The meaning of Einstein's equation

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This is a brief introduction to general relativity, designed for both students and teachers of the subject. While there are many excellent expositions of general relativity, few adequately explain the geometrical meaning of the basic equation of the theory: Einstein's equation. Here we give a simple formulation of this equation in terms of the motion of freely falling test particles. We also sketch some of the consequences of this formulation and explain how it is equivalent to the usual one in terms of tensors. Finally, we include an annotated bibliography of books, articles, and websites suitable for the student of relativity.
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I. INTRODUCTION

General relativity explains gravity as the curvature of space–time. It's all about geometry. The basic equation of general relativity is called Einstein's equation. In units where $c=8\pi G=1$, it says

$$G_{\alpha\beta}=T_{\alpha\beta}. \tag{1}$$

It looks simple, but what does it mean? Unfortunately, the beautiful geometrical meaning of this equation is a bit hard to find in most treatments of relativity. There are many nice popularizations that explain the philosophy behind relativity and the idea of curved space–time, but most of them don't get around to explaining Einstein's equation and showing how to work out its consequences. There are also more technical introductions which explain Einstein's equation in detail—but here the geometry is often hidden under piles of tensor calculus.

This is a pity, because there is an easy way to express the whole content of Einstein's equation in plain English. After a suitable prelude, one can summarize it in a single sentence! One needs a lot of mathematics to derive all the consequences of this sentence, but we can work out *some* of its consequences quite easily.

In what follows, we start by outlining some differences between special and general relativity. Next we give a verbal formulation of Einstein's equation. Then we derive a few of its consequences concerning tidal forces, gravitational waves, gravitational collapse, and the big bang cosmology. In an appendix we explain why our verbal formulation is equivalent to the usual one in terms of tensors. This article is mainly aimed at those who teach relativity, but except for an appendix, we have tried to make it accessible to students. We conclude with a bibliography of sources to help teach the subject.

II. PRELIMINARIES

Before stating Einstein's equation, we need a little preparation. We assume the reader is somewhat familiar with special relativity—otherwise general relativity will be too hard. But there are some big differences between special and general relativity, which can cause immense confusion if neglected.

In special relativity, we cannot talk about *absolute* velocities, but only *relative* velocities. For example, we cannot sensibly ask if a particle is at rest, only whether it is at rest relative to another particle. The reason is that in this theory, velocities are described as vectors in four-dimensional space–time. Switching to a different inertial coordinate system can change which way these vectors point relative to our coordinate axes, but not whether two of them point the same way.

In general relativity, we cannot even talk about *relative* velocities, except for two particles at the same point of space–time—that is, at the same place at the same instant. The reason is that in general relativity, we take very seriously the notion that a vector is a little arrow sitting at a particular point in space–time. To compare vectors at different points of space–time, we must carry one over to the other. The process of carrying a vector along a path without turning or stretching it is called "parallel transport." When space–time is curved, the result of parallel transport from one point to another depends on the path taken, which is a direct consequence of a curved space–time. Thus it is ambiguous to ask whether two particles have the same velocity vector unless they are at the same point of space–time.

It is hard to imagine the curvature of four-dimensional space–time, but it is easy to see it on a two-dimensional surface, like a sphere. The sphere fits nicely in three-dimensional flat Euclidean space, so we can visualize vectors on the sphere as "tangent vectors." If we parallel transport a tangent vector from the north pole to the equator by going straight down a meridian, we get a different result than if we go down another meridian and then along the equator as shown in Fig. 1.

Because of the analogy to vectors on the surface of a sphere, in general relativity vectors are usually called "tangent vectors." However, it is important not to take this analogy too seriously. Our curved space–time need not be embedded in some higher-dimensional flat space–time for us to understand its curvature, or the concept of a tangent vector. The mathematics of tensor calculus is designed to let us handle these concepts "intrinsically"—i.e., working solely

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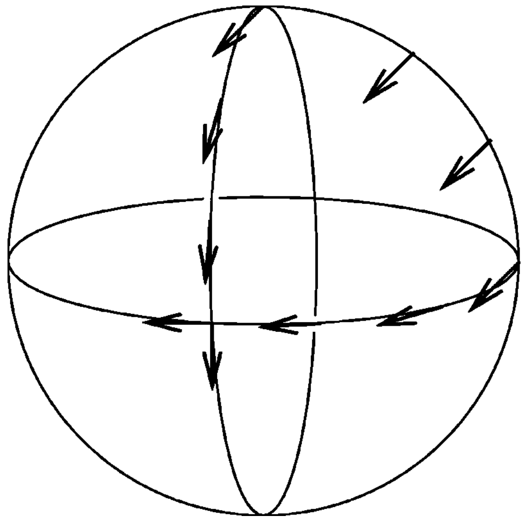

Fig. 1. Two ways to parallel transport a tangent vector from the north pole to a point on the equator of a sphere.

within the four-dimensional space–time in which we find ourselves. This is one reason tensor calculus is so important in general relativity.

In special relativity we can think of an inertial coordinate system, or "inertial frame," as being defined by a field of clocks, all at rest relative to each other. In general relativity this makes no sense, since we can only unambiguously define the relative velocity of two clocks if they are at the same location. Thus the concept of inertial frame, so important in special relativity, is *banned* from general relativity!

If we are willing to put up with limited accuracy, we can still talk about the relative velocity of two particles in the limit where they are very close, since curvature effects will then be very small. In this approximate sense, we can talk about a "local" inertial coordinate system. However, we must remember that this notion makes perfect sense only in the limit where the region of space–time covered by the coordinate system goes to zero in size.

Einstein's equation can be expressed as a statement about the relative acceleration of very close test particles in free fall. Let us clarify these terms a bit. A "test particle" is an idealized point particle with energy and momentum so small that its effects on space–time curvature are negligible. A particle is said to be in "free fall" when its motion is affected by no forces except gravity. In general relativity, a test particle in free fall will trace out a "geodesic." This means that its velocity vector is parallel transported along the curve it traces out in space–time. A geodesic is the closest thing there is to a straight line in curved space–time.

This is easier to visualize in two-dimensional space rather than four-dimensional space–time. A person walking on a sphere "following their nose" will trace out a geodesic—that is, a great circle. Suppose two people stand side-by-side on the equator and start walking north, both following geodesics. Though they start out walking parallel to each other, the distance between them will gradually start to shrink, until finally they bump into each other at the north pole. If they didn't understand the curved geometry of the sphere, they might think a "force" was pulling them together.

In general relativity gravity is not really a "force," but just a manifestation of the curvature of space–time. Note it is not the curvature of space, but of *space–time* that is involved. The distinction is crucial. If you toss a ball, it follows a parabolic path. This is far from being a geodesic in *space*. Space is curved by the Earth's gravitational field, but it is certainly not so curved as all that! The point is that while the ball moves a short distance in space, it moves an enormous distance in *time*, because one second equals about 300 000 km in units where $c=1$. Thus, a slight amount of space–time curvature can have a noticeable effect.

III. EINSTEIN'S EQUATION

To state Einstein's equation in simple English, we need to consider a round ball of test particles that are all initially at rest relative to each other. As we have seen, this is a sensible notion only in the limit where the ball is very small. If we start with such a ball of particles, it will, to second order in time, become an ellipsoid as time passes. This should not be too surprising, because any linear transformation applied to a ball gives an ellipsoid, and any transformation can be approximated by a linear one to first order. Here we get a bit more: the relative velocity of the particles starts out being zero, so to first order in time the ball does not change shape at all: the change is a second-order effect.

Let $V(t)$ be the volume of the ball after a proper time t has elapsed, as measured by the particle at the center of the ball. Then Einstein's equation says:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}\begin{pmatrix} \text{flow of } t\text{-momentum in } t \text{ direction}+ \\ \text{flow of } x\text{-momentum in } x \text{ direction}+ \\ \text{flow of } y\text{-momentum in } y \text{ direction}+ \\ \text{flow of } z\text{-momentum in } z \text{ direction} \end{pmatrix} \tag{2}$$

where these flows are measured at the center of the ball at time zero, using local inertial coordinates. These flows are caused by all particles and fields. They form the diagonal components of a 4×4 matrix T called the "stress-energy tensor." The components $T_{\alpha\beta}$ of this matrix say how much momentum in the α direction is flowing in the β direction through a given point of space–time, where $\alpha,\beta=t,x,y,z$. The flow of t-momentum in the t-direction is just the energy density, often denoted ρ. The flow of x-momentum in the x-direction is the "pressure in the x direction" denoted P_x, and similarly for y and z. It takes a while to figure out why pressure is really the flow of momentum, but it is eminently worth doing. Most texts explain this fact by considering the example of an ideal gas.

In any event, we may summarize Einstein's equation as follows:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}(\rho+P_x+P_y+P_z). \tag{3}$$

This equation says that positive energy density and positive pressure curve space–time in a way that makes a freely falling ball of point particles tend to shrink. Since $E=mc^2$ and we are working in units where $c=1$, ordinary mass density counts as a form of energy density. Thus a massive object will make a swarm of freely falling particles at rest around it start to shrink. In short: *gravity attracts*.

We promised to state Einstein's equation in plain English, but have not done so yet. Here it is:

Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball, plus the pressure in the x direction at that point, plus the pressure in the y direction, plus the pressure in the z direction.

One way to prove this is by using the Raychaudhuri equation, discussions of which can be found in the textbooks by Wald[17] and by Ciufolini and Wheeler[25] cited in the bibliography. But an elementary proof can also be given starting from first principles, as we show in the Appendix.

The reader who already knows some general relativity may be somewhat skeptical that all of Einstein's equation is encapsulated in this formulation. After all, Einstein's equation in its usual tensorial form is really a bunch of equations: the left and right sides of Eq. (1) are 4×4 matrices. It is hard to believe that the single Eq. (3) captures all that information. It does, though, as long as we include one bit of fine print: to get the full content of the Einstein equation from Eq. (3), we must consider small balls with *all possible* initial velocities—i.e., balls that begin at rest in all possible local inertial reference frames.

Before we begin, it is worth noting an even simpler formulation of Einstein's equation that applies when the pressure happens to be the same in every direction:

> Given a small ball of freely falling test particles initially at rest with respect to each other, the rate at which it begins to shrink is proportional to its volume times: the energy density at the center of the ball plus three times the pressure at that point.

This version is only sufficient for "isotropic" situations: that is, those in which all directions look the same in some local inertial reference frame. But, since the simplest models of cosmology treat the universe as isotropic—at least approximately, on large enough distance scales—this is all we shall need to derive an equation describing the big bang!

IV. SOME CONSEQUENCES

The formulation of Einstein's equation we have given is certainly not the best for most applications of general relativity. For example, in 1915 Einstein used general relativity to correctly compute the anomalous precession of the orbit of Mercury and also the deflection of starlight by the Sun's gravitational field. Both these calculations would be very hard starting from Eq. (3); they really call for the full apparatus of tensor calculus. However, we can easily use our formulation of Einstein's equation to get a qualitative—and sometimes even quantitative—understanding of *some* consequences of general relativity. We have already seen that it explains how gravity attracts. We sketch a few other consequences below.

A. Tidal forces, gravitational waves

Let $V(t)$ be the volume of a small ball of test particles in free fall that are initially at rest relative to each other. In the vacuum there is no energy density or pressure, so $\ddot{V}|_{t=0}=0$, but the curvature of space–time can still distort the ball. For example, suppose you drop a small ball of instant coffee when making coffee in the morning. The grains of coffee closer to the earth accelerate toward it a bit more, causing the ball to start stretching in the vertical direction. However, as the grains all accelerate toward the center of the earth, the ball also starts being squashed in the two horizontal directions. Einstein's equation says that if we treat the coffee grains as test particles, these two effects cancel each other when we calculate the second derivative of the ball's volume, leaving us with $\ddot{V}|_{t=0}=0$. It is a fun exercise to check this using Newton's theory of gravity!

This stretching/squashing of a ball of falling coffee grains is an example of what people call "tidal forces." As the name suggests, another example is the tendency for the ocean to be stretched in one direction and squashed in the other two by the gravitational pull of the moon.

Gravitational waves are another example of how space–time can be curved even in the vacuum. General relativity predicts that when any heavy object wiggles, it sends out ripples of space–time curvature which propagate at the speed of light. This is far from obvious starting from our formulation of Einstein's equation! It also predicts that as one of these ripples of curvature passes by, our small ball of initially test particles will be stretched in one transverse direction while being squashed in the other transverse direction. From what we have already said, these effects must precisely cancel when we compute $\ddot{V}|_{t=0}$.

Hulse and Taylor won the Nobel prize in 1993 for careful observations of a binary neutron star which is slowly spiraling down, just as general relativity predicts it should, as it loses energy by emitting gravitational radiation.[27,28] Gravitational waves have not been *directly* observed, but there are a number of projects under way to detect them.[29–32] For example, the LIGO project will bounce a laser between hanging mirrors in an L-shaped detector, to see how one leg of the detector is stretched while the other is squashed. Both legs are 4 km long, and the detector is designed to be sensitive to a 10^{-18} m change in length of the arms.

B. Gravitational collapse

One remarkable feature of this equation is the pressure term, which says that not only energy density but also pressure causes gravitational attraction. This may seem to violate our intuition that pressure makes matter want to expand! Here, however, we are talking about *gravitational* effects of pressure, which are undetectably small in everyday circumstances. To see this, let's restore the factors of c and G. Also, let's remember that in ordinary circumstances most of the energy is in the form of rest energy, so we can write the energy density ρ as $\rho_m c^2$, where ρ_m is the ordinary mass density:

$$\left.\frac{\ddot{V}}{V}\right|_{t=0}=-\frac{4\pi G}{c^4}(\rho_m c^2+P_x+P_y+P_z). \tag{4}$$

On the human scale all of the terms on the right are small, since G is very small and c is very big. (Gravity is a weak force!) Furthermore, the pressure terms are much smaller than the mass density term, since the former has an extra c^2.

There are a number of important situations in which ρ does not dominate over P. For example, in a neutron star, which is held up by the degeneracy pressure of the neutronium it consists of, pressure and energy density contribute comparably to the right-hand side of Einstein's equation. Moreover, above a mass of about two solar masses a nonrotating neutron star will inevitably collapse to form a black hole, thanks in part to the gravitational attraction caused by pressure.

C. The big bang

Starting from our formulation of Einstein's equation, we can derive some basic facts about the big bang cosmology. Let us assume the universe is not only expanding but also homogeneous and isotropic. The expansion of the universe is vouched for by the redshifts of distant galaxies. The other assumptions also seem to be approximately correct, at least when we average over small-scale inhomogeneities such as stars and galaxies. For simplicity, we will imagine the universe is homogeneous and isotropic even on small scales.

An observer at any point in such a universe would see all objects receding from her. Suppose that, at some time $t=0$, she identifies a small ball B of test particles centered on her. Suppose this ball expands with the universe, remaining spherical as time passes because the universe is isotropic. Let $R(t)$ stand for the radius of this ball as a function of time. The Einstein equation will give us an equation of motion for $R(t)$. In other words, it will say how the expansion rate of the universe changes with time.

It is tempting to apply Eq. (3) to the ball B, but we must take care. This equation applies to a ball of particles that are initially at rest relative to one another—that is, one whose radius is not changing at $t=0$. However, the ball B is expanding at $t=0$. Thus, to apply our formulation of Einstein's equation, we must introduce a second small ball of test particles that are at rest relative to each other at $t=0$.

Let us call this second ball B', and call its radius as a function of time $r(t)$. Since the particles in this ball begin at rest relative to one another, we have

$$\dot{r}(0)=0. \tag{5}$$

To keep things simple, let us also assume that at $t=0$ both balls have the exact same size:

$$r(0)=R(0). \tag{6}$$

Equation (3) applies to the ball B', since the particles in *this* ball are initially at rest relative to each other. Since the volume of this ball is proportional to r^3, and using Eq. (5), the left-hand side of Eq. (3) becomes simply

$$\left.\frac{\ddot{V}}{V}\right|_{t=0}=\left.\frac{3\ddot{r}}{r}\right|_{t=0}. \tag{7}$$

Since we are assuming the universe is isotropic, we know that the various components of pressure are equal: $P_x=P_y=P_z=P$. Einstein's equation, Eq. (3), thus says that

$$\left.\frac{3\ddot{r}}{r}\right|_{t=0}=-\frac{1}{2}(\rho+3P). \tag{8}$$

We would much prefer to rewrite this expression in terms of R rather than r. Fortunately, we can do this. At $t=0$, the two spheres have the same radius: $r(0)=R(0)$. Furthermore, the second derivatives are the same: $\ddot{r}(0)=\ddot{R}(0)$. This follows from the equivalence principle, which says that, at any given location, particles in free fall do not accelerate with respect to each other. At the moment $t=0$, each test particle on the surface of the ball B is right next to a corresponding test particle in B'. Since they are not accelerating with respect to each other, the observer at the origin must see both particles accelerating in the same way, so $\ddot{r}(0)=\ddot{R}(0)$. It follows that we can replace r with R in the above equation, obtaining

$$\left.\frac{3\ddot{R}}{R}\right|_{t=0}=-\frac{1}{2}(\rho+3P). \tag{9}$$

We derived this equation for a very small ball, but in fact it applies to a ball of any size. This is because, in a homogeneous expanding universe, the balls of all radii must be expanding at the same fractional rate. In other words, $\ddot{R}/R$ is independent of the radius R, although it can depend on time. Also, there is nothing special in this equation about the moment $t=0$, so the equation must apply at all times. In summary, therefore, the basic equation describing the big bang cosmology[36–41] is

$$\frac{3\ddot{R}}{R}=-\frac{1}{2}(\rho+3P), \tag{10}$$

where the density ρ and pressure P can depend on time but not on position. Here we can imagine R to be the separation between any two "galaxies."

To go further, we must make more assumptions about the nature of the matter filling the universe. One simple model is a universe filled with pressureless matter. Until recently, this was thought to be an accurate model of our universe. Setting $P=0$, we obtain

$$\frac{3\ddot{R}}{R}=-\frac{\rho}{2}. \tag{11}$$

If the energy density of the universe is mainly due to the mass in galaxies, "conservation of galaxies" implies that $\rho R^3=k$ for some constant k. This gives

$$\frac{3\ddot{R}}{R}=-\frac{k}{2R^3} \tag{12}$$

or

$$\ddot{R}=-\frac{k}{6R^2}. \tag{13}$$

Amusingly, this is the same as the equation of motion for a particle in an attractive $1/R^2$ force field. In other words, the equation governing this simplified cosmology is the same as the Newtonian equation for what happens when you throw a ball vertically upwards from the earth! This is a nice example of the unity of physics. Since "whatever goes up must come down—unless it exceeds escape velocity," the solutions of this equation look roughly like those shown in Fig. 2.

In other words, the universe started out with a big bang! It will expand forever if its current rate of expansion is sufficiently high compared to its current density, but it will recollapse in a "big crunch" otherwise.

D. The cosmological constant

The simplified big bang model just described is inaccurate for the very early history of the universe, when the pressure of radiation was important. Moreover, recent observations seem to indicate that it is seriously inaccurate even in the present epoch. First of all, it seems that much of the energy density is not accounted for by known forms of matter. Still more shocking, it seems that the expansion of the universe may be accelerating rather than slowing down! One possibility is that the energy density and pressure are nonzero even for the vacuum. For the vacuum to not pick out a preferred

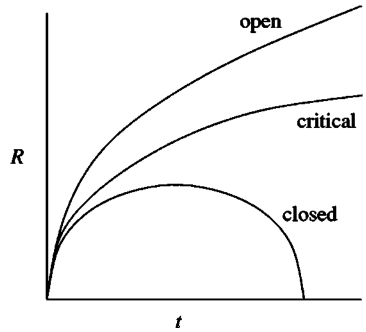

Fig. 2. The size of the universe as a function of time in three scenarios: open (where it expands forever), closed (where it recollapses), and critical (where it expands forever, but just barely).

notion of "rest," its stress-energy tensor must be proportional to the metric. In local inertial coordinates this means that the stress-energy tensor of the vacuum must be

$$T=\begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix}, \tag{14}$$

where Λ is called the "cosmological constant." This amounts to giving empty space an energy density equal to Λ and pressure equal to $-\Lambda$, so that $\rho+3P$ for the vacuum is -2Λ. Here pressure effects dominate because there are more dimensions of space than of time! If we add this cosmological constant term to Eq. (10), we get

$$\frac{3\ddot{R}}{R}=-\frac{1}{2}(\rho+3P-2\Lambda), \tag{15}$$

where ρ and P are the energy density and pressure due to matter. If we treat matter as we did before, this gives

$$\frac{3\ddot{R}}{R}=-\frac{k}{2R^3}+\Lambda. \tag{16}$$

Thus, once the universe expands sufficiently, the cosmological constant becomes more important than the energy density of matter in determining the fate of the universe. If $\Lambda>0$, a roughly exponential expansion will then ensue. This seems to be happening in our universe now.[35]

E. Spatial curvature

We have emphasized that gravity is due not just to the curvature of space, but of *space–time*. In our verbal formulation of Einstein's equation, this shows up in the fact that we consider particles moving forwards in time and study how their paths deviate in the space directions. However, Einstein's equation also gives information about the curvature of space. To illustrate this, it is easiest to consider not an expanding universe but a static one.

When Einstein first tried to use general relativity to construct a model of the entire universe, he assumed that the universe must be static—although he is said to have later described this as "his greatest blunder." As we did in the previous section, Einstein considered a universe containing ordinary matter with density ρ, no pressure, and a cosmological constant Λ. Such a universe can be static—the galaxies can remain at rest with respect to each other—only if the right-hand side of Eq. (15) is zero. In such a universe, the cosmological constant and the density must be carefully "tuned" so that $\rho=2\Lambda$. It is tempting to conclude that space–time in this model is just the good old flat Minkowski space–time of special relativity. In other words, one might guess that there are no gravitational effects at all. After all, the right-hand side of Einstein's equation was tuned to be zero. This would be a mistake, however. It is instructive to see why.

Remember that Eq. (3) contains all the information in Einstein's equation only if we consider all possible small balls. In all of the cosmological applications so far, we have applied the equation only to balls whose centers were at rest with respect to the local matter. It turns out that only for such balls is the right-hand side of Eq. (3) zero in the Einstein static universe.

To see this, consider a small ball of test particles, initially at rest relative to each other, that is moving with respect to the matter in the universe. In the local rest frame of such a ball, the right-hand side of Eq. (3) is nonzero. For one thing, the pressure due to the matter no longer vanishes. Remember that pressure is the flux of momentum. In the frame of our moving sphere, matter is flowing by. Also, the energy density goes up, both because the matter has kinetic energy in this frame and because of Lorentz contraction. The end result, as the reader can verify, is that the right-hand side of Eq. (3) is negative for such a moving sphere. In short, although a stationary ball of test particles remains unchanged in the Einstein static universe, our moving ball shrinks!

This has a nice geometric interpretation: the geometry in this model has spatial curvature. As we noted in Sec. II, on a positively curved surface such as a sphere, initially parallel lines converge toward one another. The same thing happens in the three-dimensional space of the Einstein static universe. In fact, the geometry of space in this model is that of a three-sphere. Figure 3 illustrates what happens.

One dimension is suppressed in this figure, so the two-dimensional spherical surface shown represents the three-dimensional universe. The small shaded circle on the surface represents our tiny ball of test particles, which starts at the equator and moves north. The sides of the sphere approach each other along the dashed geodesics, so the sphere shrinks in the transverse direction, although its diameter in the direction of motion does not change.

As an exercise, readers who want to test their understanding can fill in the mathematical details in this picture and determine the radius of the Einstein static universe in terms of the density. Here are step-by-step instructions:

- Imagine an observer moving at speed v through a cloud of stationary particles of density ρ. Use special relativity to determine the energy density and pressure in the observer's rest frame. Assume for simplicity that the observer is moving fairly slowly, and thus keep only the lowest-order nonvanishing term in a power series in v.
- Apply Eq. (3) to a sphere in this frame, including the contribution due to the cosmological constant (which is the same in all reference frames). You should find that the volume of the sphere decreases with $\ddot{V}/V\propto-\rho v^2$ to leading order in v.
- Suppose that space in this universe has the geometry of a large three-sphere of radius R_U. Show that the radii in the directions transverse to the motion start to shrink at a rate

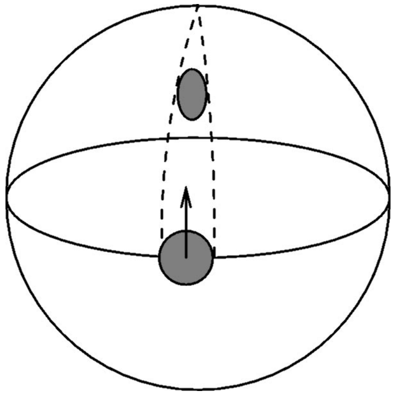

Fig. 3. The motion of a ball of test particles in a spherical universe.

given by $(\ddot{R}/R)|_{t=0} = -v^2/R_U^2$. (If, like most people, you are better at visualizing two-spheres than three-spheres, do this step by considering a small circle moving on a two-sphere, as shown above, rather than a small sphere moving on a three-sphere. The result is the same.)

- Since our little sphere is shrinking in two dimensions, its volume changes at a rate $\ddot{V}/V = 2\ddot{R}/R$. Use Einstein's equation to relate the radius R_U of the universe to the density ρ.

The final answer is $R_U = \sqrt{2/\rho}$, as you can find in standard textbooks.

Spatial curvature like this shows up in the expanding cosmological models described earlier in this section as well. In principle, the curvature radius can be found from our formulation of Einstein's equation by similar reasoning in these expanding models. However, such a calculation is extremely messy. Here the apparatus of tensor calculus comes to our rescue.[16,17]

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APPENDIX A: THE MATHEMATICAL DETAILS

To see why Eq. (3) is equivalent to the usual formulation of Einstein's equation, we need a bit of tensor calculus. In particular, we need to understand the Riemann curvature tensor and the geodesic deviation equation. For a detailed explanation of these, the reader must turn to some of the texts in the bibliography.[16,17,21–23] Here we briefly sketch the main ideas.

When space–time is curved, the result of parallel transport depends on the path taken. To quantify this notion, pick two vectors u and v at a point p in space–time. In the limit where $\epsilon \to 0$, we can approximately speak of a "parallelogram" with sides ϵu and ϵv. Take another vector w at p and parallel transport it first along ϵv and then along ϵu to the opposite corner of this parallelogram. The result is some vector w_1. Alternatively, parallel transport w first along ϵu and then along ϵv. The result is a slightly different vector, w_2 as shown in Fig. 4. The limit

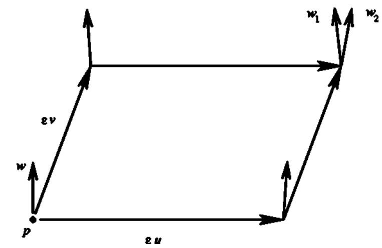

Fig. 4. Parallel transporting a vector w from one corner of a parallelogram to the opposite corner in two ways: up and then across, giving w_1, or across and then up, giving w_2.

$$\lim_{\epsilon \to 0} \frac{w_2 - w_1}{\epsilon^2} = R(u,v)w \tag{A1}$$

is well-defined, and it measures the curvature of space–time at the point p. In local coordinates we can write it as

$$R(u,v)w = R^{\alpha}_{\beta\gamma\delta} u^{\beta} v^{\gamma} w^{\delta}, \tag{A2}$$

where as usual we sum over repeated indices. The quantity $R^{\alpha}_{\beta\gamma\delta}$ is called the "Riemann curvature tensor."

We can use this tensor to compute the relative acceleration of nearby particles in free fall if they are initially at rest relative to one another. Consider two freely falling particles at nearby points p and q. Let v be the velocity of the particle at p, and let ϵu be the vector from p to q. Since the two particles start out at rest relative to one other, the velocity of the particle at q is obtained by parallel transporting v along ϵu.

Now let us wait a short while. Both particles trace out geodesics as time passes, and at time ϵ they will be at new points, say p' and q'. The point p' is displaced from p by an amount ϵv, so we get a little parallelogram, exactly as in the definition of the Riemann curvature as shown in Fig. 5.

Next let us compute the new relative velocity of the two particles. To compare vectors we must carry one to another using parallel transport. Let v_1 be the vector we get by taking the velocity vector of the particle at p' and parallel transporting it to q' along the top edge of our parallelogram. Let v_2 be the velocity of the particle at q'. The difference $v_2 - v_1$ is the new relative velocity. Figure 6 shows a picture of the whole situation. The vector v is depicted as shorter than ϵv for purely artistic reasons.

It follows that over this passage of time, the average relative acceleration of the two particles is $a = (v_2 - v_1)/\epsilon$. By Eq. (A1),

$$\lim_{\epsilon \to 0} \frac{v_2 - v_1}{\epsilon^2} = R(u,v)v, \tag{A3}$$

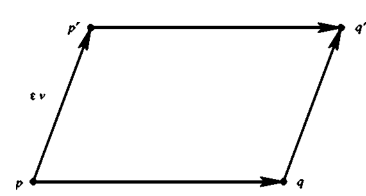

Fig. 5. Freely falling particles at p and q trace out geodesics taking them to p' and q'.

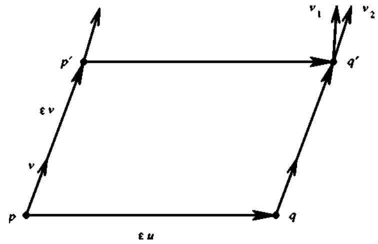

Fig. 6. Parallel transporting the velocity vector of the particle at p' to the point q' gives the vector v_1. The velocity vector of the particle at q' is v_2.

so

$$\lim_{\epsilon\to 0}\frac{a}{\epsilon}=R(u,v)v. \tag{A4}$$

This is called the "geodesic deviation equation." From the definition of the Riemann curvature it is easy to see that $R(u,v)w=-R(v,u)w$, so we can also write this equation as

$$\lim_{\epsilon\to 0}\frac{a^\alpha}{\epsilon}=-R^\alpha_{\beta\gamma\delta}v^\beta u^\gamma v^\delta. \tag{A5}$$

Using this equation we can work out the second time derivative of the volume $V(t)$ of a small ball of test particles that start out at rest relative to each other. As we mentioned earlier, to second order in time the ball changes to an ellipsoid. Furthermore, since the ball starts out at rest, the principal axes of this ellipsoid don't rotate initially. We can therefore adopt local inertial coordinates in which, to second order in t, the center of the ball is at rest and the three principal axes of the ellipsoid are aligned with the three spatial coordinates. Let $r^j(t)$ represent the radius of the jth axis of the ellipsoid as a function of time. If the ball's initial radius is ϵ, then

$$r^j(t)=\epsilon+\tfrac{1}{2}a^j t^2+O(t^3),$$

or in other words,

$$\lim_{t\to 0}\frac{\ddot r^j}{r^j}=\lim_{t\to 0}\frac{a^j}{\epsilon}.$$

Here the acceleration a^j is given by Eq. (A5), with u being a vector of length ϵ in the jth coordinate direction and v being the velocity of the ball, which is a unit vector in the time direction. In other words,

$$\lim_{t\to 0}\frac{\ddot r^j(t)}{r^j(t)}=-R^j_{\beta j\delta}v^\beta v^\delta=-R^j_{tjt}.$$

No sum over j is implied in the above expression.

Because the volume of our ball is proportional to the product of the radii, $\ddot V/V\to\Sigma_j \ddot r^j/r^j$ as $t\to 0$,

$$\lim_{V\to 0}\left.\frac{\ddot V}{V}\right|_{t=0}=-R^\alpha_{t\alpha t}, \tag{A6}$$

where now a sum over α is implied. The sum over α can range over all four coordinates, not just the three spatial ones, since the symmetries of the Riemann tensor demand that $R^t_{ttt}=0$.

The right-hand side is minus the time-time component of the "Ricci tensor"

$$R_{\beta\delta}=R^\alpha_{\beta\alpha\delta}. \tag{A7}$$

That is,

$$\lim_{V\to 0}\left.\frac{\ddot V}{V}\right|_{t=0}=-R_{tt} \tag{A8}$$

in local inertial coordinates where the ball starts out at rest.

In short, the Ricci tensor says how our ball of freely falling test particles starts changing in volume. The Ricci tensor only captures some of the information in the Riemann curvature tensor. The rest is captured by something called the "Weyl tensor," which says how any such ball starts changing in shape. The Weyl tensor describes tidal forces, gravitational waves and the like.

Now, Einstein's equation in its usual form says

$$G_{\alpha\beta}=T_{\alpha\beta}. \tag{A9}$$

Here the right side is the stress-energy tensor, while the left side, the "Einstein tensor," is just an abbreviation for a quantity constructed from the Ricci tensor:

$$G_{\alpha\beta}=R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R^\gamma_\gamma. \tag{A10}$$

Thus Einstein's equation really says

$$R_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}R^\gamma_\gamma=T_{\alpha\beta}. \tag{A11}$$

This implies

$$R^\alpha_\alpha-\tfrac{1}{2}g^\alpha_\alpha R^\gamma_\gamma=T^\alpha_\alpha, \tag{A12}$$

but $g^\alpha_\alpha=4$, so

$$-R^\alpha_\alpha=T^\alpha_\alpha. \tag{A13}$$

Plugging this into Eq. (A11), we get

$$R_{\alpha\beta}=T_{\alpha\beta}-\tfrac{1}{2}g_{\alpha\beta}T^\gamma_\gamma. \tag{A14}$$

This is an equivalent version of Einstein's equation, but with the roles of R and T switched! The good thing about this version is that it gives a formula for the Ricci tensor, which has a simple geometrical meaning.

Equation (A14) will be true if any one component holds in all local inertial coordinate systems. This is a bit like the observation that all of Maxwell's equations are contained in Gauss's law and $\nabla\cdot B=0$. Of course, this is only true if we know how the fields transform under change of coordinates. Here we assume that the transformation laws are known. Given this, Einstein's equation is equivalent to the fact that

$$R_{tt}=T_{tt}-\tfrac{1}{2}g_{tt}T^\gamma_\gamma \tag{A15}$$

in every local inertial coordinate system about every point. In such coordinates we have

$$g=\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \tag{A16}$$

so $g_{tt}=-1$ and

$$T^\gamma_\gamma=-T_{tt}+T_{xx}+T_{yy}+T_{zz}. \tag{A17}$$

Equation (A15) thus says that

$$R_{tt}=\tfrac{1}{2}(T_{tt}+T_{xx}+T_{yy}+T_{zz}). \tag{A18}$$

By Eq. (A8), this is equivalent to

$$\lim_{V\to 0}\left.\frac{\ddot{V}}{V}\right|_{t=0} = -\frac{1}{2}(T_{tt}+T_{xx}+T_{yy}+T_{zz}). \qquad \text{(A19)}$$

As promised, this is the simple, tensor-calculus-free formulation of Einstein's equation.

APPENDIX B: REFERENCES

We provide an annotated bibliography of material on relativity that we have found particularly helpful for students.

1. WEBSITES

There is a lot of material on general relativity available online. Most of it can be found starting from here:

1. **Relativity on the World Wide Web**, C. Hillman, http://math.ucr.edu/home/baez/relativity.html

The beginner will especially enjoy the many gorgeous websites aimed at helping one visualize relativity. There are also books available for free online, such as this:

2. **Lecture Notes on General Relativity**, S. M. Carroll, http://pancake.uchicago.edu/~carroll/notes/

The free online journal *Living Reviews in Relativity* is an excellent way to learn more about many aspects of relativity. One can access it at:

3. **Living Reviews in Relativity**, http://www.livingreviews.org

Part of learning relativity is working one's way through certain classic confusions. The most common are dealt with in the "Relativity and Cosmology" section of this site:

4. **Frequently Asked Questions in Physics**, edited by D. Koks, http://math.ucr.edu/home/baez/physics/

2. NONTECHNICAL BOOKS

Before diving into the details of general relativity, it is good to get oriented by reading some less technical books. Here are four excellent ones written by leading experts on the subject:

5. **General Relativity from A to B**, R. Geroch (University of Chicago Press, Chicago, 1981).
6. **Black Holes and Time Warps: Einstein's Outrageous Legacy**, K. S. Thorne (Norton, New York, 1995).
7. **Gravity from the Ground Up: An Introductory Guide to Gravity and General Relativity**, B. F. Schutz (Cambridge U. P., Cambridge, 2003).
8. **Space, Time, and Gravity: the Theory of the Big Bang and Black Holes**, R. M. Wald (University of Chicago Press, Chicago, 1992).

3. SPECIAL RELATIVITY

Before delving into general relativity in a more technical way, one must get up to speed on special relativity. Here are two excellent texts for this:

9. **Introduction to Special Relativity**, W. Rindler (Oxford U. P., Oxford, 1991).
10. **Space–time Physics: Introduction to Special Relativity**, E. F. Taylor and J. A. Wheeler (Freeman, New York, 1992).

4. INTRODUCTORY TEXTS

When one is ready to tackle the details of general relativity, it is probably good to start with one of these textbooks:

11. **Introducing Einstein's Relativity**, R. A. D'Inverno (Oxford U. P., Oxford, 1992).
12. **Gravity: An Introduction to Einstein's General Relativity**, J. B. Hartle (Addison-Wesley, New York, 2002).
13. **Introduction to General Relativity**, L. Hughston and K. P. Tod (Cambridge U. P., Cambridge, 1991).
14. **A First Course in General Relativity**, B. F. Schutz (Cambridge U. P., Cambridge, 1985).
15. **General Relativity: An Introduction to the Theory of the Gravitational Field**, H. Stephani (Cambridge U. P., Cambridge, 1990).

5. MORE COMPREHENSIVE TEXTS

To become an expert on general relativity, one really must tackle these classic texts:

16. **Gravitation**, C. W. Misner, K. S. Thorne, and J. A. Wheeler (Freeman, New York, 1973).
17. **General Relativity**, R. M. Wald (University of Chicago Press, Chicago, 1984).

Along with these textbooks, you'll want to do lots of problems! This book is a useful supplement:

18. **Problem Book in Relativity and Gravitation**, A. Lightman and R. H. Price (Princeton U. P., Princeton, 1975).

6. EXPERIMENTAL TESTS

The experimental support for general relativity up to the early 1990s is summarized in:

19. **Theory and Experiment in Gravitational Physics**, Revised ed., C. M. Will (Cambridge U. P., Cambridge, 1993).

A more up-to-date treatment of the subject can be found in:

20. "The Confrontation between General Relativity and Experiment," C. M. Will, **Living Reviews in Relativity 4** (2001). Available online at http://www.livingreviews.org/lrr-2001-4

7. DIFFERENTIAL GEOMETRY

The serious student of general relativity will experience a constant need to learn more tensor calculus—or in modern terminology, "differential geometry." Some of this can be found in the texts listed above, but it is also good to read mathematics texts. Here are a few:

21. **Gauge Fields, Knots and Gravity**, J. C. Baez and J. P. Muniain (World Scientific, Singapore, 1994).
22. **An Introduction to Differentiable Manifolds and Riemannian Geometry**, W. M. Boothby (Academic, New York, 1986).
23. **Semi-Riemannian Geometry with Applications to Relativity**, B. O'Neill (Academic, New York, 1983).

8. SPECIFIC TOPICS

The references above are about general relativity as a whole. Here are some suggested starting points for some of the particular topics touched on in this article.

a. The meaning of Einstein's equation

Feynman gives a quite different approach to this in:

24. **The Feynman Lectures on Gravitation**, R. P. Feynman *et al.* (Westview, Boulder, CO, 2002).

His approach focuses on the curvature of space rather than the curvature of space–time.

b. The Raychaudhuri equation

This equation, which is closely related to our formulation of Einstein's equation, is treated in some standard textbooks, including the one by Wald mentioned above. A detailed discussion can be found in

25. **Gravitation and Inertia**, I. Ciufolini and J. A. Wheeler (Princeton U. P., Princeton, 1995).

c. Gravitational waves

Here are two nontechnical descriptions of the binary pulsar work for which Hulse and Taylor won the Nobel prize:

27. **"The Binary Pulsar: Gravity Waves Exist," C. M. Will, Mercury, Nov-Dec 1987, pp. 162–174.**
28. **"Gravitational Waves from an Orbiting Pulsar," J. M. Weisberg, J. H. Taylor, and L. A. Fowler, Sci. Am., Oct 1981, pp. 74–82.**
Here is a review article on the ongoing efforts to directly detect gravitational waves:
29. "Detection of Gravitational Waves," J. Lu, D. G. Blair, and C. Zhao, Rep. Prog. Phys., **63**, 1317–1427 (2000).
Some present and future experiments to detect gravitational radiation are described here:
30. **LIGO Laboratory Home Page**, http://www.ligo.caltech.edu/
31. **The Virgo Project**, http://www.virgo.infn.it/
32. **Laser Interferometer Space Antenna**, http://lisa.jpl.nasa.gov/

d. Black holes

Astrophysical evidence that black holes exist is summarized in:

33. "Evidence for Black Holes," M. C. Begelman, Science **300**, 1898–1903 (2003).
A less technical discussion of the particular case of the supermassive black hole at the center of our Milky Way Galaxy can be found here:
34. **The Black Hole at the Center of Our Galaxy**, F. Melia (Princeton U. P., Princeton, 2003).

e. Cosmology

There are lots of good popular books on cosmology. Since the subject is changing rapidly, pick one that is up to date. At the time of this writing, we recommend:

35. **The Extravagant Universe: Exploding Stars, Dark Energy, and the Accelerating Cosmos**, R. P. Kirshner (Princeton U. P., Princeton, 2002).
A good online source of cosmological information is:
36. **Ned Wright's Cosmology Tutorial**, http://www.astro.ucla.edu/~wright/cosmolog.htm
The following cosmology textbooks are arranged in increasing order of technical difficulty:
37. **Cosmology: The Science of the Universe**, 2nd ed., E. Harrison (Cambridge U. P., Cambridge, 2000).
38. **Cosmology: a First Course**, M. Lachièze-Rey (Cambridge U. P., Cambridge, 1995).
39. **Principles of Physical Cosmology**, P. J. E. Peebles (Princeton U. P., Princeton, 1993).
40. **The Early Universe**, E. W. Kolb and M. S. Turner (Addison–Wesley, New York, 1990).
41. **The Large-Scale Structure of Space–time**, S. W. Hawking and G. F. R. Ellis (Cambridge U. P., Cambridge, 1975).