

Sagnac effect, twin paradox and space-time topology — Time and length in rotating systems and closed Minkowski space-times

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We discuss the Sagnac effect in standard Minkowski coordinates and with an alternative synchronization convention. We find that both approaches lead to the same result without any contradictions. When applying standard coordinates to the complete rim of the rotating disk, a time-lag has to be taken into account which accounts for the global anisotropy. We propose a closed Minkowski space-time as an exact equivalent to the rim of a disk, both in the rotating and non-rotating case. In this way the Sagnac effect can be explained as being purely topological, neglecting the radial acceleration altogether. This proves that the rim of the disk can be treated as an inertial system.

In the same context we discuss the twin paradox and find that the standard scenario is equivalent to an unaccelerated version in a closed space-time. The closed topology leads to preferred frame effects which can be detected only globally.

The relation of synchronization conventions to the measurement of lengths is discussed in the context of Ehrenfest's paradox. This leads to a confirmation of the classical arguments by Ehrenfest and Einstein.

Keywords: special relativity, Sagnac effect, twin paradox, clock synchronization, topology, Ehrenfest paradox, length measurements

1. INTRODUCTION

The interpretation of the Sagnac effect is a longstanding problem in the theory of special relativity. Although different approaches of explanation agree on the observable effects, which are in turn consistent with experiments, the interpretation in the context of special relativity is still a matter of debate, as the continuously high publication rate on the subject shows. The problem is closely related with time measurements in moving reference frames and especially with the synchronization of clocks at different positions in rotating systems.

This paper is not intended to be a review of previous work on the subject but instead tries to derive the theory and conventions from first principles in so far as it seems appropriate in the given context. It is meant to be more or less self-contained, so that some overlap with previous articles cannot be avoided. We do not claim a full axiomatic foundation of special relativity, like the one presented in the book of Reichenbach,⁽¹⁾ of course. For an overview of the Sagnac effect and closely related subjects, including experiments, theoretical analysis and philosophical interpretation, we refer to the articles of Post,⁽³⁾ Hasselbach and Nicklaus,⁽⁴⁾ Stedman⁽⁵⁾ and especially the extensive discussion of synchronization issues by Anderson et al.⁽⁶⁾ These reviews also include extended lists of references for further reading. Very recently, Rizzi and Ruggiero⁽⁷⁾ compiled a book about "Relativity in Rotating Frames" containing a number of articles about the Sagnac effect, its interpretation and related issues. The article by Rizzi and Ruggiero⁽⁸⁾ themselves comprises another review of the subject.

The outline of this paper is as follows. First we will briefly describe the problem the Sagnac effect poses for special relativity. After this introduction we will describe the principles of relativity. The Sagnac effect will be described using the standard conventions, finding that this is possible without contradictions. This proves that special relativity can be used also on the circumference of a rotating disk. However, we will see that the standard coordinates are not valid globally in this case, which leads us to the discussion of a more general approach. We will derive general coordinates and transformations which are still equivalent and compatible with standard relativity although they use a different convention. We will argue that the synchronization can be chosen arbitrarily and show that in certain situations the choice of a non-standard convention can be more convenient, although not necessary.

We will discuss why the standard coordinates are not valid globally in the situation of a rotating disk, and show that the problem can be formulated without explicitly considering the acceleration. Instead, the closed topology of the rim of a rotating disk has to be taken into account. Locally (which will be defined), the rim is indistinguishable from Minkowski space. Only if our measurement process somehow encloses the complete circle, do we notice the

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rotation and are able to detect any non-trivial effects. This in a natural way leads us to the central point of this paper, which is the topological interpretation. We will construct a (spatially) closed Minkowski space-time which will be revealed to be exactly equivalent to the rim of a rotating disk. With this model at hand, we can easily avoid some of the difficulties present in previous discussions of the Sagnac effect. This proves that acceleration is not the main reason for the problems of interpretation. Special relativity is valid, and the rim of the rotating disk can even be seen as an inertial frame, as long as the radial direction is not probed by the experiment.

After this discussion, we will turn to the twin paradox which is shown to be closely related to the Sagnac effect, especially when it is discussed on the rotating disk or equivalently in closed space-time. As before, we will propose a scenario which avoids any acceleration effects while it still leads to exactly the same physical effects as the standard situation. In this way the twin paradox is interpreted in terms of space-time topology as well.

Finally, we will discuss measurements of lengths, which are related with the synchronization problem. We will find that, although the synchronization of clocks is a matter of convention, spatial lengths of objects at rest have a meaning as invariant intervals in space-time only if these are measured along lines of standard simultaneity. The measurement of lengths of moving objects, on the other hand, is more a matter of discussion. The result will generally depend on the details of the experiment which implicitly applies a synchronization convention in order to define the lengths. This will lead to our interpretation of the Ehrenfest paradox which agrees with classical approaches but disagrees with several recent publications on the subject.

In all our presentation we will avoid any sophisticated mathematical methods in order not to hide the physical content behind the formalism. We put the emphasis on a self-consistent and logical derivation of the arguments, without hiding any implicit assumptions. Although we tried to include references to most of the relevant publications, we do not claim completeness in the discussion of previous work.

2. THE PROBLEM

The effect in question was first proposed and knowingly measured by George Sagnac,^(9,10,11) and was then interpreted as the proof of existence of the “luminiferous ether” and as a measurement of rotation relative to it. For our discussion we assume a simplified setup (see Fig. 1) which differs from real experiments in several unimportant aspects. See, e.g., Ref. 3 for a discussion of the influence of more concrete experimental setups.

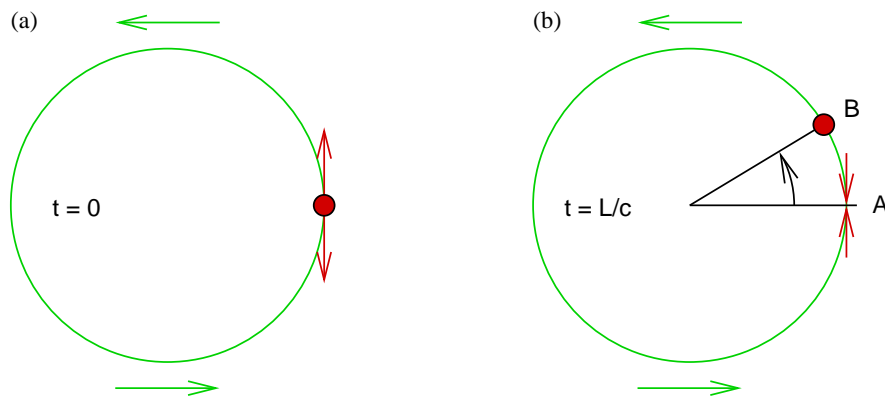


FIG. 1: Basic scenario of the Sagnac effect as seen from the laboratory frame Σ . (a) Light signals are emitted in both directions by a corotating device or observer. (b) When both signals have completed a full round, they meet again at their starting point (measured in the laboratory frame). Since the observer has moved during this time (from A to B), he detected the clockwise signal some time ago but will detect the counter-clockwise signal only in the future.

On a disk of radius R , which can rotate around its central axis, we arrange a collection of mirrors or wave guides which are able to route a ray of light in a closed path (length $L = 2\pi R$) around the rim of the disk. For the sake of simplicity, we assume that this closed path equals the circular rim of the disk. A device fixed to the disk is now sending light in both directions and combines both light paths after a complete revolution in either direction. Interference fringes are used to measure a possible phase shift between the two light paths. In this paper we discuss the equivalent situation of two light pulses traveling in both directions and the measurement of their corresponding light travel times (or their difference). The time difference, if related to the frequency of the observed light, equals the observed phase shift of interference fringes.

As long as the disk is at rest with respect to the inertial laboratory reference frame Σ , the light travel time is equal in both directions and no phase shifts are observed. We now set the disk in stationary rotation (angular velocity

ω , speed of the rim $v = \omega R$) and repeat the experiment (see Fig. 1). When viewed from the laboratory system, the interpretation is simple and does not differ between non-relativistic Galilean physics and special relativity. A time interval $\tau = L/c$ after sending, both light pulses reach the position where they originated, because the rotating mirrors do not change the speed of light in Σ . The corotating observer (at rest in the rotating reference frame Σ'), on the other hand, has moved by $v\tau$ on the circle (from A to B in Fig. 1), so that the light signal sent opposite to the rotation of the disk already crossed the observer, while the signal traveling in the same direction will reach the observer only later. The light travel time *relative to the comoving observer* (measured either in Σ or Σ') does depend on the direction of light travel. The travel times for the light moving in the same (opposite) direction as the rotation of the disk, called co- and counterrotating from now on, are called τ_+ and τ_- in the laboratory frame Σ , and are given by

$$\tau_{\pm} = \frac{L}{c \mp v} , \quad (1)$$

with a difference of

$$\Delta\tau = \tau_+ - \tau_- = \frac{2vL}{c^2} \gamma^2 . \quad (2)$$

Here we have defined $\gamma = (1 - v^2/c^2)^{-1/2}$, as usually. If $S = \pi R^2$ denotes the area of the rotating disk, this can be rewritten in a form which is often found in the literature:

$$\Delta\tau = \frac{4\omega S}{c^2} \gamma^2 \quad (3)$$

This equation can still be used if the light path is not circular but has an arbitrary shape with enclosed area S . For this paper we prefer the form of Eq. (2), which is formally related to Eq. (3) by the Stokes theorem, because we will later restrict ourself to the rim of the disk so that not S and ω but L and v will be the fundamental properties. In the general case of non-circular light paths, Eq. (2) can still be used as integral over the path. This form even holds for non-rigid rotation, as will be discussed in Sec. 7 with a variant of the experiment.

The trouble begins when we interpret this finding in the reference frame of the corotating observer Σ' . The distance traveled by both light signals measured in Σ' (in contrast to Σ) can not depend on the direction, given the symmetry of methods to measure lengths. Both signals travel the circumference length of the rim of the disk, measured in Σ' . If the total travel time, on the other hand, *does* depend on the direction, it seems as if the speed of light is not isotropic anymore which then might violate the principles of special relativity.

The easy way out of this dilemma is denying the problem altogether, by arguing that the comoving observer is not defining an *inertial* frame so that Lorentz transformations cannot be used, and a constant velocity of light cannot be expected in this frame. In the following we will show why relativity is not violated and how the comoving frame can properly be interpreted as inertial, as long as the experiments only probe the tangential but not the radial direction in which the acceleration acts. Instead of avoiding the problem, we will in this way acquire valuable insights into the theory of special relativity, not only in the case of rotating systems but also in general.

3. PRINCIPLES OF SPECIAL RELATIVITY

The result of the experiments by Michelson and Morley is often summarized in the form that *the speed of light is the same in all inertial systems*. The propagation of light in vacuum in any inertial system can therefore be described by the equation

$$ds^2 = 0 , \quad (4)$$

with the definition

$$ds^2 = c^2 dt^2 - (d\xi^2 + d\eta^2 + d\zeta^2) , \quad (5)$$

where t is the time coordinate and ξ, η, ζ are Cartesian space coordinates. Note that the principle of relativity does not postulate that this holds in all coordinates but merely that coordinates exist in which it is true. This claim extends the Galilean notion of equivalence of all inertial frames to electromagnetic phenomena. In the following we use geometrized units in which the speed of light becomes $c = 1$ for simplicity.

The same fact can be formulated in arbitrary generalized coordinates ξ^μ , which becomes more important in general than in special relativity. With a metric g , the infinitesimal interval ds^2 can be written as

$$ds^2 = g_{\mu\nu}(\xi^\alpha) d\xi^\mu d\xi^\nu . \quad (6)$$

Here we use the Einstein convention and sum implicitly over pairs of equal upper and lower indices. Index $\mu = 0$ stands for the time coordinate, $\mu = 1 \dots 3$ for the spatial ones. In the case of standard Minkowski coordinates, the metric is Minkowskian, $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Then the coordinate t measures time, and the spatial coordinates are Cartesian. The physical theory resulting from this approach we call *Special Relativity* or “SR” in the following, regardless of the coordinates that are used to describe the physical processes. This definition is important, because often the use of certain coordinate conventions is included in the notion of SR, sometimes explicitly but often only implicitly. Unclear or ambiguous definitions can then easily lead to seeming contradictions or to endless semantic discussions. For all our work, we assume that SR is a correct theory of physical reality in the absence of gravitation, but we will discuss different coordinate conventions to describe the same theory.

Since the speed of light is the same in all reference frames, the interval ds^2 must also be invariant under coordinate transformations, especially those that correspond to a change of the reference frame. Together with the principle of relativity, this means that Minkowski coordinates exist for all inertial frames, and the metric has the same form of Eq. (5) in all systems. Minkowski coordinates of different inertial frames are related through the Lorentz transformations.

An important point to discuss is the physical meaning of the invariant interval ds^2 . Let us consider a clock in uniform motion. Clearly there exists an inertial coordinate system $(t', \xi', \eta', \zeta')$ in which the clock is at rest, i.e. $\xi', \eta', \zeta' = \text{const}$. In this system, we have $ds^2 = dt'^2$ so that ds is a time interval measured by this clock. Since the interval is invariant, we can as well use any alternative coordinate system and still find that s measures the time shown by a moving clock. An ideal clock would be insensitive to accelerations so that it measures the integrated ds along its world line, the so-called proper time, regardless of whether it is accelerated or not. This is one aspect of the principle of locality which states that, even in accelerated systems (which are described by coordinates different from Minkowski), for each *event* or space-time location there exists a local comoving inertial frame. In this frame Minkowski coordinates can be used which are therefore a good description for an infinitesimally small region around this event. Local physics is described in this local *tangential space-time*. Note that in this theoretical limit the principle of locality is a part of SR, but not an additional assumption. See the discussion in Sec. 4 below for the relevance of locality in our case and for references.

Spatial lengths will be discussed in more detail later. Minkowski coordinates are constructed in a way that the spatial components measure lengths of objects *at rest* directly. We will discuss this below in Sec. 5.1 and generalize it in our discussion of the Ehrenfest paradox in Sec. 9.

4. SAGNAC EFFECT IN STANDARD MINKOWSKI COORDINATES

Let us now return to the Sagnac effect and introduce appropriate coordinates for the rim of the disk. We assume that the inertial laboratory system Σ is given by (t, ξ, η, ζ) , with a Minkowski metric according to Eq. (5). We restrict ourselves to the disk plane defined by $\zeta = 0$, in order to eliminate one coordinate. We now introduce polar coordinates r, ϕ , measured in the laboratory frame,

$$\xi = r \cos \phi \quad , \quad \eta = r \sin \phi \quad . \quad (7)$$

Written in these coordinates, the metric of the same space-time is given by the line element

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 \quad . \quad (8)$$

Since we are interested only in effects *on the rim* of the rotating disk, we furthermore restrict ourselves to a fixed radius $r = R$. It is now convenient to introduce a spatial coordinate $x = r\phi$ measured along the rim of the disk. When restricted to the rim of the disk, the metric is now

$$ds^2 = dt^2 - dx^2 \quad , \quad (9)$$

which is exactly equivalent to a 1+1 dimensional flat Minkowski space-time. This flatness would still hold with the ζ coordinate included. We learn the fundamental lesson that, as long as we are restricted to the rim of the disk, the curvature of the circumference and the corresponding acceleration when the disk is rotating do *not* influence the physics. Please note that this is more than just a manifestation of the principle of locality, stating that Minkowski coordinates can be used locally. Instead the metric is *exactly* Minkowskian not only in infinitesimally small regions but *in the whole region* where the coordinates are valid.

The fact that acceleration is not relevant here was indeed mentioned before, e.g. by Pascual-Sánchez et al.⁽¹²⁾ in a more formal mathematical way. Real clocks and rulers would always have some small extension in the radial direction so that the fixing of r might seem very artificial. However, it can be shown that the limit of infinitely small extension in radial direction is well defined and equals our approach of neglecting the coordinate altogether. This was shown

in an elementary way by Dieks⁽¹³⁾ for the case of light-clocks in accelerated frames and in a more formal but more general way by Mashhoon.^(14,15,16) As long as the measurement devices are sufficiently small (i.e. much smaller than the typical scale defined by the reciprocal acceleration), the principle of locality holds, and fixing r is well justified. A discussion including non-local effects in this sense was published by Sorge.⁽¹⁷⁾

If space-time, which defines the physics on the rim of the disk, is equivalent to Minkowski space-time, it is natural to use the standard Lorentz transformation for the moving reference frame. Let us now set the disk into stationary rotation with a velocity of the rim of v . In most what follows, we define the rotating disk as the geometrical space as explained before. We do not (yet) consider a real material disk made of solid matter. This means that we do not take into account effects of tension and elasticity. Possible expansion or contraction of the disk as a result of the motion will be discussed in the context of the Ehrenfest paradox in Sec. 9.

The standard Lorentz transformation into the comoving reference frame Σ' , defined by coordinates (t', x') , is given by

$$dt' = \gamma (dt - v dx) , \quad (10)$$

$$dx' = \gamma (dx - v dt) , \quad (11)$$

where, as before, $\gamma = 1/\sqrt{1-v^2}$. A fixed position on the disk with $dx' = 0$ moves with $dx = v dt$, as required. The metric written in the new primed coordinates is the same Minkowski metric of Eq. (9) as in the laboratory frame, with the consequence that the coordinate speed of light measured in Σ' as dx'/dt' has its universal value of $c = 1$. Now the problem of the Sagnac effect manifests itself in the following question: *How can this constant speed of light be consistent with the observed different round travel times of light in different directions?*

The solution to this problem is given by the global nature of the Sagnac effect. The round travel times do not measure *local* velocities directly, and it is not a priori clear that integrating local effects ends up with the global behavior measured in a completely different experiment. This can only be expected if the local measures (i.e. coordinates) do *match globally*. We will see that this is not the case for time on the rotating platform.

Let us return to Eq. (10) and its interpretation. The occurrence of dx as part of the expression for dt' reflects the fact that in SR simultaneity (defined as $t = \text{const}$ or $t' = \text{const}$) has no frame independent meaning but differs for different states of motion of the reference frame. This is indeed one of the central concepts of relativity. In the standard formulation of the theory, the shift of simultaneity between different systems takes exactly the form of this equation. We will see later, that even in one *given* inertial frame simultaneity is a matter of convention and can be defined in different ways. These two facts, that in SR simultaneity depends on the state of motion and that even for a fixed reference frame there is still freedom of conventions, should not be confused. The former is part of the standard formulation of SR, while the latter, if expressed in coordinates, leads to a more general formulation of the same theory. For a fundamental discussion of these issues, we refer to the work of Reichenbach,^(1,18) and Dieks⁽²⁰⁾.

If we move along a space-like curve with $dt = 0$ and $dx = R d\phi$, we return to the same event where we started after $\Delta\phi = 2\pi$ or $\Delta x = L = 2\pi R$. This is a result of the periodic nature of the coordinate ϕ or x . We can now apply Eqs. (10)–(11) to investigate how the primed coordinates of Σ' change along the same integration path:

$$\Delta t = \int dt = 0 \quad \Delta x = \int dx = L \quad (12)$$

$$\Delta t' = \int dt' = -\gamma v L \quad \Delta x' = \int dx' = \gamma L \quad (13)$$

We notice that, after going around the circle completely, we arrive at *different* coordinates in the primed system although we ended up at the *same event* in space-time. The change in spatial coordinate owing to Eq. (13) is not so surprising. We started with a periodic coordinate x and could therefore expect periodicity also in x' . We learn that the period is expanded by γ , that means the circumference L' measured in the moving frame Σ' is larger than measured in Σ ,

$$L' = \gamma L . \quad (14)$$

The discussion of this effect is deferred to Sec. 9.

Of fundamental importance for the Sagnac effect is the fact that the time coordinate t' changed as well, as can be seen from Eq. (13). We learn that, although the primed Minkowski coordinates are valid *locally* without any restrictions, they can not be used as continuous *global* coordinates for the whole disk at once. In order to use them nevertheless, we have to introduce a cut at some arbitrary position x' or x (and its periodic replications). More generally, this cut can be placed at any time-like world line $x'(t')$ or $x(t)$. On the disk with this cut excluded, the coordinates are then well-behaved and can be used to describe physics in the usual way. If we ever want to *cross* the

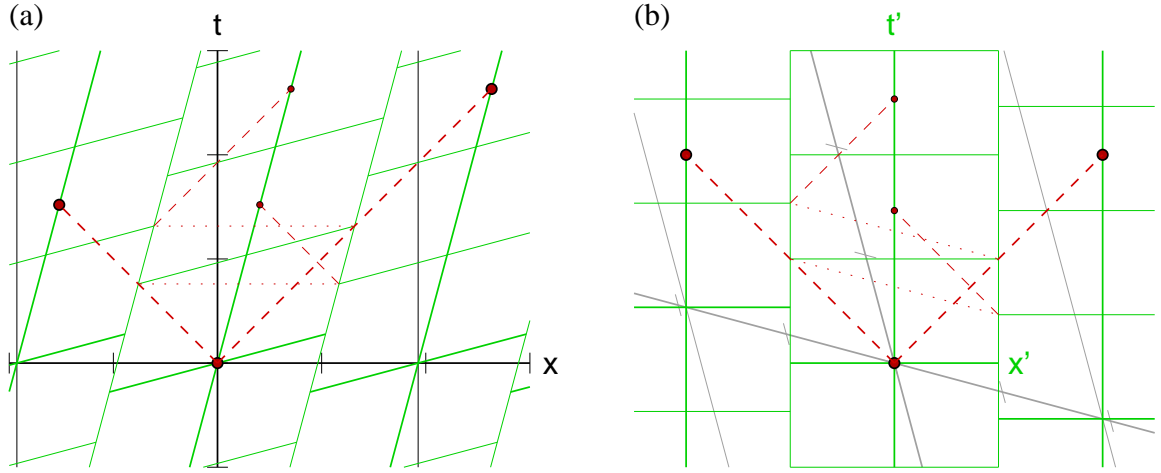


FIG. 2: Minkowski coordinates on the rim of a rotating disk. (a) Laboratory coordinates Σ (upright orthogonal axes, additional thin lines at $x = \pm L$) and comoving coordinates Σ' (tilted grid). (b) Lorentz transformation to comoving coordinates Σ' (upright orthogonal grid) with laboratory coordinates Σ (tilted axes, additional thin lines at $x = \pm L$). The discontinuous cut was placed at comoving coordinates $x' = \pm L'/2$ with $L' = 2$ units. The world lines of light signals starting at the origin and traveling in different directions are shown by dashed lines. Thick lines show the path in the periodic space-time, thin lines are folded into the fundamental coordinate interval $[-L'/2, L'/2]$. We notice that, although the local speed of light is $c = 1$ in both directions, the total round travel times differ for the both directions because of the time-lag at the cut.

cut in our physical experiment, we have to take the jump (or *time-lag*) $\Delta t'$ of the time coordinate into account. We define Θ as

$$\Theta = \gamma v L = v L' , \quad (15)$$

so that $\Delta t' = \mp \Theta$ for co- and counterrotating paths, respectively. If the walk around the disk does not happen instantaneously, but takes a time interval τ' measured by a clock at rest in Σ' , this interval has to be added to $\Delta t'$. In order to clarify the sign convention, we repeat that if we cross the cut in positive (negative) x' direction and go around completely, the proper time interval τ' measured at the starting position is the integrated coordinate time interval $\Delta t'$ *plus* the time-lag $\pm \Theta$:

$$\tau' = \Delta t' \pm \Theta \quad (16)$$

In Fig. 2 we illustrate the coordinates including the periodicity and discontinuity.

In the absence of any time-lag, we could have determined the round travel time of a light signal by integrating coordinate time intervals along the way. Because the local speed of light is $c = 1$, this amounts to a round travel time of $\tau'_0 = \Delta t' = L'$, equal to the circumference length. In reality, however, we measure the time interval at one position but the integration (or light) paths bring us around the rim of the disk in either direction. To convert the integrated local time intervals to a time interval measured at one position, we thus have to correct for the time-lag in the way defined by Eq. (16):

$$\tau'_\pm = \tau'_0 \pm \Theta \quad (17)$$

Hence we obtain

$$\tau'_\pm = (1 \pm v) L' \quad (18)$$

and a light travel time difference of

$$\Delta \tau' = 2\Theta = 2vL' . \quad (19)$$

If we compare this with the results derived in the laboratory frame Σ from Eqs. (1)–(2), we find that

$$\tau_\pm = \gamma \tau'_\pm , \quad \Delta \tau = \gamma \Delta \tau' . \quad (20)$$

The comoving time intervals in Σ' are seen *dilated* by γ in the laboratory frame Σ , just as expected from a special relativistic discussion. Seen from the moving frame Σ' , the intervals seem to be *compressed* by $1/\gamma$. This asymmetry

is not in contradiction with SR but is a direct result of the asymmetric definition of the time intervals in both frames. In Σ' they are proper time intervals measured by a clock at rest, whilst the same clock is moving in Σ so that events at different positions are compared. How this situation compares with the time dilation of a clock at rest in Σ , seen by an observer in Σ' , is discussed briefly in the context of the twin paradox in Sec. 8.1 below.

The conclusion of this section is that the Sagnac effect can be explained without any explicit physical effects of the acceleration. We confirmed that special relativity can also be used to describe effects on a rotating disk. The rim of the disk can even be seen as an *inertial system* as long as the radial degree of freedom, in which the acceleration acts, is not probed by the physical experiment. An explanation in terms of acceleration can even be excluded, because the radial acceleration cannot prefer either of the tangential directions. When inverting the sense of rotation, the Sagnac effect would change its sign, although the acceleration is unchanged.

Locally, the Sagnac effect can not be detected and the situation is equivalent to a standard relativistic inertial Minkowski frame. The only difference is that the local Minkowski coordinates cannot be extended to define a *global* coordinate system for the complete rim of the disk. Note that “locally” here does not mean “in an infinitely small region” but “in any extended region not quite surrounding the complete circle”. It is thus rather a topological notion than one of differential geometry. This already hints in the direction of our interpretation of the Sagnac effect as being a purely topological (d)effect. We will later define a physical situation which shows exactly the same effects as the rim of a rotating disk (both locally and globally) but which is by construction free of accelerations.

5. GENERALIZED MINKOWSKI COORDINATES

We found that the standard Minkowski coordinates (obtained from a Lorentz transformation of the laboratory coordinates) do not provide a valid continuous global system. We might therefore ask if there is an alternative coordinate system which is better suited for the description of global effects on the rotating disk. Generalized non-Cartesian coordinates have been regularly used in pre-relativistic physics, and there is no reason not to continue the approach of allowing non-standard coordinates in special relativity.⁽¹³⁾ This does not mean that we have to change to general relativity, because we do not discuss gravitation. In case of doubt about the correctness of certain calculations, however, we can always escape to the *formalism* used in general relativity and define arbitrary coordinates. Physical measurements are then defined in local tangential space-times. When discussing the geometry of a rotating disk in Sec. 9, we will even meet a curved Riemann space which is required to describe the spatial geometry in Σ' .

Before we will in the following define generalized coordinates (T, X) for a 1+1 dimensional space-time, we want to discuss our demands for the properties of coordinates so that they provide a useful description of space-time in a given inertial system. In contrast to many other similar derivations we do not start with a Minkowski system and determine the possible transformation to alternative inertial systems, but instead define directly the coordinates and metric. Transformations to and from standard Minkowski coordinates will be discussed in the following step.

Ideally, the T coordinate measures *time* and the X coordinate measures *length* as they are seen by an observer at rest with respect to the inertial reference frame (e.g. the rotating or non-rotating disk in our case). Without these demands, the coordinates can only be seen as labels for space-time events so that the interpretation in terms of *space* and *time* becomes difficult. Our requirements on T and X do directly restrict the coordinate freedom by referring to a given reference frame which is defined by a state of motion. Any coordinate freedom left after fixing the reference frame is then a matter of convention.

In order to understand what these notions precisely mean, we have to define the terms very carefully. As Ansatz for a general metric we use

$$ds^2 = A_{TT} dT^2 - A_{XX} dX^2 - 2A_{XT} dX dT , \quad (21)$$

with $A_{XT}^2 + A_{XX}A_{TT} > 0$ to obtain an indefinit metric, and with $A_{TT}, A_{XX} > 0$ for the convenience of simple interpretations.¹ In the following we will define the properties of the metric coefficients $A_{\mu\nu}$. In order to simplify matters, we assume that our space-time metric is homogeneous in X and stationary (i.e. homogeneous in T), so that the coefficients are constant.

Remember that we assume SR to be correct. We do not define our metric as a test theory of relativity but as a generalized formulation of relativity itself. Our approach is thus fundamentally different from the test theory of Robertson,⁽²¹⁾ who fixed the synchronization but allowed for violations of relativity. Later another very influential

¹ We will see later that $A_{TT} > 0$ is necessary while $A_{XX} > 0$ simply avoids unnecessary complications without hiding any underlying problems.

test theory was discussed by Mansouri and Sexl,^(22,23,24) which also included a free synchronization convention in addition to possible deviations from SR. Unfortunately, some confusion between both aspects rendered the discussion of this theory difficult. A test theory generalized for rotating frames is discussed by Vargas and Torr.⁽²⁵⁾ See Will⁽²⁶⁾ and references therein for a discussion of both aspects of the test theories in a number of experiments. No violations of special relativity have been detected so far.

5.1. Length

We assume that coordinate differences X can be measured with rigid rulers which are at rest in the reference frame and free of tensions. The state of motion of these rulers does indeed *define* our rest frame so that “being at rest” and “having constant X ” are equivalent statements. By using rulers, we assume (as a matter of fact *stipulate*) that the rest length (also called proper length) of a ruler does not change with its state of (inertial) motion. Given the principle of relativity, this is the most natural notion.

Absolute offsets in X have no physical meaning and can be neglected. Offsets which are a function of T would describe the transition to a different (moving) frame. In this way the spatial coordinate X is defined uniquely for a given reference frame. The constancy of the metric coefficients then guarantees non-varying distances between world lines of constant X , so that markings on rigid bodies do indeed span the spatial part of the coordinate system. Taking into account additional spatial dimensions, the situation would become more complicated, because of possible rotations, but lead to the same conclusions.

Note that, for the moment, we do not define a physical process to measure lengths. We use rigid rulers (which we assume to exist) both to define the rest frame and to measure lengths.

5.2. Time

Time intervals ΔT at constant X can be measured as proper time intervals of a clock at rest at X . We therefore demand $ds = dT$ for clocks with $dX = 0$, which with Eq. (21) leads directly to

$$A_{TT} = 1 . \quad (22)$$

Global offsets of the coordinate T are again not relevant. In contrast to the X coordinate, where differential offsets as a function of T are fixed by the choice of a given rest frame, differential offsets of T as a function of X can *not* be defined unambiguously without additional assumptions. We can put a collection of clocks at different positions, so that each clock measures T at a certain X , but the relative synchronization of clocks at different positions is a matter of convention. In non-relativistic physics, the absolute nature of simultaneity automatically fixes this synchronization, which is then even independent of the reference frame. This reflects the fact that space and time are separate entities in Galilean physics. In SR we *define* simultaneous events by $\Delta T = 0$. Simultaneity therefore inherits its conventional nature from the synchronization.

5.3. Definitions of the velocity of light

The foundation of special relativity is the fact that the speed of light is equal in all inertial reference frames. In order to measure, or even define, the *one-way* speed of light as a coordinate velocity, we have to adopt some convention for the synchronization of clocks at different places. Assuming that our coordinate system reflects this convention of time synchronization, we can now define the one-way velocities of light in the positive and negative X direction as

$$c_+ = + \left. \frac{dX}{dT} \right|_{dX>0} , \quad (23)$$

$$c_- = - \left. \frac{dX}{dT} \right|_{dX<0} , \quad (24)$$

where both derivatives are taken along the world line of a light signal, which has $ds^2 = 0$. It is clear that if we add to T an offset which is a function of X , this will generally change c_+ and c_- . Therefore the one-way velocity of light is in no way uniquely defined, neither in theory nor in experiment. The situation only changes once we do agree on a synchronization convention, i.e. a time coordinate T and the corresponding metric coefficients. This point

is not recognized in all publications on the subject but particularly stressed in the discussion of Vetharaniam and Stedman⁽²⁷⁾ and Anderson et al.⁽⁶⁾

The *two-way* velocity of light \vec{c} , on the other hand, is defined by sending a light signal to *and fro* (e.g. using a mirror at fixed distance) and dividing the total traveled distance by the total time *measured at one position*. In this way we rely on the time measurement only at one fixed position and can avoid the synchronization problem. The resulting quantity is therefore uniquely defined and does not depend on conventions. The length is the same in both directions while the travel times add up, so that we find

$$\frac{2}{\vec{c}} = \frac{1}{c_+} + \frac{1}{c_-} . \quad (25)$$

The condition for a null line can be written by setting the line element in Eq. (21) to zero. After dividing by $(dX)^2$ we obtain the quadratic equation

$$A_{TT} \left(\frac{dT}{dX} \right)^2 - 2A_{XT} \frac{dT}{dX} - A_{XX} = 0 , \quad (26)$$

with the solutions

$$\left. \frac{dT}{dX} \right|_{\pm} = \frac{A_{XT} \pm \sqrt{A_{XT}^2 + A_{XX}A_{TT}}}{A_{TT}} , \quad (27)$$

one of which is positive and one negative. These correspond to light travel in positive and negative X direction and thus to $\pm 1/c_{\pm}$. With Eqs. (23)–(25), the two-way speed of light becomes

$$\vec{c} = \frac{A_{TT}}{\sqrt{A_{XT}^2 + A_{XX}A_{TT}}} . \quad (28)$$

5.4. Constancy of speed of light

As discussed above, only the two-way speed of light has a convention independent meaning. It is indeed this \vec{c} which was measured in the Michelson-Morley experiment and found to be constant, independent of direction (we discuss only one spatial coordinate so that this does not apply) and independent of the motion of the reference frame in which it is measured. This extends the principle of relativity to electromagnetic phenomena. All inertial reference frames are equivalent and cannot be distinguished even when using light to probe the space-time.

Since we have $\vec{c} = 1$ measured in one inertial frame, the same equation has to hold in any other frame as well. In the notation of this paper, we have even *defined* our units in a way to obtain $\vec{c} = 1$. This is actually very similar to the current SI definition of the meter as the length which light travels in a certain time. In this sense the theory of special relativity can be derived from two principles: (a) equivalence of all inertial frames and (b) the finiteness of the speed of light. If the speed of light were infinite, the Lorentz transformation would degenerate to the Galilei transformation. The principle of relativity would still be satisfied then.

With Eq. (28), the condition $\vec{c} = 1$ leads directly to

$$A_{XX}A_{TT} + A_{XT}^2 = A_{TT}^2 . \quad (29)$$

Recall that we have not analyzed the measurement of lengths in detail. We simply assumed it can be done with rigid rulers in their rest frame. We then used the Michelson-Morley result of the constancy of the speed of light. This does actually not describe the propagation of light alone but the relation of the light propagation to the length of rigid rulers. In this paper we leave the question open whether this tells us anything at all about the propagation of light or if it is just a result of the fact that rigid rulers are held together by electromagnetic forces (i.e. by light) so that they naturally have to Lorentz contract when set into motion in a way which keeps the measured speed of light constant.

In any case we can now use two-way light travel times to measure lengths (“light-ruler”) and, vice versa, the periodic reflection of light between mirrors of known distance to measure time (“light-clock”). We can choose whichever measurement procedure can be analyzed most easily, because the principle of relativity ensures that they all lead to the same result. This approach will be used later in the discussion of the Ehrenfest paradox in Sec. 9.

5.5. Coordinates and transformations

The conditions that T and X measure time and length, respectively, plus the principle of relativity including the results of the Michelson-Morley experiment provide us with the conditions $A_{TT} = 1$ and $A_{XT}^2 + A_{XX} = 1$. We therefore have only one free parameter $A = A_{XT}$ with $|A| < 1$ left and can write the metric of Eq. (21) as

$$\begin{aligned} ds^2 &= dT^2 - (1 - A^2) dX^2 - 2A dX dT \\ &= (dT - A dX)^2 - dX^2 . \end{aligned} \quad (30)$$

The one-way speed of light can now be derived from Eqs. (23)–(24) with Eq. (27) and takes the particularly simple form

$$c_{\pm} = \frac{1}{1 \pm A} . \quad (31)$$

For $A = 0$, our general coordinates are equivalent to standard Minkowskian ones. Soon we will learn that the parameter A describes the synchronization convention.

The generalized Lorentz transformation with respect to a Minkowski rest frame (t, x) can now be derived from two conditions: (a) The proper time interval ds^2 is invariant, (b) a comoving point with $dX = 0$ travels with a velocity $v = dx/dt$ in the laboratory frame. This leads after some simple algebra to the following transformation and its inversion.

$$dT = \gamma [(1 - Av) dt - (v - A) dx] \quad dt = \gamma [dT + (v - A) dX] \quad (32)$$

$$dX = \gamma [dx - v dt] \quad dx = \gamma [(1 - Av) dX + v dT] \quad (33)$$

This transformation could be written even more generally as transformation between two systems which *both* have their own arbitrary synchronization parameters A and A' .

The relation to standard Minkowski coordinates in the same frame can be derived from Eqs. (32)–(33) with $v = 0$ and $\gamma = 1$:

$$dT = dt' + A dx' \quad dX = dx' \quad (34)$$

The alternative coordinates (T, X) differ from the standard Minkowskian ones (t', x') only in a position-dependent offset of time, i.e. in the synchronization convention of clocks. This justifies that we call A the “synchronization parameter”.

Without additional assumptions, the parameter A can be chosen freely without coming in conflict with experiments. However, not all synchronization conventions will lead to simple physical laws written in those coordinates. Some researchers dispute this notion, often by adding implicit and sometimes obscure assumptions, which are compatible only with specific synchronization schemes. Note that the question of whether the synchronization is conventional or not is itself a matter of convention and depends on the demands we put on a coordinate system in order to find it acceptable. The point is that if we want to find the “true” synchronization parameter, we have to define what this means. Different definitions will naturally lead to different conventions.

Einstein⁽²⁸⁾ himself argued that the time at A and B can not be compared directly, but that a general “time” can be *defined* by stipulating that the light travel time from A to B is the same as that from B to A . He was therefore well aware of the fact that this is just a convention, although he did not discuss alternative definitions. For our presentation we want to avoid any unnecessary assumptions and treat the synchronization as conventional. In some sense not even our requirements that X measures length and T measures time are truly necessary. They do, however, have a direct physical meaning in that they allow a simple interpretation of the coordinates. This is fundamentally different from the remaining coordinate freedom parameterized by A .

The synchronization can also be treated in terms of gauge theory. Minguzzi⁽²⁹⁾ presented a very clear and simple discussion and additionally⁽³⁰⁾ a far more general and formal treatment. Further thoughts about synchronization in more general systems, including gravitation and acceleration, are presented by Goy for the rotating disk,^(31,32) for clocks on earth,⁽³³⁾ and for clocks orbiting in general relativity.⁽³⁴⁾

In the following we will discuss the standard synchronization, which is especially well suited for any local effects, and an alternative convention, which simplifies the discussion on the rim of a rotating disk. Both these conventions are in some publications described as *the only possible* synchronization schemes, because different criteria for the (non-)acceptance are used. Within SR experiments cannot be used to select the *true* synchronization. They can only compare synchronization performed with different methods (see the discussion in the following section).

5.6. Standard isotropic clock synchronization

The standard method for the synchronization of clocks is the so called Einstein synchronization. To synchronize clock 1 with 2 (which for our purpose are assumed to be at rest relative to each other), we send a light signal from clock 1 (sent at t_A) to clock 2 (received at t_B) and back (received by clock 1 at t_C) and set the clocks in a way that $t_B = (t_A + t_C)/2$, so that the light travel time from 1 to 2 (measured as the difference in coordinate times at 1 and 2) is the same as the time from 2 to 1. Using this *convention*, the speed of light necessarily becomes isotropic because this very assumption was used in the definition. In terms of our generalized coordinates, this leads to $A = 0$, as Eq. (31) shows. In the following, we want to refer to the synchronization convention $A = 0$ as *standard, symmetric* or *isotropic* synchronization, no matter how this synchronization is actually achieved in an experiment.

An alternative method, which leads to the same convention, is the “slow transport of clocks”. We take a third clock, bring it to clock 1 and synchronize the two, which is trivial when they are located at the same position. This third clock is then moved *slowly* to clock 2. Clocks 1 and 2 are synchronized if the moving clock shows the same time as clock 2 when it arrives there. The limit of infinitely slow transport exists and provides a well defined synchronization procedure as we will show now.

The proper time interval Δs of the moving clock can be calculated from the metric of Eq. (30):

$$\Delta s^2 = \Delta T^2 - (1 - A^2) \Delta X^2 - 2A \Delta X \Delta T \quad (35)$$

On the way, we have $\Delta X = w \Delta T$ with w denoting the coordinate velocity of the moving clock. After moving the comparison clock to clock 2, we compare the two. Clocks 1 and 2 show coordinate time T , so that the difference between the interval measured by clocks 1+2 (ΔT) on one side and the transported clock (Δs) on the other side is

$$\Delta T - \Delta s = \frac{1 - \sqrt{1 - (1 - A^2)w^2 - 2Aw}}{w} \Delta X . \quad (36)$$

The limit of small w exists as a constant difference:

$$\lim_{w \rightarrow 0} (\Delta T - \Delta s) = A \Delta X \quad (37)$$

The clocks are called synchronized if this difference vanishes, which is equivalent to $A = 0$. We see that slow clock transport in either direction is equivalent to Einstein synchronization.

Further experimental methods can be used to perform the clock synchronization according to the same convention. These have in some cases been misinterpreted as determinations of the “objective” synchronization. A simple mechanical method will at some position send two particles of equal rest mass in opposite directions with velocities chosen in a way to obtain a total momentum of zero. This can unambiguously be accomplished by starting with the two particles at rest (zero momentum) and then use some internal process of the two-particle system to separate and accelerate them with the total momentum conserved, e.g. with the release of a taut spring. The particles have some internal clocks and after the same proper time interval, they set the clocks which they pass at this moment to the same time. This scenario can be analyzed most easily in terms of the corresponding velocity four-vectors because of their close relation to the momentum. These vectors can be defined as derivatives of arbitrary coordinates with respect to the proper time of the moving particles:

$$u^\mu = \frac{dx^\mu}{d\tau} \quad (38)$$

In terms of the coordinate velocity $w = dX/dT$, we can derive the four-velocity components using the metric of Eq. (30) and the fact that $ds = d\tau$:

$$u^T = \frac{1}{\sqrt{1 - (1 - A^2)w^2 - 2Aw}} \quad u^X = \frac{w}{\sqrt{1 - (1 - A^2)w^2 - 2Aw}} \quad (39)$$

The two separating particles would, by definition, have equal $u^T = dT/d\tau$ and, due to conservation of the total zero momentum, opposite u^X . With $w = u^X/u^T$, this leads directly to opposite coordinate velocities for the two particles and therefore, following Eq. (39) for equal u^T , to $A = 0$.

Note that the momentum conservation can be discussed in a coordinate independent (covariant) way by using projections.² If we denote the four-velocity of an observer at rest as p^μ (in our coordinates $p^T = 1$, $p^X = 0$), we can

² These projection will again be used when discussing lengths in Sec. 9.2.

write the time and space projections of u^α (\bar{u}^α and \tilde{u}^α , respectively) as

$$\bar{u}^\alpha = p^\alpha (p^\mu u_\mu) , \quad \tilde{u}^\alpha = u^\alpha - \bar{u}^\alpha . \quad (40)$$

The time projection of the four-momentum measures the energy, the space projection the spatial momentum. The two particles must have equal \bar{u}^α but opposite \tilde{u}^α .

The fact that this synchronization method leads to the symmetric convention does not come by surprise, given the fact that the four-velocity vectors live in the local tangential space-time. If the symmetry of the two vectors is inherited by the coordinates, this must necessarily lead to symmetric synchronization.

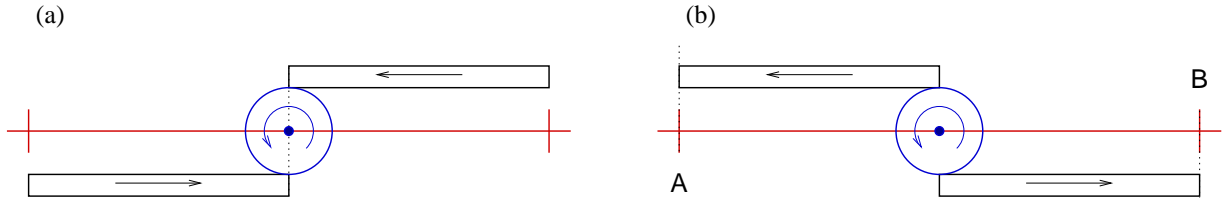


FIG. 3: The synchronization method of Spavieri.⁽³⁵⁾ (a) Two rods start to move with the same velocity, guided by contact to a rotating wheel. (b) When the ends of the rods reach the clocks at A and B , which are at equal distance from the central wheel, they are synchronized to the same time. In SR this technique is equivalent to Einstein synchronization as shown in the text.

A variant of this method was discussed in the work of Spavieri.⁽³⁵⁾ He did not consider two symmetrically separating particles but instead two rods moving in opposite directions, whose equal velocities are prescribed by a rotating wheel which is in contact with both rods (see Fig. 3). The two rods start moving at the same time at the same position, and when their ends reach markings which are at both sides at equal distances, the clocks at these markings are set to the same time. The author argued that the rods have the same velocity, so that this method unambiguously fixes the synchronization. This argument does not actually hold, however, if one takes into account that only the spatial components of the four-velocities are equal but not necessarily the coordinate velocities dX/dT . Assuming equality of the latter would take symmetric synchronization for granted and thus lead to a circular argument. Nevertheless, this setup provides a natural method to perform the isotropic synchronization in an experiment.

We found that there are many methods to perform the standard symmetric synchronization of clocks. This convention is indeed preferred in that it leads to isotropic coordinates. No other *locally* realizable convention, which does not refer to some external system (like the laboratory in the rotating disk case), can play this role as preferred convention. In terms of our parameter A this can be formulated as follows. If we assume that there is one “special” convention, this can only be $A = 0$, because for any other value $A \neq 0$ we could as well have taken $-A$. If all frames are equivalent, only the isotropic convention is preferred before all others. We will see later that special global properties of space-time may lead to the selection of preferred frames so that other conventions might be seen as more convenient in such a case.

Assuming validity of SR, several methods lead to the same isotropic synchronization and thus to the standard Minkowski metric. The equivalence of these different *methods*, which in SR lead to the same *convention*, can be tested in experiments. It should be kept in mind that these are then tests of SR but not experiments to determine the “correct” convention. See, e.g., the discussion by Croca and Selleri⁽³⁶⁾ for an example where the measurement actually compares slow clock transport with Einstein synchronization. Even the historical measurement of the speed of light by Ole Rømer, using the retardation of the observed orbits of the satellites of Jupiter, can not be interpreted as an unambiguous measurement of the one-way speed of light. It really measures this speed by comparing two slowly transported clocks, represented by the orbits of Jupiter’s satellites and the laboratory clocks on earth.

5.7. Global synchronization

Besides the symmetric convention leading to Minkowski coordinates, there is *in our scenario* another special case of the synchronization parameter. If we choose $A = v$, the generalized Lorentz transformation of Eqs. (32)–(33) takes the following form:

$$dT = \gamma^{-1} dt \quad dt = \gamma dT \quad (41)$$

$$dX = \gamma (dx - v dt) \quad dx = \gamma^{-1} dX + v \gamma dT \quad (42)$$

In the Sagnac experiment this special convention avoids the discontinuity in the time coordinate so that the coordinates are valid *globally* in the rotating frame Σ' . Simultaneity with respect to the new coordinates is equivalent to standard simultaneity in the laboratory frame Σ . The periodicity of X with L' is unchanged in this synchronization.

Note that we can define this special convention not only by referring to the external laboratory frame but also purely internal to the rim of the rotating disk. This special value of the parameter A is the only one which avoids time-lags and thus allows the definition of a *global* time coordinate. In order to achieve this, it is not sufficient to use local measurements³ but we must instead consider world lines which surround the circle completely. Concrete experimental scenarios which can be used to perform this global synchronization shall not be discussed here. Locally, this synchronization convention is in no way preferred, in agreement with the arguments presented above.

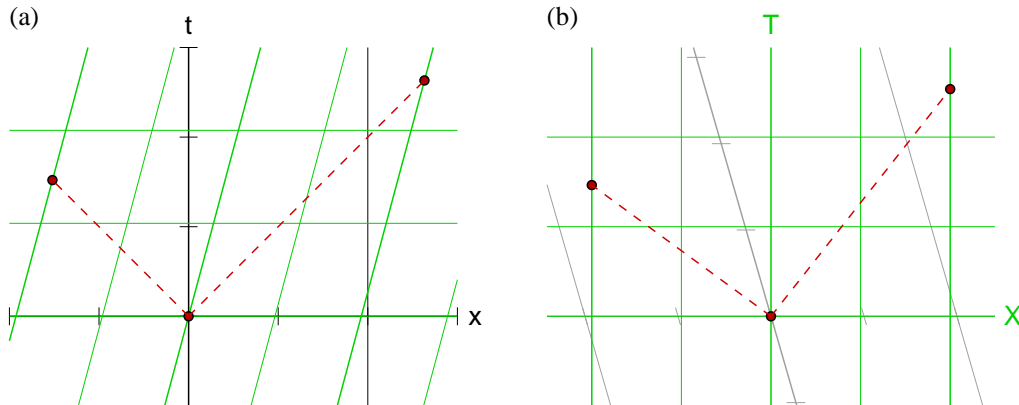


FIG. 4: Alternative coordinates on the rim of the rotating disk. (a) Laboratory coordinates Σ (upright orthogonal axes, additional thin lines at $x = \pm L$) and comoving coordinates Σ' (tilted grid, thick lines at $X = nL$). (b) Generalized Lorentz transformation to comoving coordinates Σ' (upright orthogonal grid) with laboratory coordinates Σ (tilted axes, additional thin lines at $x = \pm L$). Both coordinate systems are valid globally without discontinuities. The world lines of light signals starting at the origin and traveling in different directions are shown by dashed lines. In $\Sigma (t, x)$ the coordinate speed of light is isotropic, in $\Sigma' (T, X)$ it is anisotropic.

The price we pay for the advantage of a global coordinate system is high; the coordinates are not symmetric, and the local coordinate velocity of light is not isotropic anymore. We can calculate c_{\pm} either directly from Eq. (31) or as a special case of Eq. (39) for diverging four-velocity (u^T, u^X) . Both lead to

$$c_{\pm} = \frac{1}{1 \pm v} . \quad (43)$$

On the other hand, the global coordinates make the interpretation of the Sagnac effect much simpler (see Fig. 4 for an illustration). A globally preferred reference frame does exist, so that it is no surprise that the coordinates lose some of their symmetries when we are in a different (i.e. rotating) system. In this way the global asymmetry is formulated as a local asymmetry of coordinates. In reference frames moving with respect to the preferred frame, the coordinate speed of light is anisotropic, which leads to the different round travel times on the rotating disk, $\tau'_{\pm} = (1 \pm v)L'$. This is exactly equivalent to the former result of Eq. (18) which confirms our view that the synchronization is merely a convention and that the Sagnac effect can be explained regardless of the synchronization used.

6. TOPOLOGY

In the previous sections we showed that locally (when not surrounding the disk completely), the rim of the rotating disk is equivalent to a uniformly moving inertial system in 1+1 dimensions. No frame is preferred and standard Minkowski coordinates can be used without restrictions. Only if we allow world-lines to go around completely, are we able to detect the rotation of the disk. We find that there is a *globally* preferred reference frame, given by the system in which the disk is not rotating. This frame is preferred only as a result of the boundary conditions which are in turn an imprint of the closed topology.

³ Again not meaning infinitesimally small regions but extended regions not surrounding the circle.

It is instructive to investigate how the synchronization of clocks depends on the path along which it is performed. To simplify matters, we discuss the slow transport of clocks, which we know is equivalent to any other isotropic synchronization technique like the Einstein synchronization. Especially we want to concentrate on closed synchronization paths. In a standard open Minkowski space-time, we know that a clock transported slowly along *any* closed path beginning and ending at a fixed second clock will always stay synchronized with this latter reference clock. This implies that for open paths the result does not depend on the path followed, so that a unique and global synchronization of clocks at all positions can be achieved in this way.

In general accelerated systems like the complete rotating disk (including the radial direction), the transported clock will generally show deviations from the fixed reference clock. The measured time differences can generally have any value and will depend on the path taken. Changing the path continuously will change the measured time difference continuously as well. In such a situation it is quite natural to interpret this desynchronization as a result of the accelerations in the (non-inertial) reference frame. Even if we restrict ourselves to small but finite regions, do the clocks desynchronize. This is the reason why such systems can *at large* not be described as inertial systems in the context of SR.

The situation is very different on the rim of the rotating disk, even if we extend the discussion to the rotating cylinder by including the axial coordinate ζ which we have neglected so far. The time difference $\Delta t'$, measured after moving one of the clocks around, will always be zero as long as the path does not surround the disk (or cylinder) completely. For general paths, which can wind around arbitrarily often, we find that the time difference can take only a discrete set of values, which are all multiples of $\Theta = vL'$ as calculated in Eq. (15). The time difference (reading of the traveled clock minus reading of the fixed clock) is

$$\Delta t' = -n\Theta , \quad (44)$$

where the number $n \in \mathbb{Z}$ describes the total number of windings in the positive X direction. The situation is now fundamentally different from the general case discussed above. As long as we do not allow surrounding the rim completely, we are not able to detect the rotation and are thus allowed to describe the reference frame as an inertial system. The time difference only depends on the winding number n which can be interpreted in the context of topology as a label for the homotopy class of the clock's path in the multiply connected space-time defined by the periodic coordinate X .

This provides us with the recipe to leave the concrete scenario of the rim of the rotating disk, where some readers may still not be convinced that the acceleration has no direct physical effect, and move to a more general scenario where it is clear from the beginning that in extended regions the space-time is flat and can be described by Minkowski coordinates. Said in physical words, we will construct inertial systems which have the same (local and global) properties as the space-time defined by the rim of the rotating disk.

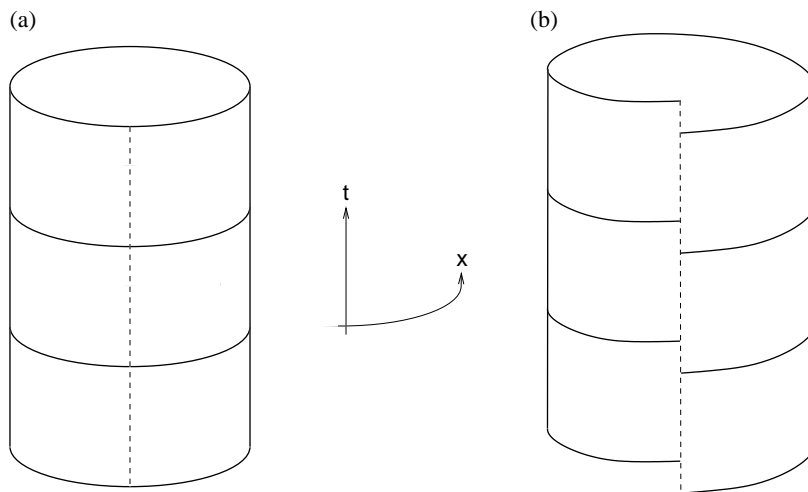


FIG. 5: Geometry of closed 1+1 dimensional Minkowski space-time. Time t is measured vertically, space coordinate x around the cylinder. (a) Identification of $(t, 0)$ with (t, L) , glued together without time jump. This corresponds to a non-rotating disk Σ . (b) Identification of $(t', 0)$ with $(t' - \Theta, L')$. This version corresponds to the rotating disk in the comoving frame Σ' . The elliptical curves/spirals are lines of constant t and t' , respectively. Note that in both cases the cylinder is embedded in a higher dimensional space-time only to illustrate the cylindrical topology. There is no intrinsic curvature.

Let us start with a standard Minkowski frame Σ in 1+1 (t and x) dimensions and compactify this space-time by making it periodic. We identify $(t, 0)$ with (t, L) , or more generally (t, x) with $(t, x + L)$, in order to obtain a space-time

which is L -periodic in x but otherwise still Minkowskian. See Fig. 5 (a) for an illustration in which the space-time is shown as a cylinder embedded in 2+1 space-time. The cylindrical shape is used only for illustration purposes. The space-time itself is by construction intrinsically flat.

We find that this space-time has the same topology and metric as the one on the rim of a disk at rest which was discussed before. Exactly the same effects can therefore be shown to exist in our periodic space-time, using the same formalism as above. A moving inertial system Σ' can again not be described globally by standard Minkowski coordinates because global isotropic clock synchronization is not possible. The spatial extension (or circumference) of the space-time in the moving frame is $L' = \gamma L$ as before (see also Sec. 9) and becomes minimal for $v = 0$. Round travel times for light signals going in different directions will (for $v \neq 0$) be different as on the rotating disk, so that a Sagnac effect will again be observed. Contrary to the scenario on the rotating disk, there is no chance to blame acceleration for the observed effects, because the reference frame is inertial by construction, with no acceleration at all. This confirms and justifies our notion of inertial frames on the rim of a rotating disk.

Instead of identifying $(t, 0)$ with (t, L) , we could also have included an additional time-lag and build an alternative geometry by identifying $(t', 0)$ with $(t' - \Theta, L')$, as shown in Fig. 5 (b). In this way we utilize the freedom the closed topology provides us with. Note that the time step is restricted to values smaller than the space periodicity, i.e. $|\Theta| < L'$, in order to avoid time-like closed curves and to preserve causality and allow a free will for beings living in this space-time. With the same argument we reject the possibility of an identification of different times at the same position. What would the effects in a geometry with time-lag be? We can compare with Eq. (13) to find that the situation in primed coordinates is the same as before on the *rotating* disk in comoving Minkowski coordinates in the frame Σ' . Remember also Fig. 2 (b) for an illustration and compare with Fig. 5 (b). Adding a time-lag in t' therefore seems to set the global system into uniform motion. We can now apply the inverse of the Lorentz transformation of Eqs. (10)–(11) in order to arrive at the frame without time-lag and with global Minkowski coordinates. This confirms that the time-lag in the compactified Minkowski space-time is a measure for its global “rotation”, equivalent to the geometry on the rim of the rotating disk. The two approaches of Fig. 5 (a) and (b) are therefore really equivalent and differ only in respect to the reference frame in which the compactification is performed.

We could now repeat the whole discussion of the Sagnac effect presented above. We would again find travel time differences (Secs. 2 and 4) and would rediscover the possibility of using alternative coordinates with a global synchronization of clocks (Sec. 5.7). We see that the situation in a closed space-time is significantly different from the scenario normally discussed in special relativity with an open topology where no frame is preferred. When applied to a spatially closed space-time, the standard relativistic formulation is restricted to be valid only locally but not globally. When going around completely, discontinuities in standard coordinates are to be expected and must be taken into account. Alternatively, the global clock synchronization can be used, but physical laws will look different when written in these anisotropic coordinates.

Additional aspects of the closed space-time will be discussed later in the context of the twin paradox (Sec. 8.4) and the Ehrenfest paradox (Sec. 9).

7. FIBER OPTIC CONVEYOR EXPERIMENTS

A modified Sagnac experiment was carried out by Wang et al.⁽³⁷⁾ We illustrate the fundamental arrangement in Fig. 6 (a). The light is guided by an optical fiber which runs around two wheels and moves with a certain velocity v . The observer is comoving with a straight part of the fiber and sends light signals in both directions. Different geometries with different enclosed areas etc. are tested in Ref. 37 with the result that the observed time difference (to first order in v) is always $\Delta t = 2vL$ for a fiber of length L , independent of the specific geometry and whether parts of the fiber are moving uniformly or circularly. This hints in the same direction as our interpretation. The time difference is not a result of acceleration (or of enclosed area) but only of velocity and length, which in this scenario is discussed in some detail by Tartaglia and Ruggiero,⁽³⁸⁾ who come to conclusions quite similar to ours, namely that acceleration is not the prime reason for the Sagnac effect, but without referring explicitly to topology.

For our analysis we neglect the reduced speed of light in the optical fiber. Everything the fiber does, is to guide the light around the wheels. Along the straight lines the fiber is not necessary. We can thus produce an equivalent situation by replacing the fiber by stationary mirrors which (disregarding the small transversal displacement introduced by the wheels) act on the light signals in the same way. See Fig. 6 (b) for an illustration. The observer is still moving with v . This situation corresponds to that of Fig. 6 (a) with infinitely small wheels. In this modified experiment, neither the observer nor the mirrors are accelerated but we nevertheless detect the Sagnac shift. This new aspect makes the discussion of this experiment worthwhile. In contrast to the standard scenario, everything can also be explained without any difficulties (like time-lags) in the standard inertial frame of the observer, using standard *continuous global* Minkowski coordinates. In this system the light is being reflected by two uniformly moving mirrors, which leads to light paths of different lengths, depending on the direction. This is shown in Fig. 6 (c). The relevant distance between

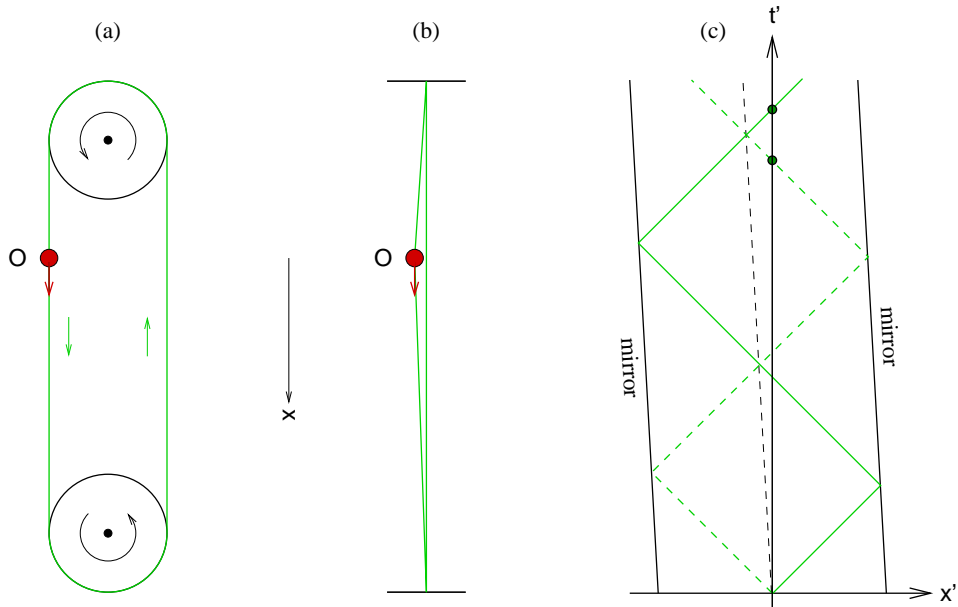


FIG. 6: The fiber optic conveyor Sagnac experiment. (a) Original setup. An optical fiber moves around two wheels. The observer O moves with the fiber along a straight line. (b) Equivalent arrangement where the light guidance by the fiber around the wheels is replaced by the reflection on two mirrors. (c) Space-time diagram in the comoving inertial frame Σ' . The light signals (solid and dashed zigzag lines) are reflected by the moving mirrors. Along the straight dashed line ($x = \text{const}$) the round travel times would be equal. The observer has instead $x' = \text{const}$, where the travel times differ in the same way as in the original Sagnac scenario.

the mirrors is the distance measured between the events of light reflection. In the combination of all light paths, this almost trivially leads to different travel times without being in (not even seemingly) contradiction to special relativity. This is no surprise, because the observer moves uniformly along a straight line and does all the measurements in this global inertial frame.

The more interesting general case, where the fiber moves along any arbitrarily curved, possibly self-crossing, path, can be interpreted very easily in terms of a closed Minkowski space-time whose spatial coordinate is measured along the fiber. It now becomes clear (and was actually tested in the experiment), that the effect depends only on the length L , the speed v and the closed topology. The specific shape of the fiber in 3-dimensional space is not relevant at all.

8. THE TWIN PARADOX

The “twin paradox” is a second example where principles of relativity *seem* to be violated if it is discussed without sufficient care. In fact there is a very close relation to the Sagnac effect as we will see below.

8.1. Time dilation

Before coming to the twin-paradox itself, we briefly review the standard meaning of time dilation in SR. Using isotropic synchronization in all inertial frames, we can apply Lorentz transformations following Eqs. (10)–(11). A clock ticking at a fixed position in the moving system Σ' has $dx' = 0$. With this we immediately obtain the time interval dt measured in the laboratory frame Σ for each interval of the moving clock dt' as

$$dt = \gamma dt' . \quad (45)$$

This is the standard form of time dilation. It is very important to keep in mind the different measurement procedures in both systems and the very different meaning of the time intervals. In the moving system Σ' , we measure the proper time of a clock along its world line, which is independent of any coordinate conventions. In the laboratory frame Σ , on the other hand, we are comparing the readings of clocks at different positions. We know from the previous discussion that such a measurement depends on the synchronization convention. Only this difference allows for the symmetry

of time dilation from one system into another and back. Reading a clock fixed in the laboratory frame Σ from the moving frame Σ' would lead to

$$dt' = \gamma dt \quad , \quad (46)$$

without contradiction to Eq. (45), because the definitions of dt and dt' are simply different in both cases.

We could instead have used the alternative convention corresponding to the global synchronization on the rotating disk from Eqs. (41)–(42) to find that time is now *dilated* in one direction but *compressed* in the other one,

$$dt = \gamma dT \quad , \quad dT = \gamma^{-1} dt \quad . \quad (47)$$

An unambiguous comparison of clocks can only be performed when their corresponding world lines cross. Such a situation was discussed above in the context of the Sagnac effect, providing invaluable insights in SR. A better known example is the situation leading to the classical twin paradox.

8.2. Standard situation

In the standard situation we have two twins, one of whom stays on earth at rest in the reference frame $\Sigma(t, x)$, and the other travels in x direction with a speed v . After some time ($\Delta t = \tau$ as measured on earth in standard convention), the traveling twin reverses the direction and returns with speed $-v$. During this experiment, the sister on earth ages by 2τ and the traveling twin only by $2\tau' = 2\tau/\gamma$, as can be calculated easily. The difference between τ and τ' may seem strange when compared with everyday intuition but, since v has to be much larger than normal travel velocities in order to produce any detectable effects, is not in contradiction with experience and especially not paradoxical so far.

The paradox becomes apparent only if we try to interpret the same situation from the reference frame Σ' of the traveling twin. If the observer on earth sees the traveler's time dilated by a factor γ , should the same not also be true in the opposite direction? In other words: Seen from the traveling twin, her sister on earth seems to travel with speed $-v$ and then v , so that (in this naive and incorrect interpretation) the effect predicted in this reference frame should be the opposite, $\tau' = \gamma\tau$ instead of $\tau' = \tau/\gamma$.

The explanation, of course, is that the traveling twin, in order to be able to return home and compare ages (or clocks), has to accelerate at some point. This means that there is no *single* inertial system Σ' associated with the traveling twin. When using the general notion of Minkowski space-time and the interpretation of the t' coordinate as “time”, the situation is nevertheless strange. The difference in ages is $\int dt(1 - 1/\gamma)$ and is thus an integral over a function which depends on the velocity but not on the acceleration.⁴ This means that there probably is no real physical acceleration effect.

We can indeed in an equivalent experiment avoid the acceleration completely, see Fig. 7(a). We now have two inertial spaceships, one traveling with $x = vt$ and the other with $x = v(2\tau - t)$. When the first spaceship passes the earth at A , clock 0 on earth and clock 1 on the spaceship are both set to zero. The spaceship travels on and when it meets the second spaceship at B , clock 2 on board spaceship 2 is synchronized to the clock 1 on the first spaceship. Since both clocks pass arbitrarily close to each other, this synchronization is unique. Now the second spaceship travels on, until it reaches the earth at C where the clock on earth and on the spaceship are compared. In this experiment the result will be exactly the same as before but no acceleration effects can be present, because only inertial world lines and reference frames are considered. An equivalent thought experiment was proposed by Dingle⁽³⁹⁾ who unfortunately used incorrect arguments in order to prove that there is *no* age difference of the twins.

If we call the proper time interval measured by the traveling twin (or on the two spaceships) $\Delta t'$, we always have $\Delta t = \gamma\Delta t'$. This should *not* be interpreted naively in the way that the moving clock is going *more slowly* than the clock on earth. At each instant we could instead consider the same situation from the moving system in which we then would say that the moving clock goes *faster* than the clock on earth. These assertions cannot both be true. The point here is that t is merely a coordinate which can be defined according to different synchronization conventions, as shown above.

In the conventional Minkowski coordinates, see Fig. 7(b), the observer on earth sees the clocks 1 and 2 being slow by the time dilation factor γ . Observers on the spaceships 1 and 2 do see the clock 0 being slow by the same factor. In the end, both clocks meet again, so that they actually can compare their readings. The two notions would therefore

⁴ If we would rewrite the integral to contain the acceleration, the integrand would depend on the position (x coordinate) or time (t coordinate). Since the space-time is homogeneous this is highly unphysical.

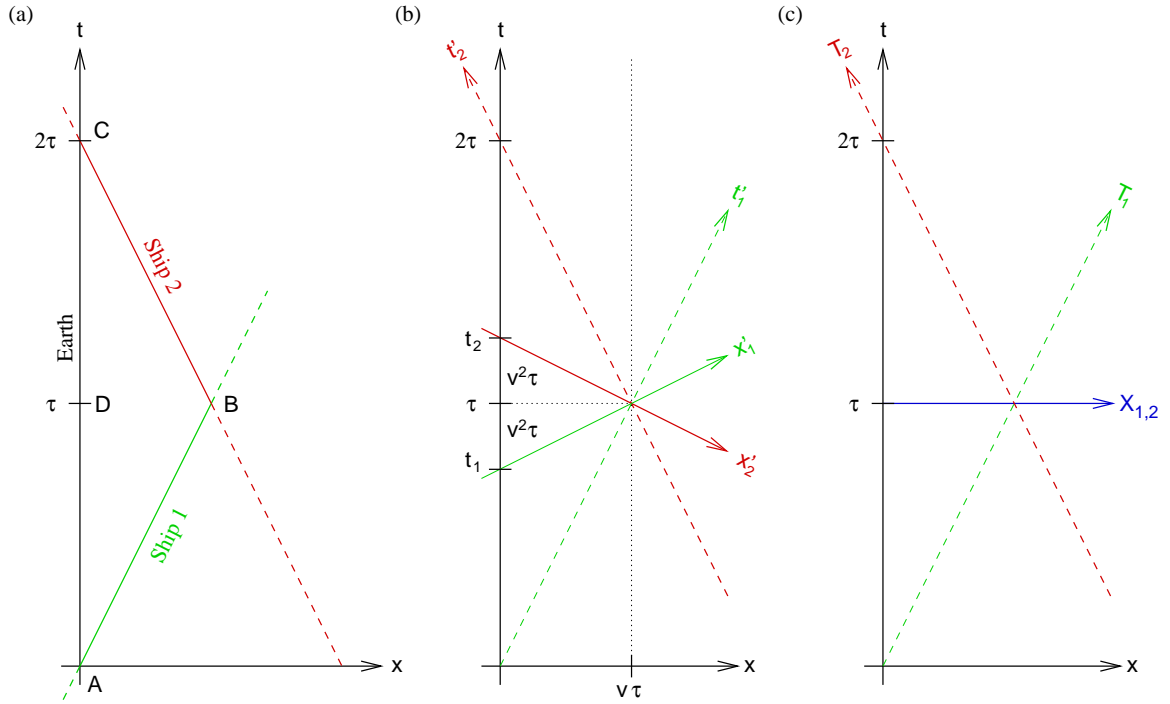


FIG. 7: (a) Scenario of the modified twin paradox experiment. Clock 0 stays on earth ($x = 0$) in Σ . Clock 1 travels with spaceship 1 and is synchronized with clock 0 when passing at A . When the two spaceships meet at B , clock 2 is synchronized to clock 1 and returns to earth. When arriving there at C , it is compared with clock 0. (b) Coordinate axes of the moving systems Σ'_1 (thin) and Σ'_2 (thick lines). The line $t'_1 = 0$ intersects $x = 0$ at $t = t_1$, the line $t'_2 = 0$ at $t = t_2$. (c) Alternative comoving coordinates. Time T is synchronized globally, $T = t/\gamma$. The spatial axes X_1 and X_2 (lines of $T_1 = 0$ and $T_2 = 0$) coincide when the spaceships meet, so that there is no jump in corresponding t .

be inconsistent if there were no discontinuity at some point. We now look at the systems of the spaceships when they meet each other at B , and determine the event $(t, x = 0)$ on earth which is simultaneous (according to isotropic convention) to this event where the transition from Σ'_1 to Σ'_2 takes place. In Σ'_1 the simultaneous time on earth is $t = t_1$, while in Σ'_2 it is $t = t_2$. This means that when we change from system 1 to 2 we miss some time interval $t_2 - t_1$ of the clock on earth. This is of course not a real physical effect. When *watching* the clock on earth from either spaceship, we will not see it instantaneous but retarded by the light travel time. Only when taking this into account and correcting for it (using standard convention), do we observe this jump. Elementary geometry (see Fig. 7) leads to $t_2 - t_1 = 2\tau v^2$. The time interval actually taken into account by the traveling twin as being simultaneous at some instant is $2\tau - (t_2 - t_1) = 2\tau(1 - v^2)$. Using standard time dilation, the traveling twin sees this interval dilated by γ , so that $\tau' = \tau/\gamma$. Taking the time gap into account, the description in the inertial frame of the traveling twin is therefore in absolute agreement with the description in the earth's rest frame, where the time dilation is seen directly and without any coordinate gaps.

As in the discussion of the Sagnac effect, there are no problems with relativistic conventions locally but only globally. Although the moving systems are inertial on both ways, we cannot combine them to one global inertial system. With the standard coordinate conventions this inevitably leads to a discontinuity (the time gap) which introduces the preferred frame effects. This time gap is equivalent to (twice) the time-lag on the rotating disk, as we will discuss in more detail below.

8.3. Discussion in alternative coordinates

As before we can use a different clock synchronization in order to avoid the discontinuity. Using the global coordinates introduced in Sec. 5.7, there is no time jump when reversing the direction of travel, see Fig. 7(c). Now the reference frame of the earth Σ *seems* to be preferred because we have $dT = \gamma^{-1} dt$ but $dt = \gamma dT$, so that the transformation is not symmetric anymore. It can now be said without contradiction that the observer on earth sees the traveling clocks dilated by γ but the traveling observer sees clocks on earth going fast by the same factor. It is all a matter of convention how the clocks are compared. Sometimes a non-conventional clock synchronization actually does simplify

the discussion of problems.

8.4. Twin paradox on rotating disk or in closed space-time

We found that, in the twin paradox as in the Sagnac effect, global inconsistencies of standard Minkowski coordinates when considering closed world lines can be interpreted without any acceleration effects and are thus more a topological than a kinematical problem. To illustrate this even better, let us now consider the twin paradox in a closed Minkowski space-time which, as we have learned before, is equivalent to the rim of a (rotating) disk. This scenario allows the repeated comparison of both twin's clocks (or ages) even if both are in inertial motion.

Let us again consider the scenario of Fig. 7. We can now avoid the acceleration and switching of reference frames if we change the space-time topology by identifying $x = 0$ with $x = L = v\tau$ with equal t coordinates corresponding to each other. The earth is then at rest in the preferred frame Σ of this space-time. Measured in Σ , the traveling twin returns to $x = 0$ at $B \equiv D$ after the time interval τ without any acceleration, simply by surrounding the closed space-time. Measured by the traveling twin in Σ' , the interval is shorter, $\tau' = \tau/\gamma$, although both twins are traveling uniformly. How can this effect now be explained in the comoving frame Σ' ?

From the discussion in Sec. 6 above we know that the closed Minkowski space-time is equivalent to the situation on the rim of a disk so that both situations can be analyzed with the same formalism. The twin at home is at rest on the rim while the traveling twin moves around once. If we interpret this from the reference frame Σ' of the traveler, we have a situation equivalent to the Sagnac effect. In this frame we would expect $\Delta t' = \gamma\tau$, which is not in agreement with the result from the laboratory frame, $\tau' = \tau/\gamma$. The difference between the two is the time-lag that was discussed before:

$$\Delta t' - \tau' = (\gamma - \gamma^{-1})\tau = \Theta \quad (48)$$

Here we used $L' = \gamma L = \gamma v\tau$ to recover the time-lag $\Theta = vL'$ according to Eq. (15). The negative sign of Eq. (16) has to be taken, because the twin at rest moves in the *negative* x' direction in Σ' . The time-lag is independent of the velocity with which the disk is surrounded. One and the same lag is observed with slow clocks, in the twin paradox and in the Sagnac effect where light is used. This is in accordance with our interpretation as a topological effect.

In this scenario, where both twins stay in their inertial frame for the whole experiment, it is obvious that acceleration cannot be blamed for the detected age difference or the asymmetry of time dilation. This means that, globally, not all inertial frames are equivalent, but there is a preferred frame which in the case of the disk is given by the non-rotating laboratory frame and in the case of the closed Minkowski space-time by the compactification without a time-lag. We learned that the preferred frame is also the one with minimal length of the circumference. The two twins do indeed age differently, although they are both in global inertial systems which are equivalent locally. However, this situation is not paradoxical at all, because in the closed space-time there *is* a unique preferred frame which accounts for the difference.

The twin paradox in flat closed space-times has been discussed before in publications by Brans and Stewart,⁽⁴⁰⁾ Low⁽⁴¹⁾ and Dray,⁽⁴²⁾ who already found that, in addition to the local time-dilation effects, global properties of space-time have to be taken into account, and that indeed these global effects are responsible for the selection of a preferred frame. Similar effects in closed Minkowski space-times were discussed by Peters,⁽⁴³⁾ without explicitly referring to the Sagnac effect or twin paradox. The situation was generalized to closed 3+1 space-times by Barrow and Levin⁽⁴⁴⁾ and Uzan et al.,⁽⁴⁵⁾ without changing the conclusions in so far as they are relevant in our context. To the knowledge of the author, the equivalence of the twin paradox in open and closed space-times has not been discussed before. This is also true for the exact equivalence of the (rotating) disk with the closed 1+1 Minkowski space-time and some of the implications for the connection between the Sagnac effect and twin paradox.

9. DEFINITION AND MEASUREMENT OF LENGTHS

In this section we want to discuss the measurement and interpretation of lengths in connection with different synchronization schemes. This is of particular interest, because several authors argued that, even in the rest frame, length measurements are a matter of convention and do depend on the adopted synchronization. We will show that only the measurement of the length of *moving* (with respect to the observer) objects reflect the conventionality of the synchronization. However, this ambiguity can easily be resolved if the measurement process is analyzed with some care. We will show that the synchronization which is relevant for the definition of length is a direct result of how the measurement is performed.

9.1. Ehrenfest paradox

A motivation to study these problems is the discussion of a seeming paradox which was first presented by Ehrenfest.⁽⁴⁶⁾ For a historical review with discussion of different approaches of solutions, we refer to the work of Rizzi and Ruggiero⁽⁴⁷⁾ and Grøn⁽⁴⁸⁾ and the references therein. In the following presentation of the problem, we use Ehrenfest's original notation, which differs from the rest of this paper.

Consider a rigid cylinder (“ein relativ-starrer Zylinder”) of radius R and arbitrary height H . When at rest, the circumference of the cylinder will have a length of $2\pi R$. The cylinder is now set into stationary rotation around its axis. The radius of the rotating cylinder measured by an observer at rest is called R' . We now have two requirements which contradict each other and thus form the paradox: (a) Seen from the laboratory frame, the circumference moves with some velocity so that it will be Lorentz contracted. Measured from outside, it will appear shorter, therefore $2\pi R' < 2\pi R$.⁵ (b) The radius moves perpendicular to its extension so that its length will not change, therefore $R' = R$.

The solution of the paradox, as we interpret it, lies in the assumption of the existence of rigid bodies which can be set in rotation. Closely related to this is the validity of Euclidean geometry in the rotating frame. If space in the rotating frame is not Euclidean, a rigid body which “fits” into Euclidean space will usually “not fit” into Non-Euclidean space. If this is the case, a real solid body will deform, so that the true effects measured in an experiment will in addition to relativistic effects also depend on the elastic properties of the body itself. In order to avoid this difficulty and to define the problem unambiguously, we formulate it differently: Consider not a rigid body but the geometrical space of a rotating disk, defined in the laboratory frame by $\zeta = 0$. For the moment, we do not care how this disk is built. A solid disk set to rotation may deform, but we assume that we can compensate this deformation so that we again have a flat platform with $\zeta = 0$ on which we can perform our experiments in the rotating frame Σ' . We now construct on this disk a circle of radius R' (measured in Σ' ; note that this definition of R' differs from the one of Ehrenfest⁽⁴⁶⁾ given above) using a collection of standard rods of known length, which are somehow supported in order to avoid tensions and deformations, and to be sure that they define proper lengths. This radius will be seen under the same length $R = R'$ in the laboratory frame because of Ehrenfest's argument (b) above, which is consistent with relativity. In the same way we measure the circumference of the circle by laying one rod behind the other until we return to the first one. The length measured in this way we call L' . The question is now: How does this comoving length L' relate to the length L measured in the laboratory frame? Note the fundamental difference of this scenario from the original Gedankenexperiment of Ehrenfest. We do not rely on Euclidean geometry and we do not need rigid bodies. Nevertheless will our discussion explain the solution of the original paradox.

Einstein^(49, 51, 53) (see also Ref. 54 for early comments on Ehrenfest's publication) analyses the problem in the following simple way. Seen from the laboratory frame, we can at a certain time (simultaneous in the standard isotropic sense) make markings on a reference platform at rest, which is close to the rotating disk, at the positions where the endpoints of all the measurement rods on the disk are located at that moment. Then we can count the markings and measure their distances in the laboratory frame. The number, of course, will be the same as seen on the disk, but the lengths will appear Lorentz contracted with the usual factor $1/\gamma$. Therefore we see the complete rotating circumference contracted by the same factor. Measured in the laboratory, the markings will span the circle with circumference $L = 2\pi R$, so that we find $L = L'/\gamma$ or

$$L' = \gamma L \quad , \quad (49)$$

exactly as in Eq. (14). There is nothing wrong with this reasoning, but since other arguments have been proposed to “prove” different effects, a deeper discussion is definitely worthwhile and will help in understanding relativity better. In particular, we will have to discuss the same problem from the perspective of the rotating reference frame. We found before that the rim of the disk can be described as a flat 1+1 Minkowski space-time (modulo a global synchronization oddity). The issue of length on the rim should therefore also be treatable in the moving frame in terms of SR. There is no reason to avoid this discussion with the argument that the system is accelerated and SR would not be valid here. We can not agree with Peres⁽⁵⁵⁾ in this point.

⁵ Note that with $2\pi R'$ the circumference in the *laboratory* system is calculated so that the use of Euclidean geometry is justified. The same is true for $2\pi R$ which is the proper circumference of the cylinder, measured at rest.

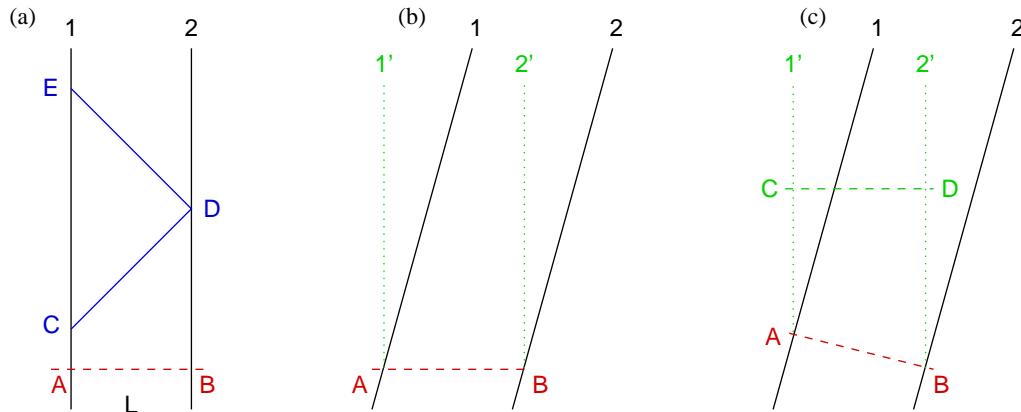


FIG. 8: (a) Definition of length (or distance) L between two world lines 1 and 2 of particles measured in their common rest frame. We use a light signal CDE to measure the distance. The invariant interval along the world line of particle 1 between C and E (proper time measured by 1) is $\Delta s = 2L$. An equivalent measure of L is given by the invariant interval along the dashed line AB , which is defined to be simultaneous to A according to standard isotropic synchronization. This interval between A and B is $\Delta s^2 = -L^2$. These measures of length are independent of coordinates. (b) Measurement of the distance between moving particles 1 and 2. At the two standard-simultaneous events A and B , markings ($1'$ and $2'$) are made. Their distance (equal to the interval AB) defines the momentary distance of 1 and 2. (c) Distance between moving particles 1 and 2 with alternative convention. Markings are made at A and B not standard-simultaneously. The distance CD (not the interval AB) between these markings is used as measurement of the distance of 1 and 2.

9.2. Lengths of objects at rest

We have *defined* our general coordinates of Sec. 5 so that X measures *proper length* in the rest frame, which is in turn defined by constant X . We can therefore start our discussion from this known fact. It is important to understand that lengths are not measured as intervals between *events* in space-time, for which an invariant ds^2 exists, but between *world lines* of objects, i.e. in this case between lines of constant X_1 and X_2 . These lines are defined independent of coordinates but only by the chosen inertial frame. In order to measure the distance without referring to coordinates, we can utilize the constancy of the two-way speed of light. We send a light signal from X_1 to X_2 and back and use the invariant proper time τ measured by a clock at rest at X_1 to determine the length or distance between the two world lines, $L = \tau/2$. The same result would be obtained by making the measurement at X_2 and sending the light to X_1 instead. In our coordinates this leads to $L = |X_1 - X_2|$ by construction. With this measurement procedure at hand, we can now try to interpret the length geometrically in our 1+1 Minkowski space-time. It is easy to see that the length equals the integrated invariant intervals in the following way,

$$L = \int_{X_1}^{X_2} dL = \int_{X_1}^{X_2} \sqrt{-ds^2} , \quad (50)$$

where the integration is performed along a straight line of standard Einstein simultaneity. See Fig. 8(a) for an illustration. This view is confirmed with the general metric of Eq. (30). In the isotropic synchronization, we have $A = 0$ so that $dT = 0$ defines a line of simultaneity. This directly leads to $ds^2 = -dX^2$, i.e. the spatial coordinate really measures length. However, we should not forget that this convention for A is only one out of many possibilities. In the general case, a line of standard simultaneity is defined by $dT = A dX$ so that ds^2 keeps its invariant value, just as required.

This shows that, although the clock synchronization is a matter of convention, the spatial length between two particles at rest is always given by the invariant interval ds^2 between the world lines measured along a line of *standard simultaneity*. Note that (for $A \neq 0$) this will generally *not* be a line of constant T . This geometrical definition of lengths can be used for rest lengths in *any* inertial frame without referring to coordinates. In order to measure the distance in an experiment, no explicit synchronization has to be performed, because the two-way light travel time measured at one position can be used. This is consistent with the standard convention of SR where length is measured by the spatial Minkowski coordinates, i.e. *orthogonal* to the local time direction.

The definition of lengths is particularly simple in this situation and should be no matter of debate. Discussions of space (in contrast to space-time) in more general situations, including accelerations and even gravitation, were presented by Rizzi and Ruggiero^(8,47) and Ruggiero⁽⁵⁶⁾ and in the references therein, in the notion of the *relative space*. They in a very convincing way use world lines as equivalence classes and define the “relative space” as quotient

space relative to these classes. In simple words, a point in relative space is a world line in space-time which is *defined* to be at rest, or actually defines the local rest frame. This is close to the real experimental situation where material bodies are used as a reference frame.

According to Refs. 8,47,56 (and references by Cattaneo therein), the temporal and spatial projections of a vector $r^\alpha = (r^T, r^X)$, \bar{r}^α and \tilde{r}^α , respectively, are⁶

$$\bar{r}^\alpha = \begin{pmatrix} r^T - A r^X \\ 0 \end{pmatrix}, \quad \tilde{r}^\alpha = \begin{pmatrix} A \\ 1 \end{pmatrix} r^X, \quad (51)$$

so that the spatial length becomes

$$\tilde{r}^\mu \tilde{r}_\mu = g_{\mu\nu} \tilde{r}^\mu \tilde{r}^\nu = -(r^X)^2. \quad (52)$$

This confirms that, irrespective of synchronization, the X coordinate measures length in space, in agreement with the definition of the coordinates in Sec. 5.1. This confirms our view which was derived in a more elementary way above. The general definition can be used not only on the rim but on the complete disk. We do not need this full treatment here and only mention that, as expected, nothing happens in the radial direction so that the radius of the disk is the same in both frames.

9.3. Lengths of moving objects

The measurement of lengths of moving objects (or distances of moving particles) is not as unique as that of objects at rest, because we have to define *when* to measure the positions. This is not a matter of interpretation but of real concrete physics. The standard convention of relativity is to use those events on the two moving particles' world lines which are simultaneous in the Einstein convention of the observer's frame and take the invariant interval between them. A practical measurement process could mark the positions of both moving particles (or endpoints of a rod) simultaneously in the observer's frame and then measure the distance between these markings with any convenient method. This is illustrated in Fig. 8 (b) and leads to the momentary distance, where "momentary" has the same ambiguity as "simultaneous". In this standard convention, the measurement L of the length of a moving rod (proper length L') leads to the well-known Lorentz contraction, $L = L'/\gamma$.

The use of a different convention of simultaneity, as illustrated in Fig. 8 (c), will lead to a different distance between two moving particles. The markings are now made at times which are not simultaneous in the standard convention, so that the distance of the markings differs from the standard measurement. The result of a measurement of the length of a moving body does depend on the exact measurement procedure. The length of a moving object is in the same way convention dependent as the time difference of clocks at different positions. This is the core of the "pole in the barn" paradox in which a pole, which at rest does not fit into a barn, is moved with such a high velocity that its Lorentz contracted version would fit in the barn. The seeming paradox is that in the system of the barn the pole now does fit, while in the system of the pole the barn is contracted so that the pole fits even less than when at rest. "Fitting in the barn" means that at some instant (synchronization dependence here!) both ends of the pole are located within the barn. The difference in the views of both reference frames lies in the synchronization of time along the pole.

Does this mean that the Lorentz contraction is only a matter of measuring conventions and is merely illusionary? Not at all, as shown by a thought experiment which was introduced by Dewan and Beran⁽⁵⁸⁾ and later used by Bell^(57, 59) for a discussion of length contractions not only in the Einstein formalism but also in terms of pre-relativistic Lorentz-FitzGerald contractions. The idea is illustrated in Fig. 9. Two spacecraft A and B are at rest in some inertial reference frame Σ at $y = z = 0$ with a distance of L along x . They are connected with a thin thread of proper length L . Both spacecraft now do accelerate for some time, using exactly the same program and starting simultaneously in Σ , so that measured in this rest frame they always have exactly the same velocity and stay at the same distance L . The question is: Does the thread break? The answer is yes. In the reference frame Σ , the thread seems to Lorentz-FitzGerald contract and becomes shorter than L . In the comoving frame Σ' , on the other hand, the distance between the spaceships does increase, so that the thread of length L cannot span the distance anymore. An observer in this system would find the reason for the increasing distance in the fact that the spacecraft do not accelerate simultaneously. Both observers will see a real dynamical effect, the breaking of the thread between the spacecraft. Note that not all authors agree on these conclusions for the experiment. A recent example is Field,⁽⁶⁰⁾ who argues that the thread will *not* break when accelerating.

⁶ See Eq. (40) for the definitions.

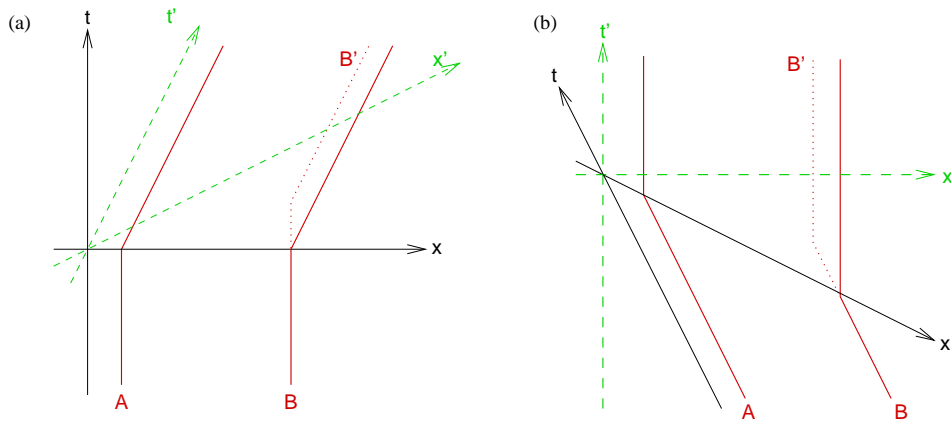


FIG. 9: Thought experiment discussed by Bell⁽⁵⁷⁾ (idealized). (a) In the laboratory rest frame Σ , two spacecraft A and B are simultaneously accelerated to the same velocity. In their final inertial frame, the separation is larger than before the acceleration, so that a thread connecting the two before will break. To prevent this, the spacecraft have to move closer together, here shown by an alternative path for B' which starts its acceleration later. (b) Same situation in the final inertial frame Σ' of the moving spacecraft. In this frame the spacecraft do not accelerate simultaneously.

9.4. Discussion of length of rotating circle

In the frame of the laboratory Σ , the discussion by Einstein presented above is appropriate. Each element of the circumference can be treated as a uniformly moving rod which thus contracts. The unambiguously defined measurement procedure clearly means that standard simultaneity in the laboratory frame must be used to measure the lengths of the rotating rim in the laboratory frame. The total length (in contrast to a total time interval) is simply the integrated local length, so that no “gaps” or other effects of the closed topology are to be expected. This means that the length of the rotating rim L' is observed contracted by a factor $1/\gamma$ in the laboratory frame Σ . Because we have $L = 2\pi R$ in Σ , this directly leads to $L' = \gamma L = 2\pi R \gamma$ and confirms the view which we presented in Sec. 4, Eq. (14), without discussion.

We now change to the rotating frame Σ' . First we want to show that the definition of L' really is a measurement of the circumference length in the rotating frame. The principle of relativity teaches us that the two-way speed of light is constant, so that we can use two-way light travel times to measure small elements of length. These can be added up without problems because, by using the two-way measurement, we are not affected by the time-lag at all. Even if we really use total round travel times, the two lags in both directions cancel exactly. This confirms that the invariant interval measured along a line of standard simultaneity really leads to the proper length of the rim of the rotating disk. It should be noted, however, that the rim of the disk is a closed curve in *space* but the line corresponding to its length is not closed in *space-time*.

In contrast to this, Klauber⁽⁶¹⁾ claimed that the length of the rim must be measured along a closed curve in space-time. We have shown that such a measurement would not be in agreement with relativity. Klauber⁽⁶¹⁾ measures lengths on the rotating disk as invariant space-time intervals not along lines of isotropic simultaneity (“non-time-orthogonal”), but along lines of constant *global* time, i.e. in our notation along $dT = 0$, where the synchronization is defined as in Sec. 5.7 with $A = v$. This leads directly to the absence of Lorentz contraction in the case of the rotating disk. It is clear that neither the concept nor the consequence is compatible with SR, and indeed the theory is meant as a testable alternative to relativity, which moves the subject outside the scope of this paper. However, the idea that the proper length of a moving ruler somehow depends on global properties of the space-time seems quite unnatural. It is not clear how this can be compatible with Lorentz contraction in inertial frames, given the fact that the rim of the disk can locally be described by inertial frames with unlimited accuracy. As long as there is no evidence for the failure of SR in rotating systems, we do not want to discuss alternative theories here.

Seen from the laboratory frame Σ , matters looked quite clear, but now another seeming paradox appears. Should the non-rotating circle, which has $L = 2\pi R$ in the laboratory frame, not contract when seen from the moving frame Σ' ? This would lead to $\Delta L' = \Delta L/\gamma$ and thus *possibly* to the same ratio for the total length, exactly the opposite of $L' = \gamma L$, which was found before. That these two views are not really in contradiction is quite easy to understand. Using the standard procedure of measuring moving lengths (at rest in Σ) in the rotating system Σ' , i.e. doing local measurements without knowing about the global nature of the rotation, would definitely lead to exactly the contraction just described. But does the length of the resting circle L measured in the moving frame Σ' at all correspond to the total length L' of the moving circle as measured in its rest frame? Naively one could expect this to be true since the

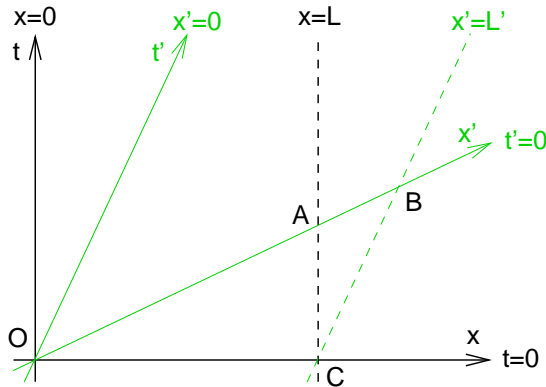


FIG. 10: Measurement of the length of the non-rotating circle in the rotating frame Σ' , shown in Σ to avoid the time-lag. For the laboratory frame Σ , lines of $t = 0$ and the start and end of the circle $x = 0$ and $x = L$ are shown. The line $t' = 0$ is according to the isotropic convention in the moving frame. “Endpoints” of the rotating circle are at $x' = 0$ and $x' = L'$. The standard measurement along isotropic simultaneity uses the invariant interval along OA (leading to L/γ) which leaves out the part AB of the complete rotating rim OB (corresponding to $L' = \gamma L$), which means that the circumference appears contracted without contradiction. With global synchronization, the interval OC would be measured (corresponding to the full circle).

rotating and non-rotating circles seem to fill the same space. Nevertheless, the answer is a clear “no”, as illustrated in Fig. 10. Measuring the resting circle in the moving frame uses only a fraction $1/\gamma^2$ of the corotating circle of rods, therefore the circle at rests seems to be contracted in the moving frame ($L'/\gamma^2 = L/\gamma$). Which part of the circle we miss is a matter of where we start the measurement. Using a globally synchronized measurement, i.e. by marking the resting rods globally simultaneous in the comoving frame, would measure a length of γL so that the rim now seems to be *expanded* by γ . This is no surprise, because the moving circle and the circle at rest do, when observed at a line of global simultaneity, fill the same interval in space-time, so that a consistent ratio must result from the comparison.

The experiment discussed by Bell⁽⁵⁷⁾ (see Sec. 9.3, especially Fig. 9) can be modified, so that a number of spacecraft is first placed at equal distances on a ring of radius R , where they can slide freely. They are then all accelerated from rest in the same way (as seen in the rest frame), so that after some time they reach a velocity of v . If they are initially tightly connected by thin threads, these threads will either expand or break, which confirms the result of our discussion. Neither the fact that the rim of the disk is radially accelerated nor the closed space-topology do play any role in the discussion of the Lorentz contraction of its length.

Tartaglia⁽⁶²⁾ claims that no Lorentz contraction occurs. That work, however, seems to be based on a circular argument. He starts with a disk at rest, filled with rods along the rim, which is then set to rotation. He assumes that the radius does not change, which means he considers the *geometrical* space of the rotating circle. Then he claims that, measured on the rotating disk, the circumference does not change, because in this system the rods do not contract “...since the rulers are laid tail to head ...” and their number stays the same. However, as pointed out by Rizzi and Ruggiero,⁽⁴⁷⁾ this is not a consistent approach because it either assumes that the geometrical length of the circumference does not change (which is exactly what should be proven with this argument, i.e. circular logic), or the rods will be pulled along the rim so that they expand under tension, which means they cannot be used as unit length rods anymore. In this sense the analysis seems to mix the geometrical (here in radius) with the material situation (here on the circumference). This is very similar to the original Ehrenfest argument but with the *hidden* assumption of the existence of a rigid disk. In the full space-time view, Tartaglia⁽⁶²⁾ says that the length of the rim measured Einstein-simultaneously (l_0), which is equal to our L' , is not directly accessible to the moving observer. We have shown how this length can be measured directly (and as integration of local measurements) using light or rigid rulers. Later Tartaglia⁽⁶³⁾ elaborated in more detail on the argument, also taking into account criticism of Ref. 47. He now uses rods which are not connected, so that gaps could build up between them if necessary. The argument now is that the space between the rods would behave in the same way as the rods themselves so that no gaps could result. However, our discussion of linear acceleration above has shown that, even in linear motion, the ends of solid rods behave differently from freely moving particles. When put into motion, the distance of free particles measured in their rest frame increases, contrary to the behavior of rigid rods. Concerning the measurement of the total 2-way round trip time, the author disputes that it can be used as measurement of length. He uses the argument that the travel time for light sent from the rim to the center and back would, when measured on the rim, also be compressed with respect to R so that it does not measure radial distance, either. This is true, but the measurement is not done in an appropriate way in this experiment. Two-way light travel times can be used only *locally* as measurements for lengths. Tangentially, the situation is homogeneous so that the time intervals can be added up directly. In the radial

direction, however, the situation is different. Let us assume that the disk is rotating with angular velocity ω . Lower case letters are used for laboratory inertial coordinates in Σ . We can use the metric of Eq. (8). If we now transform only spatial coordinates to comoving ones (primed), we get $\phi = \omega t + \phi'$ and $r' = r$. For purely radial world lines (in Σ' , $\phi' = \text{const}$), the metric then reads

$$ds^2 = (1 - \omega^2 r^2) dt^2 - dr^2 . \quad (53)$$

For a local observer comoving with the disk, we have $r = \text{const}$ so that we obtain for the proper time τ of this observer $d\tau^2 = (1 - \omega^2 r^2) dt^2$. This means that we can write the metric of Eq. (53) locally as

$$ds^2 = d\tau^2 - dr^2 , \quad (54)$$

which confirms that light travel times can be used to measure radial distances *locally*. This argument against the use of two-way light travel times as length measurements is therefore not valid.

10. DISCUSSION

Although our discussion is about the interpretation of relativity and not about the outcome of physical experiments, we should keep in mind the relevance of relativity in rotating systems for our everyday life. One aspect is the application of the Sagnac effect in technical devices like fiber-optical gyroscopes, used both in civil navigation and regrettably also in modern military systems, as discussed by Stedman.⁽⁵⁾ The more fundamental aspect is the fact that earth itself constitutes a rotating reference frame. Here the situation is much more complicated because of the influence of gravitation and the spheroidal shape of the earth. As a result of these effects, clocks on earth do generally run with different rates. Clocks close to the equator move faster and therefore run with a slower rate. On the other hand, clocks at higher altitude should run with a faster rate, as a result of gravitational time dilation. However, for clocks kept at sea level,⁷ these two effects should cancel exactly, as demanded by the equivalence principle. The earth is not exactly spherical but slightly oblate, due to the centrifugal acceleration, so that clocks close to the equator are more distant to the center, which results in the cancellation of both effects. We now have the same synchronization problems of clocks on earth as on the rotating disk. A clock slowly transported around the earth will desynchronize by some small but measurable amount. At the equator, this time-lag is of the order 200 ns, the light travel time for about 60 m. In any system of highly accurate time-keeping, this has to be taken into account, e.g. in the global positioning system GPS, as discussed by Ashby.⁽⁶⁴⁾ The official time system cannot be and in fact is not synchronized according to the Einstein convention.

10.1. Time

In the Sagnac effect we observe that, seen from a rotating disk, the round travel time for light around the rim of the disk depends on the direction. This non-existence of a *global* universal one-way speed of light c for round trips directly implies that there can be no global time synchronization in which the speed of light is isotropic. Both properties are equivalent, as shown by Minguzzi and Macdonald.⁽⁶⁵⁾

In order to understand the problems of the interpretation of the Sagnac effect and the Ehrenfest paradox in the context of SR, we derived the most general metric which is in agreement with the principles of SR. We demanded that X measures length in the rest frame (defined by constant X) and that T measures proper time of clocks at rest in this system. Together with the constancy of the two-way velocity of light, we arrived at the general metric of Eq. (30). This metric describes Minkowski space-time in coordinates that are less restricted in terms of synchronization of clocks. The desynchronization relative to the standard isotropic convention of Minkowski coordinates is quantified by the parameter A . We found that this parameter A is a matter of convention, i.e. different values do not contradict SR but lead to exactly the same physical results. Nevertheless, the standard convention, using Einstein synchronization, slow transport of clocks or other local isotropic methods, is preferred in the way that it leads to coordinates which reflect the isotropy of space-time. Generalizations of Minkowski coordinates and Lorentz transformations for different synchronization conventions have been discussed before, e.g. by Selleri,^(66,67) Minguzzi,⁽²⁹⁾ Rizzi and Serafini,⁽⁶⁸⁾ and Weber,⁽⁶⁹⁾ not always with the same conclusions as presented in this paper. In contrast to our work, most of these

⁷ Sea level is defined as a surface of constant potential, including gravitational and centrifugal contributions.

previous publications start from a Minkowski reference system and derive the generalized transformations relative to those coordinates. We instead introduced the generalized metric directly.

Closely related with the synchronization convention is the problem of one-way velocities of light in opposite directions. In order to define such a one-way velocity, clock readings at different positions must be compared, so that the result necessarily depends on the synchronization convention. If SR is valid, these one-way velocities cannot be measured unambiguously. If Einstein synchronization is used to set the clocks, the resulting speed of light must be isotropic by construction. A number of proposed experiments to detect anisotropies are thus actually comparisons of different synchronization methods. Since the standard methods have been shown to be equivalent in SR, these tests are really tests of the theory of relativity itself and cannot be used to determine “the correct” synchronization parameter or “the” one-way velocity of light. Similar discussions were presented by, e.g., Vetharaniam and Stedman,⁽²⁷⁾ Anderson et al.⁽⁶⁾ and Croca and Selleri.⁽³⁶⁾

In the standard situation of inertial frames in open Minkowski space-time, this conventionality of synchronization is not a problem. One can simply adopt the isotropic convention and work in the well-known framework of SR in its standard formulation. Things are more subtle on the rim of a rotating disk. A rotating disk is, of course, an accelerated system which can generally be distinguished from an inertial frame. However, we have shown that as long as the physical processes are restricted to constant radius R , i.e. to the *rim* of a rotating disk, there is no possibility to detect the rotation by purely local measurements. In coordinates appropriate for this situation, the metric is a standard Minkowski metric, so that the local physics is expected to be that of an inertial frame. “Locality” in this context is an extension of the standard locality of relativity (discussed, e.g., by Mashhoon,^(14,15,16) Dieks⁽¹³⁾ and Sorge⁽¹⁷⁾), where it means that in infinitesimally small regions of space-time, local comoving Minkowski frames approximate the real (possibly curved and accelerated) space-time arbitrarily well and can thus be used for the local calculations. In our case the Minkowski metric is valid not only in regions very small compared to the disk but even in extended regions.

At first view this seems to be in contradiction with the Sagnac effect, where the total round travel time of light signals, as measured by a comoving observer, depends on the direction in which the signals travel. When comparing the local effects (which are not different from inertial frames) with the global ones, we found that the time coordinate defined by the standard isotropic synchronization convention can not be used as a *global* coordinate for the complete rim of the rotating disk because of a time-lag associated with the round travel. It is this time-lag, whose sign depends on the direction of travel, which has to be taken into account when doing global measurements. The time-lag does only depend on the path along which the synchronization is performed but not on the particle velocity. Locally, the speed of light in standard coordinates is isotropic, but these coordinates do not match globally, so that the global round travel time is not isotropic anymore.

The non-uniqueness or necessity to introduce a discontinuity in order to use standard coordinates on the rim of the rotating disk can in our opinion not be seen as a contradiction but merely as a complication. A mathematically analogous situation is the complex function $f = \ln z$ which with its imaginary part measures the phase of z . For $z = r \exp(i\phi)$ we have $\ln z = \ln r + i\phi$. This function can be defined as a continuous and differentiable function in any region of \mathbb{C} as long as a cut is made along any line going from zero to infinity. This cut is necessary to account for the ambiguity of the phase ϕ . When crossing this line in either direction, the phase jumps by $\pm 2\pi$ so that $\ln z$ makes a jump as well. Without the cut, analytical continuation around zero would lead to different values, depending on the path. This is the same behavior as that of Minkowski time on the rim of the rotating disk. The function is well behaved locally but becomes ambiguous or discontinuous globally.

Another example is the measurement of time on earth. This analogy does indeed go further than mentioned by Bergia and Guidone.⁽⁷⁰⁾ Civil time is based on the mean local solar time, which in a simplified notion is the time measured by a sundial.⁸ This time coordinate conforms with our demand that T measures the proper time of a local clock. The synchronization is performed indirectly either by using sundials or in some equivalent way. This defines a very convenient time system, although it shows the same non-unique behavior as the standard Minkowski time on the rotating disk. Let us imagine that at some fixed Greenwich mean time GMT (which corresponds to the globally synchronized time) we shift our position eastwards. For each 15° of longitude, the local time will advance by one hour. When going around completely, local time has advanced by 24 hours. Taking the date into account, this convention means that a fixed clock will be desynchronized with itself by one day. In order to avoid this, a discontinuity, defined by the international date-line, must be introduced into the system. When crossing this line eastwards, the clocks have to be set back by one day. In this way the time coordinate stays unique but is not continuous anymore. Nevertheless, the local solar time is a much better time coordinate for most everyday life than the global GMT. Even the difference between global round travel times and local coordinate velocities can be seen quite well in this example. When

⁸ We neglect the discrete time-zones of full or half hours in this discussion and assume that the real local solar time is used.

Phileas Fogg and Passepartout travel *around the world in 80 days*,⁽⁷¹⁾ they go eastwards with a mean coordinate velocity of $360^\circ/81$ days. Nevertheless, they return after a time interval of only 80 days, measured locally by a fixed clock in London. If instead Phileas Fogg and Passepartout had parted, one traveling eastwards and one westwards, both with the same local coordinate velocity, their round trip times measured in London would have been 79 and 81 days, respectively. This difference of 2 days corresponds to the Sagnac effect. The physical reason, of course, is very different in this situation. Local solar time is *not* what is measured by a slowly transported clock (unless it is a sundial), and the coordinate velocity defined in this way is not directly related to momentum or locally measured speed. In addition, nobody would call twelve at noon in London “simultaneous” to twelve at noon in Yokohama. Still the similarity is quite striking.

In the specific situation of the rotating disk, an alternative convention of synchronization can be more convenient to describe the physics. We have shown (in agreement with e.g. Selleri,^(66,67) Goy^(32,34), Goy and Selleri⁽⁷²⁾ but also with Klauber^(61,73)) that the special choice of the synchronization parameter $A = v$ leads to a *globally* valid time coordinate T . Using this coordinate system, the description of local effects is in agreement with global effects. We have to remember, however, that this alternative description is not in disagreement with SR and does not lead to measurable physical effects different from the standard convention. It is merely a matter of convenience (or taste or philosophy) which coordinates are used.

Not all authors do agree on the conventionality of synchronization, in some cases because the difference between local and global effects is not being recognized. In a number of publications, the circumference length is divided by the round travel time to derive a “global velocity of light” which is then clearly anisotropic. However, in relativity velocities are defined locally as vectors in the local tangential space. If global measurements are used to derive velocities, one has to study how the local effects add up to become global effects. We have shown how this leads to time-lags on the rotating disk, which are in agreement with the anisotropic global behavior.

Selleri^(66,67,74) discusses alternative synchronization conventions in a very specific way. He starts by deriving the most general transformation, admitting that *a priori* any convention can be used. His free parameter e_1 is related to our A by the simple equation $e_1 = (A - v)\gamma$. In the case of the rotating disk, however, the author does not clearly distinguish between local and global effects. He claims that, due to symmetry, the globally anisotropic speed of light must necessarily also lead to a local anisotropy so that global synchronization *must* be used. This would only be true in the absence of any time-lag. Our discussion shows, that isotropic coordinates can explain the global anisotropy, if the time-lag is taken into account. Selleri^(66,67,74) claims that the “global” or “absolute” synchronization ($e_1 = 0$, $A = v$), which leads to what he calls “inertial transformations”, is more natural than the special relativistic convention. In order to support this view he additionally proposed an example with linear motion. The situation is very similar to the experiment we discussed in Sec. 9.3, see Fig. 9. On the two spaceships we now have one clock each which are initially synchronized in the laboratory frame Σ , using the standard convention. Even after the acceleration stops, they will remain to be synchronized in this system. Seen in the comoving inertial frame Σ' , however, the synchronization will violate the standard Einstein convention. The author uses this as an argument for the existence of absolute simultaneity and thus for the global synchronization. We do not find this very convincing, because this kind of synchronization can be achieved only by first synchronizing in the laboratory frame or by later somehow referring to that frame. It is no surprise that if we use Σ in order to synchronize clocks, they will be synchronized in exactly this frame. With measurements in the moving inertial frame Σ' , this “absolute” synchronization could not be defined. Actually, the system is exactly equivalent to the laboratory frame, so that the same conventions can be used in both. We emphasize that we do not argue against the *possibility* of using this global synchronization convention. The discussion of Selleri^(66,67,74) indeed shows very convincingly that this can be done and is in certain cases more convenient than the standard convention. We merely do not accept the absolute nature of this convention, nor the claim that SR cannot explain the Sagnac effect. If any convention plays a special role in a given inertial frame, without referring to other frames, it is the Einstein convention because of its isotropic nature. This still holds for the rotating disk as long as no global properties are probed.

Arguments similar to Selleri’s were presented by Goy and Selleri⁽⁷²⁾ and Goy,⁽³⁴⁾ who argue that if the clocks were synchronized in the beginning, they can for symmetry reasons not desynchronize when set to (circular) motion. It is true that clocks stay synchronized according to some convention, but this convention refers to the system Σ , so that it is in no way absolute for all systems. Relative to Einstein synchronization in Σ' , the clocks *do* desynchronize, but this is not violating homogeneity. No position is preferred but only directions. The desynchronization is homogeneous but not isotropic, and the anisotropy is defined by a real physical process, namely the acceleration in a certain direction.

Views similar to those of Selleri⁽⁶⁶⁾ and Goy and Selleri,⁽⁷²⁾ concerning the “absolute” nature of the global synchronization on the rotating disk, were presented by Klauber.^(61,73) Again we feel that a more thorough distinction between local and global effects would lead to a deeper understanding.

Concerning the interpretation of the Sagnac effect, our work agrees with many other publications, e.g. by Bergia and Guidone,⁽⁷⁰⁾ Rizzi and Tartaglia,^(75,76) Pascual-Sánchez et al.,⁽¹²⁾ Rizzi and Serafini,⁽⁶⁸⁾ Dieks⁽¹³⁾ and Weber.⁽⁶⁹⁾ We emphasize, however, that our discussion completely avoids effects of acceleration. In our formulation, the rim

of a rotating disk is treated as an inertial frame, not only in *infinitely* small regions (tangential space), but over *extended* regions, and in fact even everywhere if the topology is taken into account appropriately. This contrasts with and thus extends the work of Rizzi and Serafini,⁽⁶⁸⁾ who explicitly use local comoving inertial frames only for infinitesimally small regions, and others. The approach of Selleri,⁽⁶⁷⁾ who considers the limit of $R \rightarrow \infty$ with constant v and nevertheless finds a constant ratio of global round travel times τ_-/τ_+ , interpreted as ratio of one-way velocities of light c_+/c_- , is very instructive but not really necessary in view of our discussion.

In order to make the difference between local and global effects even clearer, we introduced a spatially closed Minkowski space-time which was shown to be equivalent to the space-time on the rim of a disk. In this space-time, there is *by construction* no acceleration, so that all local effects (measured by purely local means) are exactly the same as in a standard Minkowski space-time. Globally, the rotation velocity is defined by the offset in time introduced when glueing the space-time together. Local effects cannot be influenced by this offset, which confirms our view that there can be no objective local anisotropic effects, neither in this space-time nor on the rotating disk. Anisotropy becomes detectable only when world lines go around the closed space-time (or disk) completely. It is now a matter of convention to say whether the preferred frame effects are only a result of the periodic boundary conditions, or if they are “in reality” local but can not be detected in an open topology. We prefer the former interpretation, which avoids the introduction of aspects of reality which cannot be detected *by principle*.

The fiber optic conveyor experiment shows the topological nature of the Sagnac effect in a very nice way. We discussed that here it is explicitly shown that the observed effect depends only on the closed path itself and on the velocity and the length of the light paths, but not on the specific shape or on the acceleration along the path. One might imagine the closed Minkowski space-time of Fig. 5 embedded in a higher dimensional space not as a cylinder but following the shape of the fiber in the experiment. Still it is only the topology, the speed and total length which define the observable effect.

We found a close relation between the Sagnac effect and the twin paradox, especially when the latter is discussed in a closed Minkowski space-time or on a rotating disk. The paradoxical asymmetry between the two twins is exactly given by the time-lag observed when surrounding the rotating disk. Since both twins move along lines which are not topologically equivalent (homotopic), their proper time experiences different time-lags in comparison to a locally measured coordinate time. We found that the twin being at rest in the frame in which the disk is not rotating ages most quickly. This is not paradoxical, because not all inertial frames are equivalent in the closed topology case. The frame corresponding to the non-rotating disk is also the one with the smallest circumference length. This holds both on the rim of the disk and in the closed Minkowski space-time. In this way the twin-paradox can be explained as a topological effect as well.

Similar non-trivial effects imposed by topology are discussed in the context of cosmology by Barrow and Levin⁽⁷⁷⁾ and Uzan et al.⁽⁴⁵⁾ These authors show that, if the Universe is spatially closed, the preferred frame induced by this topology is the same as the preferred frame defined by the cosmic expansion.

It is somewhat ironic, that in relativistic physics global measurements on the rim of a rotating disk can detect rotation with respect to an inertial frame, which is not possible in non-relativistic physics.⁹ When sending test particles in both directions with the same velocity (measured in the rotating frame), they would take equal times for their round trips in Galilean physics, in contrast to the relativistic case, where they experience a time-lag which does not depend on their velocity but only on the speed of rotation and the length of the rim of the disk. The situation would be unchanged for a closed space(-time) in both cases. In this sense, motion in relativity seems to be “less relative” than in non-relativistic physics. In this context it is interesting to note that the situation is very different in quantum mechanics, where rotation can be detected even in the nonrelativistic limit (Schrödinger equation). This is a hint for a close connection between special relativity and quantum mechanics. We refer to Anandan,⁽⁷⁸⁾ Anandan and Suzuki,⁽⁷⁹⁾ Dieks⁽²⁰⁾ and Rizzi and Ruggiero⁽⁸⁾ for discussions of the Sagnac effect in quantum mechanics.

10.2. Space

In standard relativity, proper lengths of bodies at rest are directly given as differences of their spatial Minkowski coordinates. Formulated in a coordinate independent way, this means that lengths are measured as invariant intervals along lines of standard simultaneity, defined in the rest frame. In contrast to this, lengths of *moving* bodies are not defined uniquely. The length of a rod is defined as the distance between the world lines of its ends. The word “distance” can be translated as invariant interval, but then it must be defined along which line this interval is measured. This

⁹ Note that Foucault’s pendulum does not work on the rim of a rotating disk because it can detect the rotation only if it is allowed to move freely in the radial direction.

means it must be clear, at which time both ends of the rod are taken. This close relation to the conventions of simultaneity was our motivation to study lengths and especially the Ehrenfest paradox in some detail. The standard convention of taking isotropic simultaneity in the inertial frame where the length is to be measured leads to the well known effect of Lorentz contraction in its standard form. Generally, however, the length of a moving object depends on the exact measurement process applied in the experiment.

On the rotating disk, the measurements of lengths are less ambiguous than the measurements of time, so that the Ehrenfest paradox can be discussed without difficulty. When measuring the length of the rotating circle (L' proper length measured in the moving frame Σ') in the laboratory frame, we would typically compare it with the resting circle's circumference, which has a proper length of $L = 2\pi R$ in the laboratory frame Σ . In order to measure the complete rotating circle at once (without missing parts or counting parts doubly), standard simultaneity in Σ must be used. This is equivalent to taking a photograph from above the disk, so that the arguments by Einstein^(49, 51, 53) lead to the correct result. The rim of the disk is observed contracted by $1/\gamma$, so that $L = L'/\gamma$. This in turn means that the length of the corotating rim L' is *larger* than $2\pi R$ by a factor of γ . This also confirms the original argument of Ehrenfest.⁽⁴⁶⁾

In order to avoid the notion of rigid bodies, we discussed the geometrical space defining a rotating disk. We therefore find that in the comoving frame Σ' , the circumference of the circle is larger than in Euclidean space. This means that the rotating geometrical disk exhibits a negative curvature. A rigid disk or cylinder can therefore not be put from rest (Euclidean geometry) to rotation (curved space).

What happens to a real solid disk when it is set to rotation, depends on its elastic properties and other details of the experiment. Several simple limiting cases shall be discussed. (a) The spokes of the disk are assumed to stay rigid but the rubber circumference is allowed to expand. In this way the result would be equivalent to the discussion of the geometrical disk. The corotating circumference is expanded, so that its Lorentz contracted version observed in the laboratory corresponds to the circle at rest. The observer in Σ would not see any change of the circle, because the elastic expansion and the Lorentz contraction cancel. (b) The spokes are made of rubber but the circumference keeps its length. In this case the radius will shrink in order to decrease the geometrical circumference length. This contraction could be observed from Σ and Σ' . The cases (a) and (b) are discussed in a recent discussion by Davidović and Arsenović⁽⁸⁰⁾ under the names of “star disc” and “ring disc”, respectively.

A third special case is also of interest: (c) The disk is rigid along its two-dimensional extension but flexible in the ζ direction. When set to rotation, it would then have to bend in the vertical direction so that the positive curvature obtained in this way compensates for the negative curvature caused by the rotation. The surface would seem positively curved from Σ , but intrinsically flat when observed from Σ' .

This discussion shows that the Ehrenfest paradox is real and can only be solved by understanding that rigid bodies are not consistent with SR, or at least they cannot accelerate arbitrarily. Linear accelerations are not a problem, because the Lorentz contraction can “borrow” space from outside. On the rotating disk or in the closed Minkowski space-time, this is not possible because of the restrictions imposed by the topology.

Our analysis confirms the original arguments of Ehrenfest⁽⁴⁶⁾ and Einstein,^(49, 51, 53) and showed that an explanation consistent with those views can also be derived in the corotating frame without any difficulty. We therefore do not agree with the views of Tartaglia,^(62, 63) Klauber⁽⁶¹⁾ and Peres,⁽⁵⁵⁾ who claim that there is no Lorentz contraction and thus no Ehrenfest paradox. On the other hand, our discussion is in good agreement with the work of various other authors, e.g. Rizzi and Ruggiero,⁽⁴⁷⁾ Ruggiero,⁽⁵⁶⁾ Dieks⁽¹³⁾ and Weber.⁽⁶⁹⁾

11. CONCLUSIONS

All the discussions in this paper have in common the importance of clear definitions of the concepts used. Once *time* and *length/space* are defined unambiguously, the measurements in all the (real and thought) experiments can be translated relatively easily into their corresponding space-time concepts. The rim of a rotating disk can be treated as an inertial system without any contradiction, as long as the radial coordinate is not probed. Unless the disk is surrounded, the situation is exactly equivalent to that of linear motion as discussed in standard relativity textbooks. When the global structure, i.e. the topology, of the rotating circle becomes relevant, one has to ensure that the locally valid coordinates and space-time descriptions really match or “mesh” globally. On the rotating disk, we found that local Minkowski coordinates can not be extended globally, which gives the explanation for the Sagnac effect. In order to overcome any lack of confidence in our notion that the acceleration is not important for the discussion of effects on the rim of the rotating disk, we showed the equivalence with the situation of a topologically closed Minkowski space-time, which can physically not be distinguished from the rim of a rotating disk. We have to keep in mind that the standard notion of special relativity using Lorentz transformations leads to coordinates which are valid locally. In standard open space-time situations, these coordinates can be extended to form globally valid systems. Periodic boundary conditions or closed space-time topology, on the other hand, will restrict the allowed range of these

coordinates. Additional discontinuities or time-lags can appear when matching the local coordinates globally. This leads to preferred frame effects which are purely global and of topological nature. Although most fundamental physics is usually formulated in terms of partial differential equations, we have to keep in mind that such a local description can naturally not explain the world completely. Boundary conditions and thus the topology of space-time can play an equally important role.

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1. H. Reichenbach, *Axiomatik der relativistischen Raum-Zeit-Lehre* (Vieweg, Braunschweig, 1924), english version in Ref. 2.
 2. H. Reichenbach, *Axiomatization of the theory of relativity* (Berkeley University Press, 1969), english version of Ref. 1.
 3. E. J. Post, *Rev. Mod. Phys.* **39**, 475 (1967).
 4. F. Hasselbach and M. Nicklaus, *Phys. Rev. A* **48**, 143 (1993).
 5. G. E. Stedman, *Rep. Prog. Phys.* **60**, 615 (1997).
 6. R. Anderson, I. Vetharaniam, and G. E. Stedman, *Phys. Rep.* **295**, 93 (1998).
 7. G. Rizzi and M. Ruggiero, eds., *Relativity in Rotating Frames* (Kluwer, 2004).
 8. G. Rizzi and M. L. Ruggiero, *The relativistic Sagnac effect: two derivations*, in Ref. 7 (2004), gr-qc/0305084.
 9. G. Sagnac, *C. R. Acad. Sci.* **157**, 708 (1913).
 10. G. Sagnac, *C. R. Acad. Sci.* **157**, 1410 (1913).
 11. G. Sagnac, *J. Phys. (Paris)* **4**, 177 (1914).
 12. J.-F. Pascual-Sánchez, A. San Miguel, and F. Vicente, *Isotropy of the velocity of light and the Sagnac effect*, in Ref. 7 (2004).
 13. D. Dieks, *Space and Time in a Rotating World*, in Ref. 7 (2004).
 14. B. Mashhoon, *Phys. Lett. A* **143**, 176 (1990).
 15. B. Mashhoon, *Phys. Lett. A* **145**, 147 (1990).
 16. B. Mashhoon, *The hypothesis of locality and its limitations*, in Ref. 7 (2004).
 17. F. Sorge, *Local and global anisotropy in the speed of light*, in Ref. 7 (2004).
 18. H. Reichenbach, *Philosophie der Raum-Zeit-Lehre* (de Gruyter, Berlin & Leipzig, 1928), english version in Ref. 19.
 19. H. Reichenbach, *The philosophy of space & time* (Dover, New York, 1958), english version of Ref. 18.
 20. D. Dieks, *Europ. J. Phys.* **12**, 253 (1991).
 21. H. P. Robertson, *Rev. Mod. Phys.* **21**, 378 (1949).
 22. R. Mansouri and R. U. Sexl, *Gen. Relativ. Gravit.* **8**, 497 (1977).
 23. R. Mansouri and R. U. Sexl, *Gen. Relativ. Gravit.* **8**, 515 (1977).
 24. R. Mansouri and R. U. Sexl, *Gen. Relativ. Gravit.* **8**, 809 (1977).
 25. J. G. Vargas and D. G. Torr, *Phys. Rev. A* **39**, 2878 (1989).
 26. C. M. Will, *Phys. Rev. D* **45**, 403 (1992).
 27. I. Vetharaniam and G. E. Stedman, *Phys. Lett. A* **183**, 349 (1993).
 28. A. Einstein, *Ann. Phys. (Leipzig)* **17**, 891 (1905).
 29. E. Minguzzi, *Found. Phys. Lett.* **15**, 153 (2002).
 30. E. Minguzzi, *Class. Quant. Gravit.* **20**, 2443 (2003).
 31. F. Goy, *ArXiv Gen. Relat. Quant. Cosmol. e-print* (1996), London conference on Physical Interpretations of Relativity Theory, gr-qc/9607047.
 32. F. Goy, *ArXiv Gen. Relat. Quant. Cosmol. e-print* (1996), gr-qc/9607042.
 33. F. Goy, *ArXiv Gen. Relat. Quant. Cosmol. e-print* (1997), gr-qc/9702042.
 34. F. Goy, *ArXiv Gen. Relat. Quant. Cosmol. e-print* (1997), Talk of the Athen's Conference on Relativistic Physics, gr-qc/9707005.
 35. G. Spavieri, *Phys. Rev. A* **34**, 1708 (1986).
 36. J. R. Croca and F. Selleri, *Nuovo Cimento B* **114**, 447 (1999).
 37. R. Wang, Y. Zheng, A. Yao, and D. Langley, *Phys. Lett. A* **312**, 7 (2003).
 38. A. Tartaglia and M. L. Ruggiero, *ArXiv Gen. Relat. Quant. Cosmol. e-print* (2004), gr-qc/0401005.
 39. H. Dingle, *Proc. Phys. Soc. London, Sect. A* **69**, 925 (1956).
 40. C. H. Brans and D. R. Stewart, *Phys. Rev. D* **8**, 1662 (1973).
 41. R. J. Low, *Europ. J. Phys.* **11**, 25 (1990).
 42. T. Dray, *Am. J. Phys.* **58**, 822 (1990).
 43. P. C. Peters, *Am. J. Phys.* **51**, 791 (1983).
 44. J. D. Barrow and J. Levin, *Phys. Rev. A* **63**, 044104 (2001).
 45. J. Uzan, J. Luminet, R. Lehoucq, and P. Peter, *Europ. J. Phys.* **23**, 277 (2002).
 46. P. Ehrenfest, *Phys. Z.* **10**, 918 (1909).
 47. G. Rizzi and M. L. Ruggiero, *Found. Phys. Lett.* **32**, 1525 (2002), gr-qc/0207104.
 48. Ø. Grøn, *Space geometry in rotating reference frames: A historical appraisal*, in Ref. 7 (2004).
 49. A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie* (Vieweg, Braunschweig, 1917), english in Ref. 50.
 50. A. Einstein, *Relativity; the special and general theory* (P. Smith, New York, 1931), english version of Ref. 49.
 51. A. Einstein, *The Meaning of Relativity — A collection of four lectures delivered at Princeton University* (Princeton Uni-

- versity Press, 1921), german version in Ref. 52.
52. A. Einstein, *Vier Vorlesungen über Relativitätstheorie, gehalten im Mai 1921 an der Universität Princeton* (Vieweg, Braunschweig, 1922), german version of Ref. 51.
 53. A. Einstein, *Grundzüge der Relativitätstheorie* (Vieweg, Braunschweig, 1956), 3. edition of Ref. 52.
 54. A. Einstein, Phys. Z. **12**, 509 (1911).
 55. A. Peres, ArXiv Gen. Relat. Quant. Cosmol. e-print (2004), submitted to Am. J. Phys., gr-qc/0401043.
 56. M. L. Ruggiero, Europ. J. Phys. **24**, 563 (2003).
 57. J. S. Bell, Progress in Scientific Culture **1** (1976), also in Ref. 59, pp. 67–80.
 58. E. Dewan and M. Beran, Am. J. Phys. **27**, 517 (1959).
 59. J. S. Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, 1987).
 60. J. H. Field, ArXiv Phys. e-print (2004), physics/0403094.
 61. R. D. Klauber, Found. Phys. Lett. **11**, 405 (1998), gr-qc/0103076.
 62. A. Tartaglia, Found. Phys. Lett. **12**, 17 (1999), physics/9808001.
 63. A. Tartaglia, *Does anything happen on a rotating disk?*, in Ref. 7 (2004).
 64. N. Ashby, *The Sagnac effect in the Global Positioning System*, in Ref. 7 (2004).
 65. E. Minguzzi and A. Macdonald, Found. Phys. Lett. **16**, 593 (2003).
 66. F. Selleri, Found. Phys. **26**, 641 (1996).
 67. F. Selleri, Found. Phys. Lett. **10**, 72 (1997).
 68. G. Rizzi and A. Serafini, *Synchronization and Desynchronization on Rotating Platforms*, in Ref. 7 (2004).
 69. T. A. Weber, *Elementary Considerations of the Time and Geometry of Rotating Reference Frames*, in Ref. 7 (2004).
 70. S. Bergia and M. Guidone, Found. Phys. Lett. **11**, 549 (1998).
 71. J. Verne, *Tour du monde en quatre-vingts jours* (Pierre-Jules Hetzel, Paris, 1873), English: *Around the world in 80 days*.
 72. F. Goy and F. Selleri, Found. Phys. Lett. **10**, 17 (1997), gr-qc/9702055.
 73. R. D. Klauber, Found. Phys. Lett. **16**, 447 (2003).
 74. F. Selleri, *Sagnac effect: end of the mystery*, in Ref. 7 (2004).
 75. G. Rizzi and A. Tartaglia, Found. Phys. **28**, 1663 (1998).
 76. G. Rizzi and A. Tartaglia, Found. Phys. Lett. **12**, 179 (1999).
 77. J. D. Barrow and J. Levin, Mon. Not. R. Astron. Soc. **346**, 615 (2003).
 78. J. Anandan, Phys. Rev. D **24**, 338 (1981).
 79. J. Anandan and J. Suzuki, *Quantum Mechanics in a Rotating Frame*, in Ref. 7 (2004).
 80. D. M. Davidović and D. Arsenović, Found. Phys. Lett. **17**, 183 (2004).