A simple relativistic paradox about electrostatic energy

Wolfgang Rindler and Jack Denur
Department of Physics, The University of Texas at Dallas, Richardson, Texas 75083-0688

(Received 15 September 1987; accepted for publication 13 December 1987)

A charged parallel-plate vacuum capacitor moves uniformly through an inertial frame. Its field energy alone does not transform according to the familiar law "energy $= \gamma \times$ rest energy." However, when the stresses in the supports are taken into account, the entire system *does* satisfy this relation.

A charged parallel-plate vacuum capacitor forms the two opposite ends, of area A, of a rectangular box of length l and negligible mass. This box is at rest in an inertial frame S_0 whose x axis may be chosen parallel to the electric field lines, i.e., normal to the capacitor plates (see Fig. 1). We shall also look at this box from a "lab frame" S, through which it moves uniformly at speed v in the direction of the field lines.

Now it is one of the well-known properties of the Lorentz transformation of electromagnetic fields[1] that a parallel electric field E in one inertial frame S_0 transforms into an identical parallel electric field E in any other inertial frame S moving relative to S_0 in a direction parallel to the field. Since the energy density of an electric field is given by $\rho = E^2/8\pi$ (in Gaussian units), ρ is an invariant among these frames. Thus the energy density of the electric field of our capacitor is the same in S and in S_0. In S, however, the gap between the plates is shortened by length contraction to $l/\gamma, \gamma = (1-v^2)^{-1/2}$ (we work in units such that $c=1$), whereas, of course, the area of the plates is still A. Consequently, the energy (mass!) of the box appears to be *decreased* by a factor γ. Yet we know from general theory[2] that the mass of any isolated system transforms like that of a particle, i.e., it *increases* by a γ factor when moving through the lab.

That this should be so in our particular case can also be seen directly by discharging the capacitor through a "point resistance" (of sufficient ohmage to avoid radiation losses), which will thereby gain extra rest mass, say Δm_0. The total energy of the system cannot change; since Δm_0 and its measure Δm in S are clearly related by $\Delta m = \gamma \Delta m_0$, the original energies of the system must be so related too.

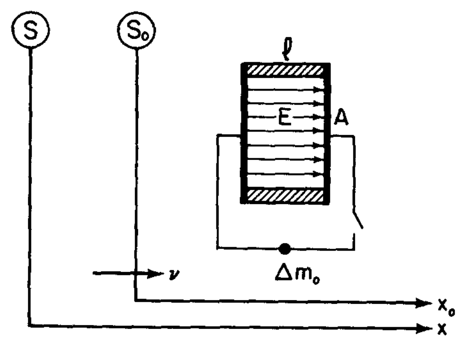

Fig. 1. A parallel-plate vacuum capacitor, which can be discharged through a point resistance, is at rest in an inertial frame S_0. It moves through the lab frame S with velocity v.

What is wrong, then, with our original argument? What is wrong is that at least the *sides* of the box (shaded in Fig. 1) cannot be infinitely light, because they must provide the pressure to counterbalance the tension of the field to hold the plates apart. This pressure in the sides of the box entails that the mass of these sides transforms differently from the mass of a single particle or of an unstressed body[3]: The density ρ_0 and the pressure in the x direction, t_0^{11}, both measured in the rest frame, contribute as follows to the density ρ in the lab frame (recall that $c=1$):

$$\rho = \gamma^2(\rho_0 + v^2 t_0^{11}) . \tag{1}$$

If the total cross-sectional area of the sides of the box is ΔA, then the pressure force exerted by the sides on the plates in the rest frame S_0 is $t_0^{11}\Delta A$. But this must balance the electric force $E^2A/8\pi$ between the plates, whence

$$t_0^{11} = E^2A/8\pi\Delta A . \tag{2}$$

Thus, by (1) and (2), the mass M of the sides of the box in the lab frame S is given by

$$\begin{aligned} M &= (l/\gamma)\Delta A\rho = l\Delta A\gamma[\rho_0 + v^2(E^2A/8\pi\Delta A)] \\ &= \gamma[M_0 + v^2(E^2lA/8\pi)] , \end{aligned}$$

where M_0 is the mass of the sides in S_0. Consequently, the mass of the entire system in the lab frame is given by

$$\begin{aligned} M + U &= \gamma[M_0 + v^2(E^2lA/8\pi)] + E^2lA/\gamma 8\pi \\ &= \gamma M_0 + (v^2\gamma + 1/\gamma)U_0 \\ &= \gamma(M_0 + U_0) , \end{aligned} \tag{3}$$

where $U = (E^2/8\pi)(l/\gamma)A$ and $U_0 = (E^2/8\pi)lA$ are the energies in S and S_0, respectively, of the electric field. Equation (3) shows that all is well: The mass (energy) of the *entire* system transforms like that of a particle.

If one wants, one can take into account also the finite masses of the capacitor plates; since these transform "normally," the satisfactory nature of the result (3) is not altered thereby.

[1] See, for example, W. Rindler, *Introduction to Special Relativity* (Clarendon, Oxford, 1982), p. 120.
[2] Reference 1, Sec. 50.
[3] Reference 1, pp. 150, 151.

