

Relative space: space measurements on a rotating platform

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Abstract

We introduce here the concept of relative space, an extended 3-space which is recognized as the only space having an operational meaning in the study of the space geometry of a rotating disc. Accordingly, we illustrate how space measurements are performed in the relative space, and we show that an old-aged puzzling problem, Ehrenfest's paradox, is explained in this purely relativistic context. Furthermore, we illustrate the kinematical origin of the tangential dilation which is responsible for the solution of Ehrenfest's paradox.

1. Introduction

In the special theory of relativity (SRT) the rotation of the reference frame, contrary to the translation, has an absolute character and can be locally measured by Foucault's pendulum or by the Sagnac experiment. This peculiarity of rotation, inherited from Newtonian physics, is difficult to understand in a relativistic context. As a matter of fact, many authors who opposed the SRT found, in the relativistic approach to rotation, important arguments against the self-consistency of the theory. In 1909 Ehrenfest [1] pointed out an internal contradiction in the SRT, applied to the case of a rotating disc; few years later, in 1913, Sagnac [2, 3] demonstrated an apparent contradiction in the SRT with respect to the experimental data.

Over the years, those seminal papers influenced discussions on the foundations of the SRT, though the 'paradoxes' disappear when a careful analysis is undertaken, following the axioms of the theory. So, it is perhaps surprising that, one century later, in some papers and textbooks, misleading or even uncorrect arguments are given to explain Ehrenfest's paradox and the Sagnac effect. For instance, Klauber in 1998 [4, 5] proposed a 'new theory of rotating frames' in order to address the contradictions which, according to him, appear in the SRT when applied to rotating frames.

In [6] we studied Ehrenfest's paradox, showing that it can be solved on the basis of purely kinematical arguments in the SRT. We adopted a geometrical approach based on (i) a precise definition of the concept of 'space of the disc' and (ii) a precise choice of the 'standard rods' used by the observers on the platform. The space of the disc has been formally identified with

what we called the ‘relative space’; its geometry has been recognized to be non-Euclidean¹ and its metric coincides with the one which is found in classic textbooks of relativity, in spite of the shift of context, as we stressed. We adopted the projection technique introduced by Cattaneo [7–11] to approach the problem, since this allows an elegant and straightforward description of the geometry of any reference frame.

Even though a global *isotropic* 1 + 3 splitting of space–time is not possible when we deal with rotating observers, the introduction of the relative space allows well defined procedures for space measurements that can be actually performed globally by the observers in rotating frames and which reduce to the standard measurements in any locally co-moving inertial frames.

Here we show that these procedures allow a systematic study of the space measurements, causing Ehrenfest’s paradox to disappear. Moreover, we illustrate in a clear way (with the aid of a pictorial representation) the kinematical origin of the dilation which leads to the solution of Ehrenfest’s paradox.

We believe that the concept of relative space can help to clear up many misunderstandings, often caused by the lack of proper definitions of some crucial concepts in the theoretical apparatus used to describe these physical problems: indeed, they only appear puzzling or contradictory in the SRT when ambiguous entities and procedures are adopted.

2. Ehrenfest’s paradox

The formulation of Ehrenfest’s paradox [1] is as follows.

Let R , R' be the radii of the rotating disc, as measured, respectively, by the inertial and rotating observer; ω is the constant angular velocity of the disc, as measured in the inertial frame. The paradox arises when the following contradictory statements are taken into account:

- (a) The circumference of the disc must show a contraction relative to its rest state, $2\pi R < 2\pi R'$, since each element of the circumference moves in its own direction with instantaneous speed ωR .
- (b) If one considers an element of a radius, its instantaneous velocity is perpendicular to its length; thus, an element of the radius cannot show a contraction with respect to the rest state. Therefore $R = R'$.

On the basis of this contradiction, Ehrenfest pointed out the apparent inconsistency of the kinematics of bodies which are rigid, according to the definition of rigidity given by Born (see below).

Many authors have attempted to explain the paradox. The most popular attempts at solution can be summarized in this way:

- (1) neither the rods along the rim nor the circumference contract; neither the rods along the radius nor the radius itself contract; as a consequence, the space of the rotating disc is Euclidean (see [4, 5, 12]);
- (2) both the rods along the rim and the circumference contract; neither the rods along the radius nor the radius itself contract; as a consequence the surface of the disc bends, because of rotation (see [13, 14]);
- (3) rods along the rim contract, while the circumference does not; neither the rods along the radius nor the radius itself contract. As a consequence the space of the disc is not Euclidean: $\implies 2\pi R = L < L' = \gamma 2\pi R'$, where γ is the Lorentz factor (see [15–21]).

Let us say a few words about why these solutions are not completely satisfactory².

Approach (1) is the one which is in agreement with common sense, since people can find it difficult to grasp how non-Euclidean features may appear on a disc because of its rotation.

¹ The non-null curvature of the space of the disc has nothing to do with the space–time curvature, which is always null if gravitation is not taken into account.

² See [6] for a historical and detailed analysis of attempts at solution of the paradox.

However, to maintain the statement ‘nothing happens on a rotating disc’ a remarkable price must be paid. For instance, Klauber’s analysis is based on a deep criticism of the foundations of the SRT: he claims that the SRT cannot be applied to rotating frames, but must be significantly amended to take into account the non-inertial motion of rotating observers. In particular, he maintains that the ‘hypothesis of locality’ (see [22]), which states the local equivalence of an accelerated observer with a momentarily co-moving inertial observer³, is not valid in a rotating frame. Accordingly, his approach appears to challenge the very foundations of relativity, because the hypothesis of locality is one of the most important axioms of the theory. Dropping this axiom (which ultimately is justified by experimental observation) forces us to abandon the idea that the theory of relativity can describe the physical world, since, actually, there are no perfectly inertial frames in the real world⁴.

On the other hand, Tartaglia’s result can be justified and accepted if and only if a specific choice of the measuring rods is made. However, this choice is questionable when applied to the case of a rotating disc: Tartaglia’s measuring rods are not the standard rods of the SRT. As we show in section 3, in the SRT we can locally substitute the light rays for the standard rigid rods; furthermore, we shall show in section 6 that a careful study of the effects of an acceleration process explains the kinematical origin of the Ehrenfest paradox.

Approach (2) introduces difficulties in the explanation of the paradox (for instance the dynamical properties of the rotating disc must be taken into account), and inconsistencies: a non-symmetric deformation with respect to the plane of the non-rotating disc should determine a screw sense in space, thus violating spatial parity *in a purely kinematical context*.

Finally, approach (3) formally agrees with ours: however, those authors do not define explicitly the geometric context in which measurements are made. Moreover, some of them refer to the ‘space of the disc’ as if it were a submanifold or a subspace embedded in space–time: this is not the case, since the lack of time-orthogonality (see below) does not allow a global $1 + 3$ splitting. Here, we shall show that the relative space is the only extended space, having a clear operational meaning, that can be formally defined as the physical space of the disc.

3. The space–time geometry of a rotating platform

3.1. Parametrizing the rotating frame

The physical space–time is a (pseudo)Riemannian manifold \mathcal{M}^4 , that is a pair (\mathcal{M}, g) , where \mathcal{M} is a connected four-dimensional Hausdorff manifold and g is the metric tensor⁵. Let the signature of the manifold be $(1, -1, -1, -1)$. Suitably differentiability condition, on both \mathcal{M} and g , are assumed.

Let K be the inertial laboratory frame, in which the platform (with its measuring apparatus) rotates with a constant angular velocity ω . Let K be parametrized by a set of coordinates $\{x^\mu\} = (ct, r, \vartheta, z)$, where t is the inertial time of K and (r, ϑ, z) are the cylindrical spatial coordinates. In this frame, let us consider the equations

$$\begin{aligned} r &= r_0 \\ \vartheta &= \vartheta_0 + \omega t \\ z &= z_0. \end{aligned} \tag{1}$$

If $r_0 \in [0, R]$, these equations describe the points of a cylinder with radius R , rotating with constant angular velocity ω . When z_0 is the same for each point of the system, we deal with a rotating disc, whose points have cylindrical coordinates (r_0, ϑ_0, z_0) , at the initial time $t = 0$.

³ Provided that standard rods and clocks are used.

⁴ As pointed out by Selleri [23] ‘because of the terrestrial rotation, the orbital motion of the Earth around the Sun, the Galactic rotation... all of our knowledge about inertial systems has been obtained in frames having a small but non-zero (centripetal) acceleration’.

⁵ The Riemannian structure implies that \mathcal{M} is endowed with an affine connection compatible with the metric, i.e. the standard Levi-Civita connection.

The world-lines of each point of the disc are time-like helices (whose pitch, depending on ω , is constant), wrapping around the cylindrical surface $r = r_0 = \text{constant}$, with $r \in [0, R]$. These helices fill, without intersecting, the whole space-time region defined by $r \leq R$;⁶ they constitute a time-like ‘congruence’ Γ which defines the rotating frame K_{rot} , at rest with respect to the disc⁷.

Let us introduce the coordinate transformation

$$\begin{aligned} x'^0 &= ct' = ct \\ x'^1 &= r' = r \\ x'^2 &= \vartheta' = \vartheta - \omega t \\ x'^3 &= z' = z. \end{aligned} \quad (2)$$

This coordinate transformation defines the passage from a chart $\{x^\mu\}$ adapted to the inertial frame K to a chart $\{x'^\mu\}$ adapted to the rotating frame K_{rot} .⁸

In the chart $\{x'^\mu\}$ the metric tensor is written in the form [6]

$$g'_{\mu\nu} = \begin{pmatrix} 1 - \frac{\omega^2 r'^2}{c^2} & 0 & -\frac{\omega r'^2}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega r'^2}{c} & 0 & -r'^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (3)$$

This is the so-called ‘Born metric’.

In the chart $\{x'^\mu\}$ the time t' is equal to the coordinate time t of the inertial frame K . In this way, we label each event P in K_{rot} using the time of a clock at rest in K , whose world-line (a straight line parallel to the time axis) intersects P , and not by means of a clock at rest on the disc. The transformation (2) has a Galilean character, and this is due to the peculiarity of angular velocity which, contrary to translational velocity, has an absolute value that can be locally measured.

Remark. The parametrization of the rotating frame K_{rot} by the coordinate t of the inertial frame K is the only way to synchronize the clocks (globally) on the platform: in fact their proper times cannot be synchronized by Einstein’s convention, because of rotation. We recall here that a physical frame is said to be *time-orthogonal* when there exists at least one adapted chart in which $g_{0i} = 0$: this means that the lines $x^0 = \text{variable}$ are orthogonal to the 3-manifold $x^0 = \text{constant}$. This is an intrinsic property of the physical frame, i.e. it does not depend on the coordinates used to parametrize the frame. A way to characterize this feature is the introduction of the ‘spatial vortex tensor’, which is a tensor for coordinate transformation ‘internal’ to the physical frame. The spatial vortex tensor of the rotating frame K_{rot} is not null: hence this is not a time-orthogonal frame⁹. In particular, a global clock synchronization is impossible in these non-time-orthogonal frames [24].

⁶ The condition $R < c/\omega$ is usually imposed: this simply means that the velocity of the points of the disc cannot reach the velocity of light.

⁷ The concept of ‘congruence’ refers to a set of world-lines filling the manifold, or some part of it, smoothly, continuously and without intersecting.

⁸ If $\{x^\mu\} = (x^0, x^1, x^2, x^3)$ is a system of coordinates in a subset $U \subset \mathcal{M}$, these coordinates are said to be ‘adapted’ to the physical frame if

$$g_{00} > 0 \quad g_{ij} dx^i dx^j < 0.$$

(Greek indices run from 0 to 3, Latin indices run from 1 to 3.) Accordingly, the couple $(\{x^\mu\}, U)$ is said to be a ‘chart’ adapted to the physical frame. The subset U is the coordinate domain of this chart; in our case, U is defined by the condition $r \in]0, R]$. In what follows, we shall always refer to this domain, even if it will not be explicitly declared.

⁹ For the definitions of the ‘spatial vortex tensor’ and of the ‘internal’ coordinate transformations, together with their applications to the rotating disc, see [6].

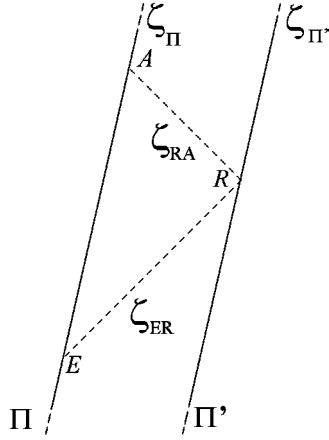


Figure 1. ζ_Π and $\zeta_{\Pi'}$ are the world-lines of the two neighbouring points Π and Π' in the rotating frame, between which a light signal propagates. The event E corresponds to the emission of the light signal (at time $t'_R + dt'_E$) which propagates along the null world-line ζ_{ER} . The events R corresponds to the reflection of the signal (at time t'_R) which, after propagating along the null world-line ζ_{RA} , is absorbed at the space-time event A (at time $t'_R + dt'_A$).

3.2. The local spatial geometry of the rotating frame

Using the metric (3) the line element, in the chart $\{x'^\mu\}$, is written in the general axis-symmetric form:

$$ds^2 = g'_{00}c^2 dt'^2 + g'_{rr} dr'^2 + g'_{\vartheta\vartheta} d\vartheta'^2 + g'_{zz} dz'^2 + 2g'_{t\vartheta}c dt' d\vartheta'. \quad (4)$$

We can introduce the local spatial geometry of the disc, which defines the proper spatial line element, on the basis of the local optical geometry. To this end we can use the radar method [18, 24].

Let Π be a point in the rotating frame, where a light source, a light absorber and a clock are lodged; let Π' be a near point where a reflector is lodged. The world-lines of these points are the time-like helices ζ_Π and $\zeta_{\Pi'}$ (see figure 1). A light signal is emitted by the source in Π and propagates along the null world-line ζ_{ER} toward Π' ; here it is reflected back to Π (along the null world-line ζ_{RA}), where it is finally absorbed. Let $d\tau$ be the proper time, read by a clock in Π , between the emission and absorption events: then, according to the radar method, the proper distance between Π and Π' is defined by

$$d\sigma = \frac{1}{2}c d\tau. \quad (5)$$

Now, we are going to parametrize these events, using the coordinates adapted to the rotating frame, in order to obtain the explicit expression of the proper spatial line element. Let x'^i and $x'^i + dx'^i$ be, respectively, the spatial coordinates of Π and Π' . The space-time intervals between the events of emission E and reflection R , and between the events of reflection R and absorption A , are null. Hence, by setting $ds^2 = 0$ in (4), we can solve for dt' , and obtain the two solutions

$$dt'_E = \frac{1}{cg'_{00}} \left(-g'_{t\vartheta} - \sqrt{(g'_{t\vartheta} d\vartheta')^2 - g'_{00}(g'_{rr} dr'^2 + g'_{\vartheta\vartheta} d\vartheta'^2 + g'_{zz} dz'^2)} \right) \quad (6)$$

$$dt'_A = \frac{1}{cg'_{00}} \left(-g'_{t\vartheta} + \sqrt{(g'_{t\vartheta} d\vartheta')^2 - g'_{00}(g'_{rr} dr'^2 + g'_{\vartheta\vartheta} d\vartheta'^2 + g'_{zz} dz'^2)} \right) \quad (7)$$

which correspond to the propagation along the two directions between Π and Π' .¹⁰ So, if t'_R is the coordinate time of the reflection event, the coordinate times of the emission and absorption

¹⁰ That is along the null world-lines ζ_{ER} and ζ_{RA} .

events are, respectively, $t'_R + dt'_E$ and $t'_R + dt'_A$. Consequently, the coordinate time elapsed between these two events turns out to be

$$\begin{aligned}\delta t' &\doteq (t'_R + dt'_A) - (t'_R + dt'_E) \\ &= dt'_A - dt'_E \\ &= \frac{2}{cg'_{00}} \sqrt{(g'_{t\vartheta} d\vartheta')^2 - g'_{00}(g'_{rr} dr'^2 + g'_{\vartheta\vartheta} d\vartheta'^2 + g'_{zz} dz'^2)}.\end{aligned}\quad (8)$$

The corresponding proper time difference is

$$d\tau = \sqrt{g'_{00}} \delta t'. \quad (9)$$

Hence, using (9), (8) and the definition (5) of the radar spatial proper distance, we obtain

$$d\sigma = \frac{1}{\sqrt{g'_{00}}} \sqrt{(g'_{t\vartheta} d\vartheta')^2 - g'_{00}(g'_{rr} dr'^2 + g'_{\vartheta\vartheta} d\vartheta'^2 + g'_{zz} dz'^2)}. \quad (10)$$

Explicitly, by inserting the elements of the metric tensor (3), the proper spatial line element is written as

$$d\sigma^2 = dr'^2 + \gamma^2 r'^2 d\vartheta'^2 + dz'^2 \quad (11)$$

where $\gamma = \gamma(\omega, r) \doteq 1 / \sqrt{1 - \frac{\omega^2 r^2}{c^2}}$.

In general this method leads to defining the proper spatial line element in the form

$$d\sigma^2 \doteq \gamma'_{ij} dx'^i dx'^j \quad (12)$$

where γ'_{ij} can be thought of as a ‘spatial metric tensor’. According to equation (11) the components of this tensor are

$$\gamma'_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 r'^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (13)$$

If we are concerned with a disc, where the z coordinate is constant, we get

$$d\sigma^2 = dr'^2 + \gamma^2 r'^2 d\vartheta'^2. \quad (14)$$

To complete the description of the rotating frame, it is easy to check that this frame is rigid according to Born’s definition of rigidity: a body moves rigidly if the spatial distance $d\sigma = \sqrt{\gamma'_{ij} dx'^i dx'^j}$ between neighbouring points of the body, as measured in their successive (locally inertial) rest frames, is constant in time. This is the case of the rotating frame K_{rot} , since the spatial metric tensor γ'_{ij} does not depend on time.

4. The ‘relative space’ of a rotating disc

In the previous section we have outlined the local spatial geometry of a rotating disc on the bases of the local optical geometry¹¹. Namely, when light rays are used, locally, as standard rods, the line element which allows measurements of lengths is given by (11). This local spatial geometry is defined at each point of the rotating frame. However, in order to have the possibility of confronting measurements performed at different points in the frame, a procedure to extend all over the disc the local spatial geometry is required. But this cannot be done in a straightforward way, because a rotating frame is not time-orthogonal and hence it is not possible to choose an adapted chart in which, globally, the lines $x^0 = \text{variable}$ are orthogonal to the three-manifold $x^0 = \text{constant}$ (each of which is described by the metric (13)). In other

¹¹ It is worthwhile to stress that the use of the *optical congruence* is meaningful only in any *local* Minkowskian (tangent) frame, whose space geometry is actually the geometry of an optical space.

words, a global foliation, which would lead ‘naturally’ to the definition of the space of the disc, is not allowed.

Nevertheless, if we shift the context, from the ill defined notion of space of the disc thought of as a subspace or a submanifold embedded in space–time, to a definition which has a well defined and operational meaning, we are led to the concept of the ‘relative space’. To this end, first of all let us start from a more formal description of the local splitting of space–time that allows us to write the local spatial metric (13). Let ζ be a time-like helix, describing the evolution of a point of the disc, and let P be an event which belongs to ζ . We can identify a one-dimensional vector space Θ_P (‘time direction’), which is spanned by the vector tangent to ζ in P , and a three-dimensional vector space Σ_P (‘space platform’), normal to Θ_P ; Σ_P is endowed with the metric (13). Hence the space tangent to P has a splitting in the form $T_P = \Theta_P \oplus \Sigma_P$.

This local splitting has an important property: *the splitting $T_P = \Theta_P \oplus \Sigma_P$ and the spatial metric tensor $\gamma'_{ij}(P)$ are invariant along the lines of congruence Γ , i.e. they are dragged along the world evolution.* This property is strictly related to the rigidity of the rotating frame, namely it depends on the fact that the metric (3) is globally stationary and the metric (13) is locally static. In particular, it can be explained in terms of isometries, i.e. Killing fields in the submanifold $r = \text{constant}$ of the space–time (see [6]). As a consequence it is possible to define a one-parameter group of diffeomorphisms with respect to which both the splitting $T_P = \Theta_P \oplus \Sigma_P$ and the space metric tensor $\gamma'_{ij}(P)$ are invariant. The lines of Γ constitute the trajectories of this ‘space \oplus time isometry’.

This fact suggests a procedure to define an *extended 3-space*, which we shall call *relative space*. Naively, it can be thought of as the union of the infinitesimal (local) space platforms, but a more rigorous definition can be given.

Definition. *Each element of the relative space is an equivalence class of points and of space platforms, which verify this equivalence relation:*

RE: ‘Two points (two space platforms) are equivalent if they belong to the same line of the congruence’.

That is, the *relative space* is the ‘quotient space’ of the world-tube of the disc, with respect to the equivalence relation RE, among points and space platforms belonging to the lines of the congruence Γ .

This definition simply means that the relative space is the manifold whose ‘points’ are the lines of the congruence. Our definition emphasizes the role of the space platforms: the reference frame defined (as above) by the relative space coincides *everywhere* with the local rest frame of the rotating disc.

In other words, the relative space is a formal tool which allows a connection among all the local optical geometries that are defined in the neighbourhood of each point of the space. As a consequence, space measurements globally defined in the relative space reduce, immediately, to standard measurements in any local frame, co-moving with the disc. In this way, a natural procedure to make a comparison between observations performed by observers at different points is available. The physical context in which these distant observations are made is defined, both from a mathematical and operational point of view, by the relative space.

We want to stress, again, that it is not possible to describe the relative space in terms of space–time foliation, i.e. in the form $x^0 = \text{constant}$, where x^0 is an appropriate coordinate time, because the space of the disc, as we saw before, is not time-orthogonal. Hence, thinking of the space of the disc as a sub-manifold or a subspace embedded in the space–time is misleading: instead, the space of the disc, defined by the relative space, must be thought of as a quotient space. If we long for some kind of visualization, we can think of the relative space as the union of the infinitesimal space platforms, each of which is associated, by means of the requirement of M -orthogonality, to one and only one of lines of the congruence.

5. Measurements in the relative space

After having introduced the relative space and described its geometry, we come back to our original purposes, i.e. the solution of Ehrenfest's paradox.

Space measurements are locally performed by a rotating observer by means of the spatial metric tensor γ'_{ij} :

$$\gamma'_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 r'^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

As a consequence, the length of an infinitesimal segment on the rim of the circumference is

$$dl'_\Sigma = \left(\sqrt{\gamma'_{ij}(\omega, r') dx'^i dx'^j} \right)_{r'=R, z'=\text{const}} = \gamma(\omega, R) R d\vartheta'. \quad (16)$$

From the third equation of (2) it follows that at fixed coordinate time of the inertial frame K , $d\vartheta' = d\vartheta$. Consequently, the angle all around the periphery of the disc, measured on it, is equal to 2π : $\vartheta' \in [0, 2\pi]$.

Hence, the measurement of the circumference on the rim of the disc, performed by the rotating observer, turns out to be

$$l'_\Sigma = 2\pi R\gamma. \quad (17)$$

We want to stress that the dilation appearing in (17) has a pure kinematical origin and by no means ought it to be interpreted as a dynamical process, involving the structure of the disc, as some authors have claimed [25]. The meaning of this dilation will be made clear in the next section.

For the observer in the inertial frame the length given in equation (16) appears to be contracted by the standard factor γ^{-1} :

$$dl = \gamma^{-1} dl'_\Sigma. \quad (18)$$

Since

$$dl = \gamma^{-1} \gamma R d\vartheta' = R d\vartheta' \quad (19)$$

we obtain that, in correspondence to the measure of the circumference given in equation (17), performed by the rotating observer, the inertial observer measures a length $2\pi R$, as expected, since the space of the inertial frame K is Euclidean.

Expression (17) is in agreement with the fact that the space geometry of the disc is not Euclidean: a curvature tensor can be computed and it is not null¹².

Remark. In the past, different authors, such as Berenda [16], Arzeliès [17] and Grøn [19], have calculated the curvature of the space of the disc. It is worthwhile to note that they did not define the proper geometrical context in which their calculations were performed; moreover, their calculations do not rely upon the use of a splitting technique: they just computed the components of the curvature tensor of the *space-time* which have all spatial indices (R_{ijkl}), and they referred to it as the *space* curvature tensor. On the other hand, we have shown that the relative space is the natural context in which space measurements are performed and we computed its curvature (see [6]) using Cattaneo's splitting technique. Nevertheless, their results are equal to ours, and this is due to the fact that, for the rotating disc in uniform motion, the physical frame is stationary; however, things are different for those physical reference frames which lack symmetries, such as the axis-symmetry and the stationarity of the rotating disc.

¹² It is interesting to note that this confirms Einstein's early intuition [15]: he suggested that rotation must distort the Euclidean geometry of the platform, so that the geometry of the inertial frame remains Euclidean. Indeed the rotating disc was just a heuristic tool in order to investigate the possibility that the geometry of the Minkowskian space-time could be distorted by a gravitational field [26].

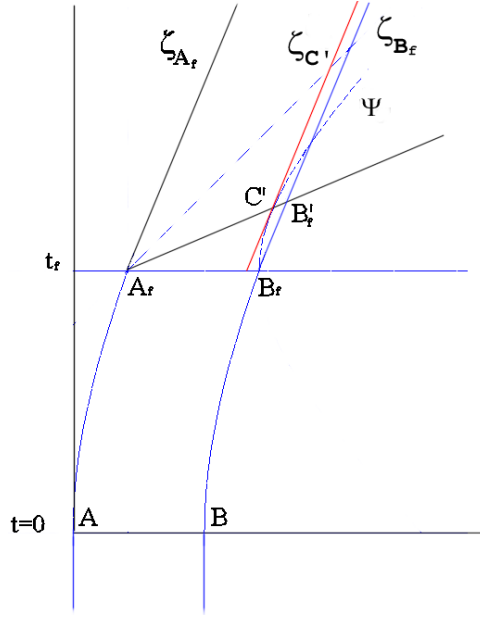


Figure 2. The world-strip $(\zeta_{A_f}, \zeta_{C'})$ of a length $\lambda_0 = AB = A_f B_f$, at rest on the rim, does not entirely cover the world-strip $(\zeta_{A_f}, \zeta_{B_f})$ of this length, as measured in the inertial frame.

(This figure is in colour only in the electronic version)

6. Lengths in the relative space

We want to clarify, here, the origin of the dilation appearing in (17), which, as we showed, is ultimately responsible for the solution of Ehrenfest's paradox. In order to do that, it is useful to analyse what happens to 'standard lengths' when they undergo an acceleration process.

Let us consider the world-strip of an infinitesimal piece of the rim of the disc, which is at rest until $t = 0$ in the inertial frame K (see figure 2).

When $t = 0$, the disc starts being accelerated in such a way that all points of its rim have identical motion, as observed in K . If $I = [0, t_f]$ is the interval representing the period of time during which acceleration acts, $\forall t \in I$ the acceleration distribution of all points of the rim is the same, as observed in the inertial frame K . From a pictorial point of view, this means that the world-lines of all points of the rim are congruent (i.e. superimposable). During the acceleration period, the disc is not Born-rigid although it appears rigid in K . This means that, depending on the simultaneity criterion in the inertial frame, the length of the infinitesimal piece of the rim is always congruent with the starting segment AB ; in particular, when $t = t_f$, it is represented by the segment $A_f B_f$. On the other hand, from the point of view of the local observer at rest on the rim, whose world-line passes through A when $t = 0$, the simultaneity criterion is not defined by the family of straight lines parallel to AB , but it varies at each instant, depending on the velocity (in K) of the rim itself. That is, when the acceleration period finishes, the piece of the rim is represented by the segment $A_f B'_f$, in the local co-moving frame associated with A_f . Let us put $A_f B_f = AB = \lambda_0$, where λ_0 is the wavelength of a monochromatic radiation emitted by a source at rest in K .¹³ The M -circumference of radius λ_0 , with centre in A_f , whose equation is

$$\Psi \equiv \{P \in M^2 : A_f P = \lambda_0\} \quad (20)$$

¹³ Since AB is infinitesimal, the monochromatic radiation must be chosen in such a way that λ_0 is very small when compared with the length of the circumference.

can be built by considering, in each reference frame, the wavelength of the given radiation, emitted by a source at rest in that frame. This M -circumference, which is a hyperbola in the Minkowskian plane, intersects the segment $A_f B'_f$ in C' , and we obtain

$$A_f B'_f = A_f C' \gamma = \lambda_0 \gamma > \lambda_0. \quad (21)$$

This relation means that the world-strip $(\zeta_{A_f}, \zeta_{C'})$ of a length λ_0 , at rest on the rim, does not entirely cover the world-strip $(\zeta_{A_f}, \zeta_{B'_f})$ of this length, as measured in K (see figure 2). From a physical point of view, equation (21) shows that each element of the periphery of the disc, of proper length λ_0 , is stretched during the acceleration period. This is a purely kinematical result of our acceleration program.

However, this result remains valid if one takes into account the interactions among the physical points of the rim (for instance, those interactions which ensure rigidity in the phase of stationary motion). In particular, during and after the acceleration period, each point of the disc is subject to both radial and tangential stresses; the former maintain each point on the circumference $r = r_0$, while the latter ought to give zero resultant, because of the axial symmetry: *each point is pulled in the same way by its near points, in both directions*. As a consequence, the elongation of every element of the rim, due to tidal forces experienced during the acceleration period, remains even when acceleration finishes¹⁴.

Remark. The arguments given treat the disc as a set of non-interacting particles. The only constraint is that every particle must move along a circular trajectory, with a given law of motion, according to a kinematical definition of rigidity in K .

From the considerations above, it follows that the dilation which is responsible for the solution of Ehrenfest's paradox has a pure kinematical origin. The enlargement of the rod (assumed to be a standard rod), in the rest frame at the end of acceleration phase, is due to the change of the simultaneity criterion. In figure 2 this is represented by the change in the slope of the infinitesimal space platforms which are associated, by means of the requirement of M -orthogonality, with the lines of the congruence Γ .

7. Conclusions

We introduced the concept of relative space in order to clarify the fundamental processes of measurements that take place on a rotating platform, or more generally, in a rotating frame. In particular, space measurements are involved in the solution of Ehrenfest's paradox. Often, in the literature on this subject, misunderstandings arise, and we believe that these are based on a lack of clear and self-consistent definition of the fundamental concepts used.

The concept of relative space clarifies, in a mathematical way, the physical context in which measurements are performed by rotating observers. Even though a global *isotropic* 1 + 3 splitting of the space-time is not allowed when dealing with rotating observers, space measurements are defined globally in the relative space, and reduce in a straightforward way to the standard measurements in any local frame, co-moving with the disc.

In the relative space we have been able to outline the solution of Ehrenfest's paradox, demonstrating its kinematical origin, which is related to the way of measuring lengths in a rotating platform. Furthermore, the geometry of the space of the disc, which follows from our assumptions, turns out to be non-Euclidean, in line with Einstein's early intuition. The solution of the Ehrenfest's paradox outlined in this paper is strongly dependent both on a proper definition of the physical space of the disc, i.e. the relative space, and on a proper choice of the congruence adopted to perform the measurements in such a space, i.e. the (local) optical congruence.

¹⁴ A shortening of the elements of the rim, due to Hooke's law, cannot be invoked, since these elements do not have free endpoints.

In conclusion, it appears that the SRT, even when applied to rotating platforms, is self-consistent and does not raise paradoxes, provided that proper definitions of geometrical and kinematical entities are adopted.

We believe that our approach to the study of these apparently paradoxical problems leads to a deeper understanding of the foundations of the theory and demonstrates, in a clear way, some operational aspects of the measurement processes involved.

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