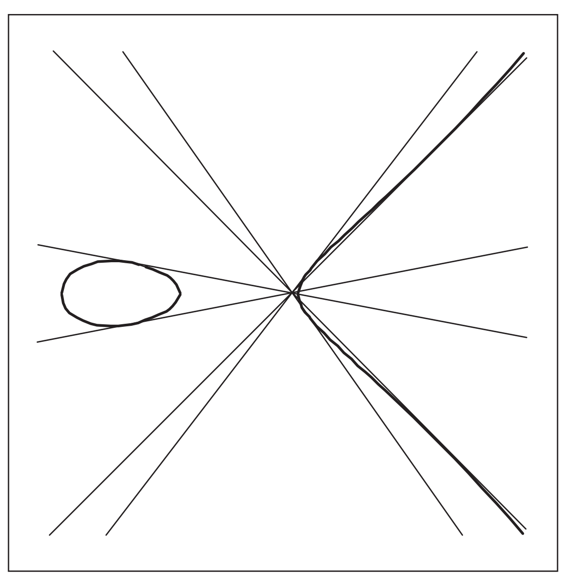
This diagram represents a polynomial expression that is fourth order in C and sixth order in L.

Because it's sixth order in L, it's reasonable to expect that it's possible to find a situation where there are six tangents to a cubic from a given point. This seems excessive but it's possible as Figure 2 shows.

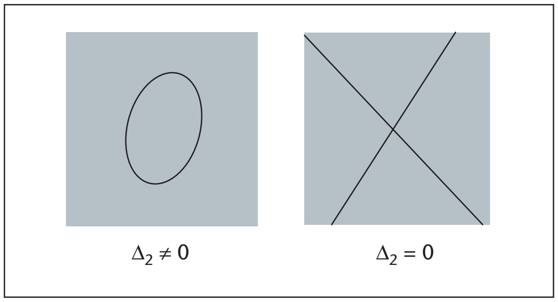
Finally, it's pretty easy to imagine the 2DH diagram that tells whether a line L is tangent to a fourth order curve F now that we have the 1DH diagram for the discriminant of a quartic.

Discriminants

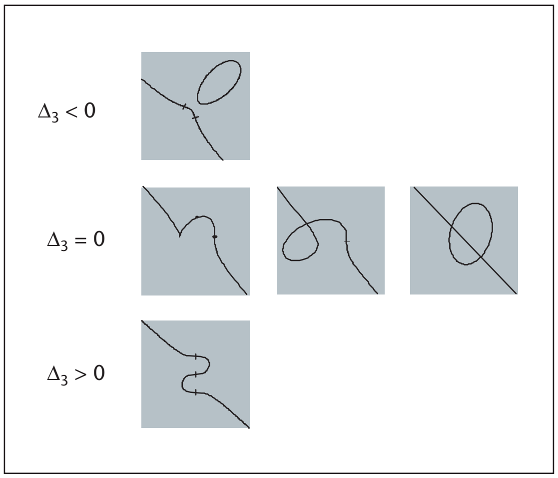
The concept of the discriminant also bumps up from 1DH land to 2DH land. Again, the discriminant being zero tells us that there are places where both the function and its derivatives are zero. Geometrically this means that there are places on the curve (function = 0) where the tangent isn't defined (derivative = 0). This can happen if the curve is factorable into lower order curves—the points in question are the points of inter-



2 Six tangents from a point to a cubic curve.



3 Relation between quadratic discriminant and geometry.

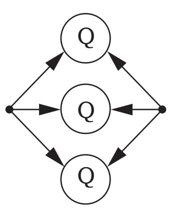


4 Relationship between cubic discriminant and geometry.

section of the lower order curves. Or it can mean that there are cusps or self-intersections in the curve. We'll see examples of all these below.

Quadratic. The discriminant of a quadratic curve is just the determinant of the matrix $\boldsymbol{Q}$. In diagram notation, this looks like

$$\Delta_2 = -\frac{1}{6}$$



If this discriminant is zero it means that the quadratic is factorable into two linear terms. Geometrically, it means that the curve isn't a simple conic section, but a degenerate one consisting of two intersecting straight lines (see Figure 3).

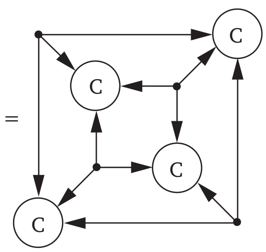
Cubic. An equivalent expression for the cubic curve case is considerably more complicated. Paluszny and Patterson[1] describe the cubic discriminant as a polynomial that's degree 12 in the coefficients $A \ldots F$ and that has more than 10,000 terms. Manipulating this thing explicitly is inconvenient. Actually, it's not that complicated. George Salmon[2] showed that the discriminant is a function of two simpler quantities:

$$\Delta_3 = T^2 + 64S^3$$

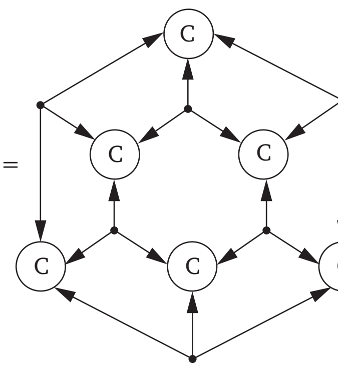
where S is degree 4 in $A \ldots K$ and has 25 terms, and T is degree 6 in $A \ldots K$ and has 103 terms. Salmon worked out all these terms by hand (it's amazing what people had time to do before the invention of television). Figure 4 shows the relation between the cubic discriminant and the geometry of the cubic curve. Notice the varieties of cusp, self-intersection, and lower-order-curve intersection that can happen when $\Delta_3 = 0$. (The tick marks on the figure show the locations of inflection points on the curves.)

How can we express Δ_3 as a tensor diagram? Let's work backwards and see what sort of simple diagrams we can make out of C nodes and epsilons. After some fooling around I came up with the following two:

$C_4 =$



$C_6 =$



Again, using the program described in my March/April 2001 column, I've been able to verify that

$$C_4 = -24S$$
$$C_6 = -6T$$

Pastafazola! Salmon's invariants correspond to the two simplest tensor diagrams we can make for cubic curves! It's things like this that make me believe that I'm really onto something with all this tensor diagram nonsense. Some more fiddling with constants gives us

$$\Delta_3 = \left(\frac{C_6}{-6}\right)^2 + 64\left(\frac{C_4}{-24}\right)^3$$
$$= \frac{1}{6^3}\left(6\left(C_6\right)^2 - \left(C_4\right)^3\right)$$

Relationships

There's something even more interesting going on here. Notice the similarity between the formula for the discriminants of a 1DH quartic polynomial and a 2DH cubic curve:

$$\text{1DH:}\ \ \Delta_4 = \frac{1}{2^3}\left(6\left(F_3\right)^2 - \left(F_2\right)^3\right)$$

$$\text{2DH:}\ \ \Delta_3 = \frac{1}{6^3}\left(6\left(C_6\right)^2 - \left(C_4\right)^3\right)$$

This means that a relationship exists between the possible root structures of a fourth-order polynomial and the possible degeneracies of a third-order curve. That's one of the things I'm currently trying to understand. I'll update you as I learn more. ■

Acknowledgment

I'd like to thank Matt Klaasen of Digipen for turning me on to David Hilbert's book, *Theory of Algebraic Invariants*.

References

1. M. Paluszny and R Patterson, "A Family of Tangent Continuous Cubic Algebraic Splines," *ACM Trans. Graphics*, vol. 12, no. 3, July 1993, p. 212.
2. G. Salmon, *Higher Plane Curves*, Hodges, Foster, and Figgis, Dublin, 1879.

Readers may contact Jim Blinn by email at blinn@microsoft.com.

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