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Limitation on proper length in special relativity

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The proper length L^* of an accelerating object is limited to the values $L^* \leqslant c^2/a_F^*$ where a_F^* is the proper acceleration of the front end. For the maximum length, the acceleration of the rear end is infinite. The effect rests on the relativity of simultaneity, one of the most difficult relativity concepts for students. We explore some consequences of this result.

I. THE PARADOX

Warnick[1] has proposed a paradox whose solution reveals a physical limitation on the definition of proper length ("rest length") in special relativity. Briefly, the paradox envisions a spaceship of proper length L^* whose front end accelerates from rest over distance x_F in time t_F to a speed at which the ship is contracted to half its rest length. Then the rear end has moved a distance $x_F + (L^*/2)$ in time t_F with an average speed $[x_F + (L^*/2)]/t_F$. Since L^* can be arbitrarily long, this average speed can be arbitrarily great, even greater than the speed of light. *Sic transit relativitas!*

In the statement of the paradox and in what follows, all quantities are measured with respect to the Earth frame unless otherwise noted.

II. PARADOX RESOLVED: SHIFTING SIMULTANEITY

Analyze the motion of the spaceship as a series of paired impulses giving equal speed boosts $d\beta$ to the front and rear ends, where $\beta = v/c$. In order that proper length be preserved, each pair of speed boosts must be simultaneous (as well as equal) in the spaceship frame: $\Delta t' = 0$. They also take place a distance L^* apart in the spaceship frame:

$\Delta x' = L^*$. Then there is a time difference between the paired boosts in the Earth frame, given by the Lorentz transformation equation $t = \gamma(t' + \beta x'/c)$:

$$\Delta t = \beta\gamma L^*/c, \tag{1}$$

where $\gamma = (1-\beta^2)^{-1/2}$. In the Earth frame the rear end receives each speed boost *before* the paired boost is given to the front end. The accumulated Lorentz contraction from speed zero to β is accounted for by the rear end "catching up" with the front during the time lapses between the paired boosts. (See the Appendix for a derivation.)

Now as the spaceship accelerates, the rear boost occurs at a time increasingly earlier than the paired front boost, according to the equation

$$\frac{d(\Delta t)}{d\beta} = \frac{d}{d\beta}\left(\frac{L^*}{c}\beta\gamma\right) = \frac{L^*}{c}\gamma^3. \tag{2}$$

A consequence is that the boosts occur closer together in time at the rear than the front. By an extension of this reasoning, one can derive a limiting condition for which "sequential" rear-end boosts occur not just closer together in time but actually *at the same time*. Let dt_F be the time between sequential boosts at the front. Set this equal to the *change* in Δt between rear boosts using Eq. (2):

$$dt_F = d(\Delta t) = (L^*/c)\gamma^3\, d\beta.$$

But the acceleration of the front end a_F is equal to $c\, d\beta/dt_F$, so that the condition that sequential rear-end boosts occur at the same time in the Earth frame is just

$$(L^*/c^2)\gamma^3 a_F = 1 \quad \text{(limiting case)}. \tag{3}$$

Under these conditions, the rear experiences two speed boosts at the same time ($dt_R = 0$). That is, *the acceleration $c\, d\beta/dt_R$ of the rear end is infinite*. Any more rapid acceleration of the rear end is physically impossible. Thus Eq. (3) sets a limit on the front-end acceleration for any given proper length, or conversely a limit on the proper length for any given front-end acceleration.

We can further simplify the form of Eq. (3) by noting that $\gamma^3 a_F$ is the *proper* acceleration a_F^* experienced by a rider in the front of the spaceship at any Earth speed β. (See the Appendix.) Then the condition on the proper length or on the front-end acceleration of the spaceship may be expressed as an inequality:

$$L^* \leqslant c^2/a_F^*; \quad a_F^* \leqslant c^2/L^*. \tag{4}$$

For a spaceship 2 km long, the maximum proper acceleration of the front end is 4.5×10^{13} m/s^2. For a spaceship of length equal to the distance between Earth and moon, the maximum front-end acceleration is "only" 24 million times the acceleration of gravity—hardly anything to worry about in any practical case, at least on the macroscopic scale. [On the subatomic scale, however, accelerations corresponding to the limit given by Eq. (4) are quite conceivable. An intriguing special case arises if we postulate two particles of charge e, mass m, and diameter L^* being placed in contact and beginning to accelerate under their mutual repulsion. The initial acceleration would be equal to e^2/mL^{*2}. If we set this equal to the limiting acceleration c^2/L^* of Eq. (4), we arrive at the condition $L^* = e^2/mc^2$, which for $m = m_e$ gives us the "classical electron radius." Thus each of these electrons has, in some sense, achieved the limiting acceleration according to the analysis above. The actual value of the initial acceleration in this case would be 3.2×10^{30} *g*!]

What happens if we demand that the front end of the spaceship increase speed at a faster rate than the limit given in Eq. (4)? Since greater-than-infinite acceleration of the rear end is impossible, the result will be speed boosts that are no longer simultaneous in the spaceship frame. In fact, the speed boost for the front end will occur before that for the rear in this frame and the spaceship will be pulled apart: It will no longer have a fixed proper length.

The limit on proper length in Eq. (4) resolves the paradox proposed by Warnick. Fundamentally the solution depends on the relativity of simultaneity (according to which the simultaneous speed boosts in the spaceship frame imply rear-first boosts in the earth frame) and the *change* in this relativity of simultaneity as the spaceship accelerates (so that sequential speed boosts can occur at the same time for the rear, leading to infinite acceleration). Since the relativity of simultaneity is one of the most difficult relativity concepts for students, we explore here some of the implications and consequences of our result.

III. LIMITING SPEED HISTORY

Warnick's statement of the paradox is powerful in part because it is not limited to a particular program of front-end acceleration. Our solution to the paradox has a similar character, pointing out a limit on the instantaneous acceleration of the front end for a given speed β and proper length L^*. One can carry this analysis to its logical limit and derive a program of front-end acceleration such that the rear-end acceleration is infinite for *all* speeds up to (nearly) the limiting speed of light ($\beta = 1$). In this case, the rear end receives at one time *all* the speed boosts necessary to carry it (nearly) to this speed.

The result is shown in Fig. 1. From Eq. (4), the required limiting front-end acceleration is simply constant *proper* acceleration in the instantaneous spaceship frame: the front-end observer experiences a steady constant acceleration c^2/L^*. This leads to a single value for the maximum proper length; call this maximum value L_m^*. Then Eq. (1) tells us that, with respect to the Earth frame, the front end follows the implicit speed curve $\beta\gamma = t/(L_m^*/c) = t/t_m$, the curve of increasing rear–front anticipation that makes all rear-end speed boosts occur at the same time. By measuring time in units $t_m = L_m^*/c$, the time for light to travel the maximum proper length, we generalize the results to spaceships of all L_m^* (see Appendix).

All less strenuous histories of front and rear speeds can

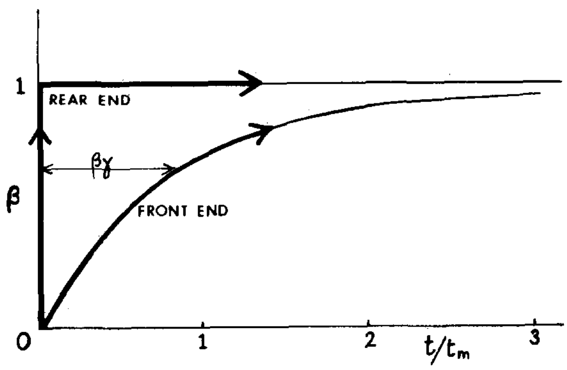

Fig. 1. Limiting-case speed-versus-time curves for front and rear of spaceship if rest length is to be preserved.

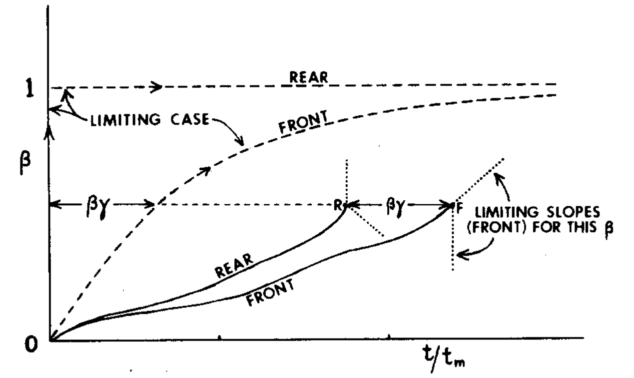

Fig. 2. Curves *OF* and *OR* are less strenuous histories of the speeds of front and rear ends of spaceship. Dashed lines are limiting case for comparison. Dotted lines show limits of front- and rear-end accelerations. The maximum acceleration of the front end (or deceleration of the rear end) is equal to the magnitude of the slope of the limiting-case front end curve for that speed.

be derived graphically from this limiting case, as shown in Fig. 2. Let *OF* be some arbitrary speed history of the front end of a spaceship of the same proper length L^*_m (which, for this history, is not the longest possible length). The corresponding curve *OR* for the rear can be simply constructed: The rear end achieves each speed a time $t/t_m = \beta\gamma$ earlier than the front end. This is the same anticipation as in the limiting case at the same speed. Moreover, the slope of the front-end speed-versus-time curve cannot be greater than the slope of the "limiting" front end curve at the same speed, as shown for point *F*.

In a *deceleration* the roles of front and rear ends are exchanged. If the *rear* end is given a particular deceleration, the proper length of the spaceship has a greatest value such that the deceleration of the *front* end is infinite. This symmetry is consistent with the principle of relativity, according to which the occupants of the spaceship cannot detect by internal observations their velocity with respect to the Earth. Therefore when the velocity changes they cannot distinguish between decrease of forward velocity and increase of backward velocity with respect to the Earth. The resulting description in the Earth frame is asymmetric in the limiting acceleration and deceleration for the front end (and also asymmetric for the rear) as shown by the dotted lines in Fig. 2.

IV. EXPERIENCES OF THE RIDERS

For a spaceship of maximum possible proper length L^*_m the front-end rider experiences a constant proper acceleration $a^*_F = c^2/L^*_m$ in the limiting-case speed history shown in Fig. 1. The rear-end rider suffers an infinite acceleration. What do riders between these two extremes experience?

The answer to this question is readily obtained, because the type of motion that we have been discussing has been familiar since the earliest days of relativity theory under the name "hyperbolic motion."[2] By direct integration of our condition $t/t_m = \beta\gamma$ (see Appendix), one can deduce that the front end of the spaceship follows a hyperbolic worldline in the Earth space-time diagram:

$$x^2 - c^2t^2 = L^{*2}_m.$$

This is shown in Fig. 3, where we measure distance along

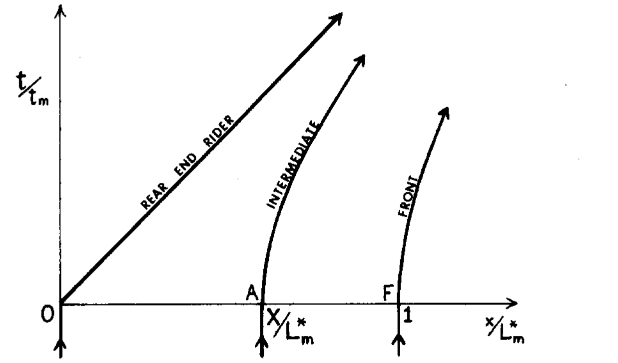

Fig. 3. Worldlines of front, rear, and intermediate riders in Earth spacetime diagram for limiting-case spaceship acceleration.

the spaceship in units of L^*_m by plotting

$$(x/L^*_m)^2 - (t/t_m)^2 = 1. \tag{5}$$

We asume that acceleration begins at $t = 0$. At this instant, the rear end receives *all* the speed boosts and thereafter follows the light line $x = ct$ or $x/L^*_m = t/t_m$. Point *A* on the spaceship having the coordinate $x = X$ at $t = 0$ (and proper distance X from the rear for all time) follows a hyperbolic world line given by

$$(x/L^*_m)^2 - (t/t_m)^2 = (X/L^*_m)^2. \tag{6}$$

A rider at this point on the spaceship experiences a constant proper acceleration c^2/X.

We can now imagine a program for the constant proper acceleration of a spaceship of *any* proper length $L^* < L^*_m$ as shown in Fig. 4. Set the desired acceleration of the front end equal to $a^*_F = c^2/L^*_m$, thus defining a maximum proper length, plotted as before on the spacetime diagram. In this case the rear end also follows a hyperbolic world line, beginning at *R*, a distance L^* back from the front end. The rear end has proper acceleration $c^2/(L^*_m - L^*)$. Points in between have proper acceleration $c^2/(L^*_m - Y)$, where Y is the proper distance from the front of the spaceship.

If the spaceship is accelerated according to the above

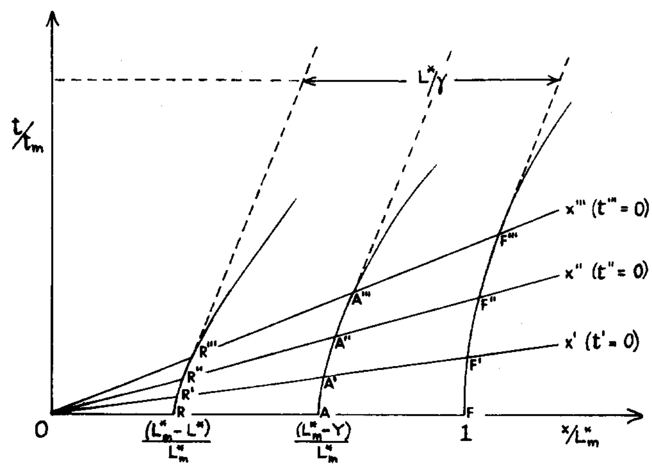

Fig. 4. Worldlines of front, middle, and end of spaceship of proper length L^*, less than the maximum, undergoing constant proper acceleration. The axes Ox', Ox'', Ox''' are sequential lines of simultaneity in the spaceship frame. Dashed world lines represent coasting that begins simultaneously in the frame S'''.

conditions, it finds itself instantaneously at rest in a succession of inertial frames S', S'', S''', etc. In Fig. 4 each of the points F', A', and R' has an instantaneous velocity relative to the Earth frame S that is exactly the same as the velocity of S' itself. Moreover, the events F', A', and R' lie on a line of simultaneity for S'; it is in fact the line $t'=0$. Thus, if riders on the accelerating spaceship were passing a line of clocks all synchronized in S', the clock nearest to each rider would read $t'=0$ at the moment when the spaceship came to be instantaneously at rest with respect to that set of clocks. Further acceleration would lead to similar observations on clocks reading $t''=0$ in S'', $t'''=0$ in S''', and so on. This does *not*, however, imply that the journey involves no elapsed time for the riders themselves. Clocks carried on the spaceship itself will (see Appendix) record elapsed time (proper time) T_X^* proportional to their constant proper distances X from the spacetime origin of Fig. 4:

$$T_X^* = (X/c)\ln[(1+\beta)/(1-\beta)]^{1/2}. \tag{7}$$

This effect ($t'=0$ but nonzero reading on an accelerating clock) is, once again, due to the shifting simultaneity of the different inertial frames through which the accelerating clock passes. Only the rear-end clock in the limiting case receives all speed boosts simultaneously at $t=t'=t''=0$.

If the acceleration of the spaceship is not continuous, but is brought about by discrete equal speed boosts (as judged, for example, by the observed change in blue shift of light from a distant star in the forward direction), then the proper time intervals ΔT_X^* must be made proportional to $X=L_m^*-Y$, so that the pulse rate for the front rider is slower than for riders farther back. The required program of acceleration can thus be carried out on the basis of observations and actions by operators at different places along the spaceship, without any need for communication between them (see Appendix).

V. MEANING OF "PROPER LENGTH"

How can it be that a spaceship remains "rigid" while riders at different points along its length experience different accelerations? The answer, once again, depends on shifting simultaneity as the spaceship accelerates. Figure 4 shows sequential lines of simultaneity in the spaceship frame, labeled $t'=0$ and $t''=0$ and $t'''=0$. One may think of events F, F', F'', etc. as the speed boosts for the front end, A, A', A'', etc. as the speed boosts for the middle, and R, R', R'', etc. as the speed boosts for the rear end. As we have just seen, however, the front-end rider experiences the boosts in slower sequence than do the other observers. Nevertheless, the first (say) ten speed boosts give as great a total speed to the front as to the middle, even though the middle rider experiences them in a shorter total elapsed time, and the rider in the rear in a still shorter time.

Complications of this kind have led some writers[3] to declare that the concept "rigid body" has no meaning in relativity. And indeed the structure of the spaceship cannot transmit (at the speed of sound) the speed boosts from one part to another instantaneously. Thinking rigorously, one can envision the spaceship as a row of unconnected mass points that are individually accelerated. Inspectors riding along with the spaceship can verify the constancy of the (proper) distance between these mass points and can collect data usable later to verify that each paired front and rear boost (and those in between) occurred simultaneously in the spaceship frame. But of course the interpretation of the data must involve a recognition that the definition of simultaneity was changing continuously during the acceleration process.

Under these circumstances a proper length of the accelerating spaceship can be defined, subject to the limitation on total proper length (or acceleration) derived earlier.

VI. LORENTZ CONTRACTION

Early in this paper we said that the cumulative Lorentz contraction of the spaceship as observed in the Earth frame can be accounted for by the time lapses between paired rear boost and front boost, time lapses during which the rear is moving faster than the front. The greater speed of the rear compared with the front at the same time in the Earth frame is evident in all the figures, even for the nonlimiting case shown in Fig. 4. But how can we speak of Lorentz contraction if, at a given time in the earth frame, different parts of the spaceship are moving with different speeds, so that the "contraction factor" $(1-\beta^2)^{1/2}$ cannot even be defined?

Romain[4,5] has cautioned us that "rest length" must be defined in a nonaccelerating frame. At points R''', A''', and F''' in Fig. 4 the rear, middle, and front have received equal numbers of speed boosts and are moving at the same speed. If the spaceship begins to coast at these events, simultaneous in its frame, proper length will be preserved. Thereafter the front, middle, and rear follow parallel world lines in the earth spacetime diagram, shown dashed in the figure. These parallel world lines *do* represent equal speeds, so a Lorentz contraction factor appropriate to this common velocity can be defined and is the correct one. Even if the spaceship itself continues to accelerate, measurements made at the events R''', A''', and F''' defined by the line of simultaneity $t'''=0$ in the instantaneous spaceship frame can be used to infer the Lorentz-contracted length that the spaceship would have if it did continue to coast at that speed.

VII. A DIFFERENT CASE

The acceleration process discussed here must be distinguished from the case of identical small spaceships that start simultaneously from rest a distance L_0 apart and use identical thrusts for all Earth times, examined in a well-known series of articles in this Journal.[6] Such motion preserves Earth distance between ships (for which where is no limit) but does not preserve proper distance between them.

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APPENDIX

A. Progressive contraction of accelerated spaceship (Sec. II)

For simultaneous speed boosts in spaceship frame, rear boost precedes front boost in Earth frame by Δt [Eq. (1)]:

$$\Delta t = \beta\gamma L^*/c.$$

Speed of rear exceeds speed of front by $c\,d\beta$ during Δt,

$$\therefore\ dL = -c\,d\beta\,\Delta t = -L^*\gamma\beta\,d\beta,$$

$$L(\beta) - L^* = -L^*\int_0^\beta (1-\beta^2)^{-1/2}\beta\,d\beta$$

$$= L^*[(1-\beta^2)^{1/2} - 1],$$

$$\therefore\ L(\beta) = L^*(1-\beta^2)^{1/2} = L^*/\gamma,$$

which is the conventional Lorentz contraction formula. See Sec. VI for a commentary on this result.

B. Proper acceleration in instantaneous rest frame (Sec. II)

The general transformation equation for acceleration along the direction of relative motion of two frames S and S' is[7]

$$a_x = a'_x/\gamma^3(1 + vu'_x/c^2)^3,$$

where v is the velocity of S' with respect to S, γ is $(1 - v^2/c^2)^{-1/2}$, and u'_x is the instantaneous velocity of the object with respect to frame S'. Putting $u'_x = 0$, we have[8]

$$a'_x = \gamma^3 a_x,$$

or, in the notation of the present article, Sec. II,

$$a_F^* = \gamma^3 a_F.$$

C. Limiting speed–time curve for front end (Sec. III)

For infinite rear-end acceleration, the implicit speed–time curve for the front end is given by

$$\beta\gamma = t/t_m,$$

where $t_m = L_m^*/c$, i.e.,

$$\beta/(1-\beta^2)^{1/2} = t/t_m.$$

From this it follows that the curve of β vs t/t_m is given by

$$\beta = (t/t_m)/[1 + (t/t_m)^2]^{1/2},$$

which is shown in Fig. 1.

D. Hyperbolic motion (Sec. IV)

Motion of front end of spaceship for maximum acceleration is defined by $t/t_m = \beta\gamma$, with $t_m = L_m^*/c$, i.e.,

$$ct/L_m^* = \beta/(1-\beta^2)^{1/2},$$

$$\therefore\ \beta = \frac{1}{c}\frac{dx}{dt} = \frac{ct}{(L_m^{*2} + c^2t^2)^{1/2}}.$$

Integrating,

$$x = (L_m^{*2} + c^2t^2)^{1/2}$$

or

$$x^2 - c^2t^2 = L_m^{*2}.$$

E. Clock readings aboard spaceship (Sec. IV)

Consider an invariant space-time interval ds as measured in spaceship frame and in Earth frame:

$$ds^2 = c^2\,dT^{*2} - dx^{*2} = c^2\,dt^2 - dx^2.$$

For a clock at rest in the spaceship, $dx^* = 0$. For hyperbolic motion of a point on the spaceship with proper coordinate X,

$$x^2 = X^2 + c^2t^2,$$

giving

$$dx = \frac{c^2t\,dt}{x} = \frac{c^2t\,dt}{(X^2 + c^2t^2)^{1/2}},$$

$$\therefore\ ds^2 = c^2\,dT^{*2} = c^2\,dt^2\left(1 - \frac{c^2t^2}{X^2 + c^2t^2}\right)$$

$$= \frac{X^2c^2\,dt^2}{X^2 + c^2t^2},$$

$$\therefore\ dT_X^* = \frac{X}{c}\frac{c\,dt}{(X^2 + c^2t^2)^{1/2}}.$$

Integrating,

$$T_X^* = (X/c)\ln\{[ct + (X^2 + c^2t^2)^{1/2}]/X\}.$$

But for hyperbolic motion, speed is given by

$$\beta = \frac{1}{c}\frac{dx}{dt} = \frac{ct}{x} = \frac{ct}{(X^2 + c^2t^2)^{1/2}},$$

$$\therefore\ ct = X\beta(1-\beta^2)^{-1/2}$$

and

$$(X^2 + c^2t^2)^{1/2} = X(1-\beta^2)^{-1/2},$$

$$\therefore\ T_X^* = (X/c)\ln[(1+\beta)/(1-\beta)]^{1/2}.$$

F. Velocity increments in spaceship and Earth frames (Sec. IV)

For velocity increments $d\beta$ as measured in Earth frame, the above equation for T_X^* gives

$$dT_X^* = \frac{X}{c}\frac{d\beta}{(1-\beta^2)} = \frac{X}{c}\gamma^2\,d\beta.$$

The proper acceleration a_X^* at X is c^2/X. Hence

$$a_X^*\,dT_X^* = c\gamma^2\,d\beta = \gamma^2\,dv.$$

This is the correct result, as given by the velocity addition theorem, for the change of velocity from v to $v + dv$ (Earth frame) as observed from a frame itself traveling with speed v (i.e., the instaneous rest frame of the spaceship):

$$dv' = \frac{(v + dv) - v}{1 - (v + dv)v/c^2} \to \gamma^2\,dv.$$

If we consider acceleration by a succession of sudden speed boosts $\Delta\beta$, their spacings in time at at given position along the spaceship are given by

$$\Delta T_X^* = (X/c)\gamma^2\Delta\beta = [(L_m^* - Y)/c]\gamma^2\Delta\beta.$$

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