# Uniformly accelerated reference frame and twin paradox

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Some simple concepts of general relativity are presented via two animated films and an elementary treatment that emphasizes the event horizon and the gravitational frequency shift. The twin paradox is illustrated from the standpoint of each twin.

## I. INTRODUCTION

Historically, advanced topics such as special relativity and quantum mechanics have gradually moved into earlier and earlier phases of physics instruction, until they are now introduced in first college courses. General relativity has largely resisted this trend because of the mathematical maturity required in dealing with Riemannian geometry. An important purpose of this paper and the two films described here is to place the concepts and formulas pertaining to one aspect of general relativity, the uniformly accelerated reference frame, within reach of anyone who has mastered the Lorentz transformation and elementary calculus.

## II. FILMS

The pair of animated Super-8 cartridges deals with simple general-relativistic concepts.[1] They are suitable for physics majors and for students who have achieved some understanding of special relativity. They should be preceded by a discussion of the frequency shift[2–4] in a uniform gravitational field $g$; higher clocks run faster by the factor $f/f_0$, with

$$f/f_0 = 1 + gx/c^2, \tag{1}$$

where $x$ is height, positive upward; $g$ is positive; $f_0$ is frequency of a clock at zero height, $f$ is frequency of an identical clock at height $x$, and $c$ is the speed of light.

The first film, *Uniformly Accelerated Reference Frame*, begins with the effect of gravitational potential difference on clock rates, illustrated schematically first near the Earth's surface (Fig. 1) and then, using the principle of equivalence, in the pseudogravitational field of a uniformly accelerated space ship (Fig. 2). The event horizon appears and its inaccessibility is illustrated (Figs. 3–5).

For pictures in which the observer is in an inertial frame of reference, objects (stars, light pulses, and so forth) are all shown at the same instant of time in that frame. For an inertial frame this presents no conceptual difficulty; but it should be noted that this is not necessarily the same as the visual appearance of these objects as literally "seen" by the observer. For pictures in which the observer is in an accelerated frame, objects are shown at the location they would have, again at a given instant, if the observer were to turn off his acceleration momentarily, thus entering an inertial frame. (The event horizon, described in more detail below, is represented by a dashed line drawn at its location in the uniformly accelerated frame.)

The second film, *Twin Paradox*, shows a round trip to a star 4.5 light-years away, on the following schedule:

(i) Starting from rest at Earth, the ship undergoes constant proper acceleration of about one-half the acceleration due to gravity at the surface of the Earth, for two years as measured on Earth.

(ii) The ship coasts at velocity $0.71c$ for four years.

(iii) The ship decelerates to rest as in (1) above; then, after a pause of two years at the star, the ship returns to Earth in the same way. Elapsed time is 18 years on Earth and 14.7 years on the ship.

The trip is shown first in the frame of reference of the Earth, then in that of the space ship (Fig. 6). Lorentz contraction and other phenomena are pointed out along the way. Yearly light pulses emitted by the Earth serve to mark the passage of time. An inset dial shows how fast time is passing in the "nonobserver's" frame: for example, in the frame of reference of the ship, Earth clocks may go fast or slow at various points in the journey, and the dial thus shows the rate of passage of time on Earth in the ship frame.

## III. DERIVATION OF EQUATIONS

"Accelerated observers can be analyzed using special relativity...general relativity is built on special relativity."[5] In

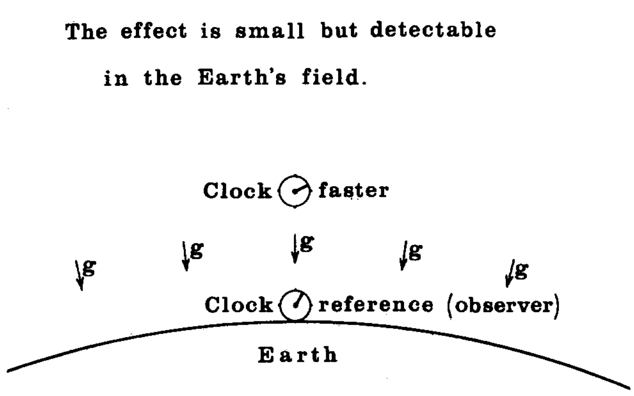


Fig. 1. Gravitational frequency shift in the Earth's field.

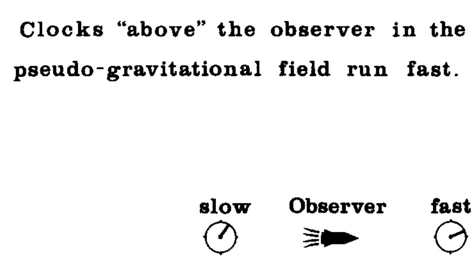


Fig. 2. Gravitational frequency shift in the noninertial reference frame of an accelerating spaceship.



 

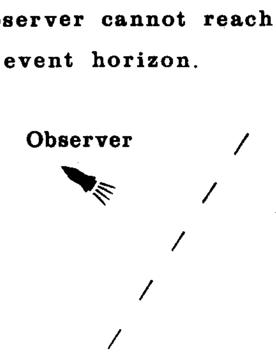


Fig. 3. Whatever gyrations he may cause the spaceship to perform, the observer can never reach the event horizon. If he stops accelerating, it vanishes; if he then starts toward its former position, it reappears behind him.

order to make our treatment accessible to those not familiar with the tensor formalism of general relativity, we will derive all relevant equations here starting from the Lorentz transformation and employing only elementary calculus. (We will nevertheless arrive at some results that properly belong to general relativity; students of relativity will find it instructive to compare this approach to that of Perrin,[6,7] and to consider the difficulties in extending it to fields of gravitating masses.)

From the standpoint of Earth, at rest in the "unprimed" inertial reference frame, the outward trip is shown schematically in Fig. 7. In Part 1 of the trip the ship accelerates at constant proper acceleration for distance $D$; in Part 2 it coasts at velocity $v(\max)$ for distance $L$; and in Part 3 it decelerates to rest after a further distance $D$, the endpoint in this case being marked by a star. The situation in the "primed" frame[8] is shown in Fig. 8. The primed frame is the *instantaneously co-moving inertial frame* whose position, velocity, and time differ little from those of the ship[5,9] (it is somewhat analogous to the tangent to a curve at a point). Earth positions and times in the co-moving frame are essentially the same as in the ship frame; but, as we shall find, their time derivatives are not.

Distances and times corresponding to the various labeled parts of the trip are listed in Table I, for the particular parameters used in the *Twin Paradox* film.

We now examine, in order, Parts 1, 2, and 3 of the trip in the unprimed frame, and then I, II, and III in the primed frame. Finally, equations establishing the positions of the yearly light pulses will be produced.

*Part 1.* The Lorentz transformation for the separation $\Delta x$, $\Delta t$ of two events in the unprimed frame is[10]

$$\Delta x' = \gamma \Delta x - \beta\gamma \Delta ct, \tag{2}$$

$$\Delta ct' = \gamma \Delta ct - \beta\gamma \Delta x, \tag{3}$$

for motion confined to the $x$ direction. The two events are separated by $\Delta x', \Delta t'$ in the primed frame. We use

$$\beta = v/c, \tag{4}$$

where $v$ is the velocity of the primed frame in the unprimed frame, and

$$\gamma = 1/(1-\beta^2)^{1/2}. \tag{5}$$

We rewrite the Lorentz transformation in the form

$$\Delta x = \gamma \Delta x' + \beta\gamma \Delta ct', \tag{6}$$

$$\Delta ct = \gamma \Delta ct' + \beta\gamma \Delta x'. \tag{7}$$

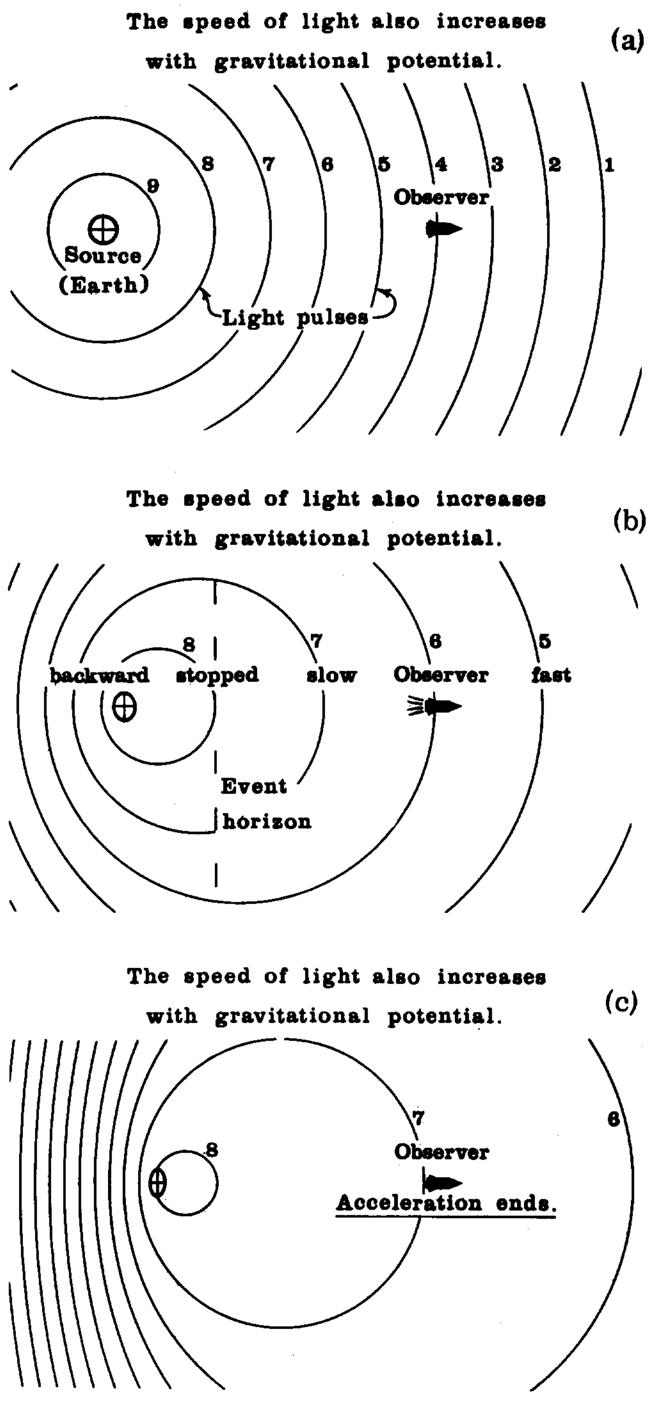


Fig. 4.(a) Earth is stationary relative to the observer and is producing numbered annual light pulses. The observer is about to begin uniform acceleration $g$ such that $c/g = 4$ years, directed away from the Earth. (b) After 2.7 ship years, the acceleration period is half over. Light speeds relative to the observer are indicated in four regions; clock rates behave in the same manner. The observer receives pulses in their order of production: 4,5,6; but he cannot receive pulse number 8 while acceleration continues (see discussion of event horizon). (c) The acceleration has just ended. The observer will receive pulse number 8 in 4 years. The pulses display a pronounced Doppler effect.

Dividing,

$$\frac{\Delta x}{\Delta ct} = \frac{\Delta x' + \beta \Delta ct'}{\Delta ct' + \beta \Delta x'}. \tag{8}$$

If the two events occur on the ship, then Eq. (8) becomes

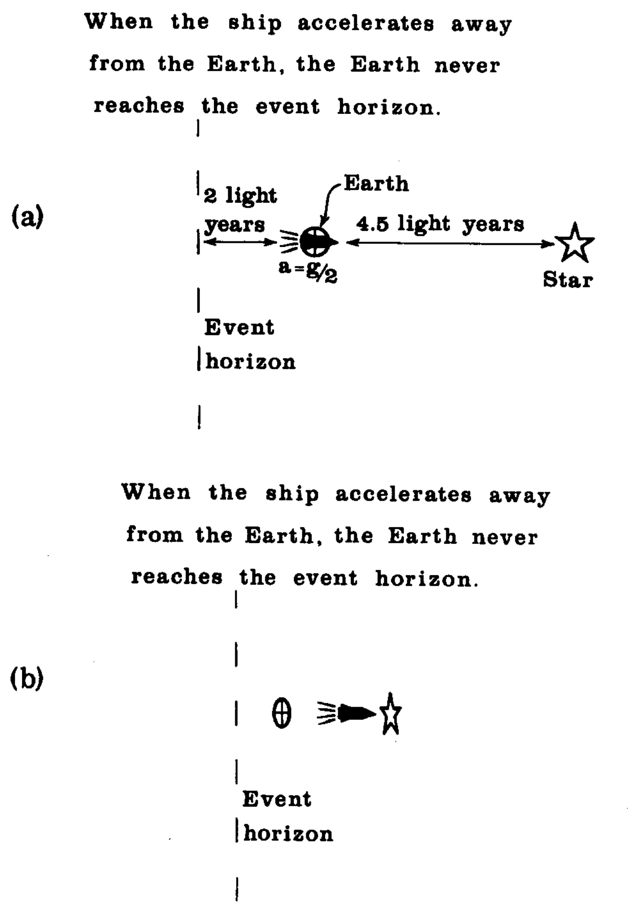


Fig. 5.(a) Earth and star are stationary; acceleration is just beginning. (b) Later. The Earth–star distance becomes increasingly Lorentz contracted.

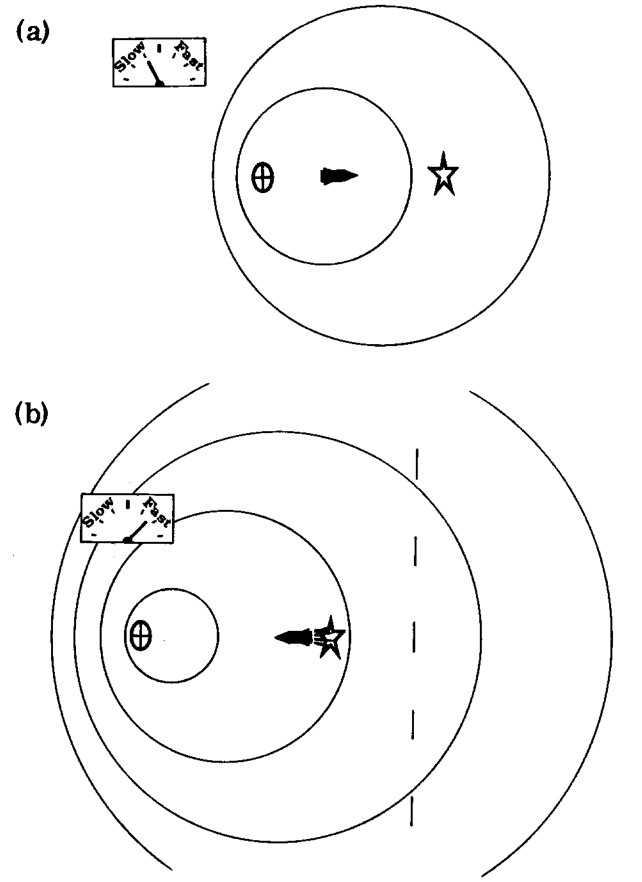


Fig. 6.(a) In the ship frame of reference, Earth and the star are shown moving to the left and Lorentz-contracted. Earth clocks are running slow because of time dilation. There is no acceleration. (b) Somewhat later, the ship is decelerating to rest at the star. Gravitational frequency shift now outweighs time dilation, and Earth clocks are running fast.

$$\beta_s = \frac{\beta'_s + \beta}{1 + \beta'_s \beta}, \tag{9}$$

the usual *velocity addition* formula, where $\beta_s$ and $\beta'_s$ refer to velocity of the ship in the unprimed and primed frames, respectively.

Consider an acceleration of the ship relative to the instantaneously co-moving inertial frame. Differentiation of Eq. (9), with $\beta$ constant, yields

$$d\beta_s = \frac{1 - \beta^2}{(1 + \beta'_s \beta)^2} d\beta'_s. \tag{10}$$

Since $\beta'_s$ is small, this may be written

$$d\beta_s = (1 - \beta^2) d\beta'_s, \tag{11}$$

$$d\beta_s = (1 - \beta^2)(g/c) dt', \tag{12}$$

where we have introduced the constant proper acceleration $g = c\, d\beta'_s/dt'$, the acceleration of the ship relative to the co-moving inertial frame.

Now, for two events at the co-moving frame origin, separated by $\Delta t'$, we have $\Delta x' = 0$; Eq. (7) then becomes

$$\Delta t = \gamma \Delta t' \tag{13}$$

the usual *time dilation*. Substituting in Eq. (12),

$$d\beta_s = (1 - \beta^2)(g/c) dt/\gamma, \tag{14}$$

$$d\beta_s = (1 - \beta^2)^{3/2} (g/c) dt. \tag{15}$$

This implies $a_s = g/\gamma^3$; if the ship's proper acceleration $g$ remains constant, then the ship's acceleration in the Earth's frame of reference, $a_s$, must decrease as $\gamma$ increases. (Otherwise the ship velocity in the Earth's frame would ultimately surpass the speed of light.)

Integration of Eq. (15), with $\beta = \beta_s$ ("different inertial frames at different instants," Footnote 5), yields

$$\boxed{\beta_s = 1/[1 + (c/gt)^2]^{1/2}.} \tag{16}$$

As $t$ increases, ship velocity approaches the speed of light. In view of Eq. (5),

$$\gamma_s = [1 + (gt/c)^2]^{1/2}. \tag{17}$$

Thus

$$\beta_s \gamma_s = gt/c \tag{18}$$

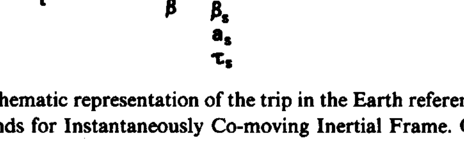


Fig. 7. Schematic representation of the trip in the Earth reference frame. ICIF stands for Instantaneously Co-moving Inertial Frame. Quantities pertaining to Earth, ICIF, and ship are shown beneath corresponding symbols. Quantities are positive to the right (except time). For purposes of illustration, the ship is shown for Part 2 of the trip; during Part 1 it is in the section to the left labeled 1, and so on.

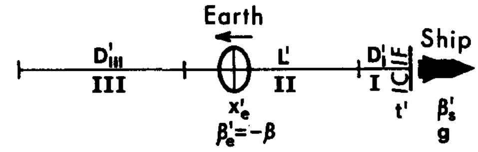


Fig. 8. As Fig. 7, in the ICIF. In the ship frame, time is $t_s$ and Earth quantities are $x_e$, $\beta_e$, and $\tau_e$.

which implies that, in the Earth frame, ship momentum[11] $p = m_0\beta_s\gamma_s c$ increases uniformly with time; here $m_0$ is ship rest mass. Since force $F = dp/dt$, constant $g$ implies constant force in the frame of reference of the Earth, just as in that of the ship.

We now set $\beta_s = dx_s/c\,dt$, where $x_s$ is ship position in the Earth frame, and integrate Eq. (16), obtaining

$$x_s = \{[1 + (gt/c)^2]^{1/2} - 1\}c^2/g \qquad (19)$$

so

$$\boxed{x_s = (\gamma_s - 1)c^2/g.} \qquad (20)$$

Equation (19) implies that, for all $t$, $x_s > ct - c^2/g$; so a light pulse starting at $t = 0$ at a distance $c^2/g$ or more behind the ship can never catch up with it.

Ship clock rate is given by the time dilation relationship [cf. Eq. (13)]:

$$\boxed{d\tau_s = dt/\gamma_s,} \qquad (21)$$

where $\tau_s$ is ship proper time. (*Proper time* of a particle is time measured by a standard clock attached to that particle.) While the ship moves, its clocks run slow in the Earth frame. Ship proper time is obtained by substituting Eq. (17) and integrating:

$$d\tau_s = dt/[1 + (gt/c)^2]^{1/2}, \qquad (22)$$

$$\boxed{\tau_s = (c/g)\sinh^{-1}(gt/c).} \qquad (23)$$

If we solve Eq. (23) for $gt/c$ and substitute in Eq. (19), we obtain

$$x_s = (c^2/g)[\cosh(g\tau_s/c) - 1]. \qquad (24)$$

This equation gives position of the ship in the Earth's frame, as a function of ship time. It is plotted in Fig. 9, with $c/g = 2$ years, so $g = 4.76\ \mathrm{m/sec}^2$, approximately one-half of the acceleration due to gravity at the surface of the Earth. For comparison, the nonrelativistic equation $x_s = g\tau_s^2/2$ is also shown. It is observed that if the acceleration were to continue for 46 years on board ship, a space traveler would reach a distance of $10^{10}$ light-years from Earth, beyond the farthest stars now known. Nonrelativistically, in 46 years the traveler would be only 529 light-years from Earth, even though in this case the velocity would greatly exceed that of light. The relativistic advantage may be ascribed to the time dilation, in the Earth frame of reference, or equivalently to the Lorentz contraction, in the ship frame. (As will be apparent from the discussion of the event horizon, below, in the ship frame of reference the Earth never gets more than two light-years from the ship during the acceleration; see Fig. 5.)

*Part 2.* The ship moves at constant speed: $\beta_s = \beta\,(\max)$. Its position is given by $x_s - D = c\beta\,(\max)t$, where $x_s = D$ and $t = 0$ at the start of Part 2. Ship time is given by the time dilation relation: $\tau_s = t/\gamma(\max)$.

*Part 3.* For convenience we set $t = 0$ at the end of Part 3, where $x_s = L + 2D$. Results are then similar to those of Part 1 and will merely be listed here. Note that $g$ is negative now, the acceleration has changed direction:

$$\beta_s = \pm 1/[1 + (c/gt)^2]^{1/2}, \qquad (25)$$

$$x_s = L + 2D + (\gamma_s - 1)c^2/g, \qquad (26)$$

$$d\tau_s = dt/\gamma_s, \qquad (27)$$

$$\tau_s = (c/g)\sinh^{-1}(gt/c). \qquad (28)$$

*Part I.* Equations (24) and (20) indicate that

$$\gamma_s = \cosh(g\tau_s/c) \qquad (29)$$

in the Earth frame of reference. In the primed co-moving frame, when ship clocks read $\tau_s = t' = t_s$, time in the ship frame (since $\beta'_s$ is small), then because of the reciprocal nature of the Lorentz transformation,

$$\beta'_e = -\beta = -\beta_s, \qquad (30)$$

where $c\beta'_e = dx'_e/dt'$, Earth velocity, and $x'_e$ is Earth position, in the primed frame. In this frame, in view of Eqs. (29), (30), and (5),

$$\gamma'_e = \cosh(gt_s/c), \qquad (31)$$

$$\boxed{\beta'_e = -\tanh(gt_s/c).} \qquad (32)$$

[$\beta'_e$ is negative; see Figs. 7 and 8. Equations (31) and (32) are more useful to us written in terms of $t_s$ rather than $t'$.]

We now derive Earth coordinate velocity and clock rate in the ship frame of reference, starting with the Lorentz transformation.

We suppose that the ship initially coincides with the primed instantaneously co-moving inertial frame in location and velocity. At time $t' = t_s = 0$, the ship undergoes an impulse acceleration bringing it to a small speed, with $\beta'_s \ll 1$ relative to the primed frame (see Fig. 8). Then the ship rests in its new inertial frame for time $\Delta t_s \approx \Delta t'$ (since $\beta'_s$ is small). The average acceleration is $g = \beta'_s c/\Delta t_s$.

At time $t'_0 = 0$ in the primed frame, Earth clocks are at position $x'_{e0}$ and read $\tau_{e0}$; they move at speed $\beta'_e c$ (not necessarily small). Then position $x_{e0}$ and time $t_{s0}$ of the event "Earth clocks read $\tau_{e0}$" are given, in the inertial frame which coincides with the ship frame after the acceleration, by the Lorentz transformation (subscript 0 refers to the

Table I. Lengths and times for the various parts of the trip as labeled in Figs. 7 and 8, based on the exact values: proper acceleration of space ship such that $c/g = \pm 2$ years; time of acceleration or deceleration 2 years in the Earth frame; and coasting time 4 years in the Earth frame.

| Part | Earth reference frame 1 | 2 | 3 | Ship reference frame I | II | III | Either total |
|---|---|---|---|---|---|---|---|
| Length (lt-yr) | 0.83 | 2.83 | 0.83 | 0.59 | 2.00 | 1.90 | 4.49 |
| Earth time (yr) | 2.00 | 4.00 | 2.00 | 1.41 | 2.00 | 4.59 | 8.00 |
| Ship time (yr) | 1.76 | 2.83 | 1.76 | 1.76 | 2.83 | 1.76 | 6.35 |

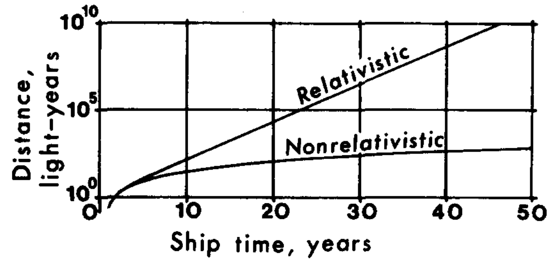


Fig. 9. Distance a space ship goes, in the reference frame of the Earth, plotted versus time on board ship, for constant $g = 4.76$ m/sec$^2$.

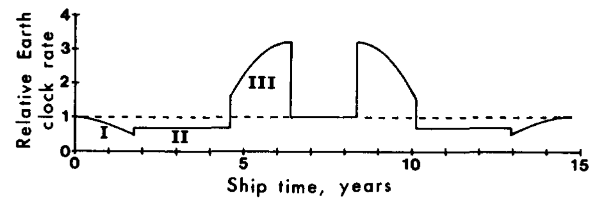


Fig. 10. Earth clock rate relative to that of the ship, in the ship reference frame, plotted versus ship time. Roman numerals refer to stages in the voyage, Fig. 8. The fact that the area between the curve and the dashed line is positive shows that Earth clocks advance, relative to ship clocks, for the trip as a whole.

event):

$$x_{e0} = \gamma'_s x'_{e0} - \beta'_s \gamma'_s c t'_0 \approx x'_{e0}, \tag{33}$$

$$c t_{s0} = \gamma'_s c t'_0 - \beta'_s \gamma'_s x'_{e0} \approx -\beta'_s x_{e0}, \tag{34}$$

where we have used $\gamma'_s \approx 1$ and $t'_0 = 0$. We see that the event does not occur at the same time in the two frames: $t'_0 \neq t_{s0}$.

At $t_s = 0$, in the inertial frame that coincides with the ship frame after the acceleration, the Earth has moved at speed approximately $\beta'_e c$ for time $-t_{s0} = \beta'_s x_{e0}/c$ [Eq. (34)] since the event in question, and so it is located at $x_{e0} + \beta'_e \beta'_s x_{e0}$; and Earth clocks have advanced from $\tau_{e0}$ by only $-t_{s0}/\gamma'_e = \beta'_s x_{e0}/c\gamma'_e$ due to the time dilation. This is the situation in the ship frame immediately after the acceleration.

In other words, in the noninertial frame of the ship during the brief acceleration, the Earth moves rapidly from $x_{e0}$ to $x_{e0} + \beta'_e \beta'_s x_{e0}$, and its clocks advance rapidly from $\tau_{e0}$ to $\tau_{e0} + \beta'_s x_{e0}/c\gamma'_e$.

After the ship coasts for time $\Delta t_s$, Earth position and proper time have advanced still further, to $x_e$ and $\tau_e$, respectively:

$$x_e = x_{e0} + \beta'_e \beta'_s x_{e0} + \beta'_e c \Delta t_s, \tag{35}$$

$$\tau_e = \tau_{e0} + \beta'_s x_{e0}/c\gamma'_e + \Delta t_s/\gamma'_e. \tag{36}$$

Dividing by $\Delta t_s$, we obtain

$$\frac{x_e - x_{e0}}{\Delta t_s} = c\beta'_e \left(1 + \frac{g x_{e0}}{c^2}\right), \tag{37}$$

$$\frac{\tau_e - \tau_{e0}}{\Delta t_s} = \frac{1}{\gamma'_e}\left(1 + \frac{g x_{e0}}{c^2}\right). \tag{38}$$

Going over to continuous acceleration,

$$\boxed{\beta_e = \beta'_e \left(1 + \frac{g x_e}{c^2}\right),} \tag{39}$$

$$\boxed{\frac{d\tau_e}{dt_s} = \frac{1}{\gamma'_e}\left(1 + \frac{g x_e}{c^2}\right),} \tag{40}$$

in the ship frame, with $x_e$, $\tau_e$, and $\beta_e c = dx_e/dt_s$ being Earth position, proper time, and coordinate velocity respectively. ($x_e$, $\beta_e$, and $\beta'_e$ are negative; see Fig. 8.)

Equations (39) and (40) imply the existence of an *event horizon* at

$$x_e = -c^2/g, \tag{41}$$

where motion and time stop ($\beta_e = 0$ and $d\tau_e/dt_s = 0$). On the other side of the event horizon, where $1 + g x_e/c^2$ is negative, clocks and light go backwards. Since all motion stops at the event horizon, no signal can pass it, and consequently regions beyond the event horizon are not immediately accessible to observation; however, the event horizon may be removed at any time that acceleration stops, and thus bodies which have in the past been on the other side may affect the observer in the future (as in Fig. 4).

According to Eq. (39), when the ship initiates acceleration, the Earth displays a discontinuity in velocity of a factor of $1 + g x_e/c^2$. Light behaves in a similar fashion. This phenomenon is mentioned by Møller.[12] We may ascribe it to a change in the rate of passage of time[13] by the same factor, as described by Eq. (40). At positions of large gravitational potential the velocity may exceed $c$; an example is found in Table I, Part III.

Equation (40) involves both special and general relativistic effects: Earth clocks in the co-moving inertial frame are slowed by the factor $1/\gamma'_e$ due to the time dilation; and in addition, in the ship frame their rate may be increased or decreased by the pseudo-gravitational field $g$. In Part I, the two effects are equal in direction and magnitude; both tend to slow down the Earth clocks in the ship frame of reference.

Earth clock rates in the ship frame are plotted in Fig. 10 for the whole trip.

Earth position in the ship frame is given by

$$\boxed{x_e = -(1 - 1/\gamma'_e)c^2/g,} \tag{42}$$

$$x_e = -\left(1 - \frac{1}{\cosh(g t_s/c)}\right)\frac{c^2}{g}, \tag{43}$$

in view of Eq. (31). This is established by two observations: it yields $x_e = 0$ at $t_s = 0$, and its derivative with respect to $t_s$ is found to be Eq. (39). [The derivative of Eq. (43) is $dx_e/dt_s = -c \tanh(g t_s/c)/\cosh(g t_s/c)$; substitution from Eqs. (32) and (43) gives Eq. (39).]

The Earth–ship distance in the ship frame is plotted in Fig. 11 for the whole trip.

Earth time in the ship frame is given by

$$\tau_e = (c/g)\tanh(g t_s/c) \tag{44}$$

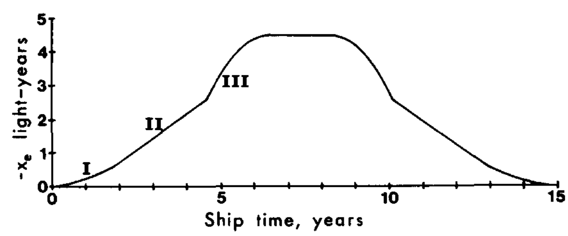


Fig. 11. Earth–ship distance in the ship reference frame, plotted versus ship time.

or, in view of Eq. (32),

$$\boxed{\tau_e = -\beta'_e c/g,} \tag{45}$$

since it is zero at $t_s = 0$ and its derivative is given by Eq. (40) [using Eqs. (31) and (43)]. [$\beta'_e$ is negative. It is merely a coincidence that Eq. (45) resembles the nonrelativistic formula $t = -v/g$.] As $t_s$ increases, $\tau_e$ approaches the limit $c/g$; in other words, as Earth approaches the event horizon, Earth clocks slow toward a stop at a particular time, namely, the distance between the event horizon and the initial (rest) position of Earth, divided by $c$.

*Part II.* The Earth moves at constant speed $-c\beta(\max)$. Its position is given by $(x_e + D'_I) = -c\beta(\max)t_s$, where $x_e = -D'_I$ and $t_s = 0$ at the start of Part II. Earth time is given by the time dilation relation: $\tau_e = t_s/\gamma(\max)$.

*Part III.* For convenience we set $t_s = 0$ at the end of Part III, where $x_e = -(L + 2D)$. Results are then similar to those of Part I. As in Part 3, $g$ is negative; and $x_e$ remains negative (Fig. 8).

$$\beta'_e = -\tanh(gt_s/c), \tag{46}$$

$$\beta_e = \beta'_e(1 + gx_e/c^2), \tag{47}$$

$$\frac{d\tau_e}{dt_s} = \frac{1}{\gamma'_e}\left(1 + \frac{gx_e}{c^2}\right), \tag{48}$$

$$x_e = -(L + 2D)/\gamma'_e - (1 - 1/\gamma'_e)c^2/g, \tag{49}$$

$$\tau_e = (L + 2D - c^2/g)\beta'_e/c \tag{50}$$

(see footnote 14).

*Light Pulses.* In the Earth's frame of reference, the yearly light pulses form uniformly spaced circles centered at the Earth. The radius of each circle, in light-years, is $t$-$n$, with $n = 0,1,\ldots,\mathrm{INT}(t)$. Here $\mathrm{INT}(t)$ means "greatest integer in $t$," and $t$ is total elapsed time on Earth in years.

In the ship frame, since the Earth does not "actually" accelerate (see Sec. IV D), the light pulses are always in an array characteristic of the Doppler effect, with uniformly spaced centers and uniformly decreasing radii. The center $x_c$ of each circle is the "retarded" Earth position:

$$x_c = x_e - \beta'_e\gamma'_e(\tau_e - n_e) \tag{51}$$

in light-years, where $x_e$ is current Earth position, $\tau_e$ is total elapsed time on Earth in the ship frame, and $n_e = 0,1,\ldots,\mathrm{INT}(\tau_e)$. Radius of each circle is $\gamma'_e(\tau_e - n_e)$.

## IV. DISCUSSION

### A. Gravitational frequency shift

Derivations of the gravitational frequency shift are based on such solid ground as conservation of energy and the equivalence principle.[2–4] Existence of the effect has been experimentally verified in the redshift of spectral lines from white-dwarf stars and from the sun.[15] Terrestrially, Pound and Rebka[16] observed it in a 22-m tower using the Mössbauer effect. Hafele and Keating[17] demonstrated both the time dilation of special relativity $(1/\gamma)$ and the gravitational frequency shift of general relativity $(1 + gx/c^2)$ by carrying atomic clocks around the world twice, once east and once west, in commercial airlines. Equation (40) shows that the two effects are multiplicative, but they may in a sense be treated as additive where both are small; using Eq. (39) for $\beta'_e$, and suppressing the $e$ subscript that appears on all variables here:

$$\left(1 + \frac{gx}{c^2}\right)\Big/\gamma' = \left(1 + \frac{gx}{c^2}\right)\left[1 - \left(\frac{\beta}{1 + (gx/c^2)}\right)^2\right]^{1/2} \tag{52}$$

$$\approx 1 + gx/c^2 - \beta^2/2. \tag{53}$$

For an airplane observed from the ground directly below, the Earth's field is approximately that of a uniformly accelerated reference frame, and the second and third terms of Eq. (53) have opposite signs and may be comparable in magnitude.

### B. Event horizon

The event horizon [see Eq. (41)], which is always found in a uniformly accelerated frame of reference, is similar to that found at the Schwarzschild radius of a black hole. It is inaccessible to observers in either case. In the frame of reference of accelerated observer $A$, in the case of the accelerated frame, or of remote observer $A$, in the case of a black hole, person $B$ may approach but never reach the event horizon. From the standpoint of person $B$ as observer, the event horizon just is not there, and $B$ may pass through without incident.[7,18]

An object may be located at an event horizon only if the event horizon moves to the object (e.g., due to change of rocket thrust by the observer); the object cannot move to a stationary event horizon, because all motion stops there. No signal can pass through an event horizon. As illustrated in Figs. 4 and 5, objects on both sides of the event horizon fall freely toward it, but they never reach it. Figure 4 shows how light may accelerate away from the event horizon.

There are of course differences between the event horizon of the uniformly accelerated frame and that of the black hole. In the former, any freely falling object of finite rest mass must sooner or later succumb to the pseudogravitational field and approach the event horizon; whereas objects outside of the Schwarzschild radius can escape entirely if they have sufficient velocity (magnitude and direction). Also, in the black hole tidal forces are significant and space–time exhibits curvature; neither is involved in the uniformly accelerated frame.

Behind the event horizon of a black hole, all detailed information is irretrievably lost, and all that remains is total mass, charge, and angular momentum.[19] By contrast, in the uniformly accelerated reference frame, the event horizon may be removed at any time that the observer terminates the acceleration; in this case no information is lost, and consequently it is meaningful to describe events behind the event horizon. In Fig. 4, for example, an event horizon springs up between the ship and the Earth, but the shipboard observer who plans to return can continue to calculate Earth position and time.

Event horizons are by no means rare, but they usually go unnoticed. For example, an observer at the equator undergoes a centripetal acceleration of $g = 3.4 \times 10^{-2}$ m/sec$^2$ due to rotation of the Earth. The corresponding event horizon is straight overhead at a distance of $c^2/g = 280$ light-years. (Other effects, such as solar and terrestrial gravitation, modify this number only slightly.)

Another paradox now suggests itself: how is it that the equatorial observer can see stars that lie beyond the event horizon at 280 light-years?

The answer is that the event horizon moves relative to the stars, and starlight slips through when the event horizon is on the other side of the Earth. If the observer were to maintain his acceleration in the same direction for hundreds of years, then stars in the direction of the event horizon would appear strikingly altered: they would be redshifted and approaching invisibility.

### C. Twin paradox

If one twin remains on Earth while the other takes a trip at high speed to a distant star, then, in the Earth frame of reference, the traveling twin displays the time dilation of special relativity, and when he returns he has aged less than the terrestrial twin. The twin paradox involves the following consideration: since all motion is relative, one could assert that the Earth took the trip at high speed, and the twin in the space ship actually remained stationary. Then the twin on Earth should show the time dilation and emerge younger than the twin on the ship. This is a paradox, i.e., an *apparent* contradiction (the phrase "apparent paradox" is a tautology); there is of course no real contradiction. The traveling twin must undergo an acceleration, as otherwise he can never return; and during this acceleration special relativity no longer applies in his reference frame. Thus the twins' trips are not of the same kind, which resolves the paradox.

When general relativity is included in analysis of the traveling twin's trip, it is found that the twins predict the same final result. Table I shows that the total Earth time in the Earth frame is the same as the total Earth time in the ship frame, although that time is distributed differently among the parts of the trip. (Ship times for corresponding parts of the trip are the same, because the events that define transition from one part to another occur at the ship.)

### D. Absolute acceleration

In principle the twin paradox is resolved with discovery of the essential asymmetry between the twins, namely, the acceleration of the traveling twin. However, there remains an intriguing puzzle to solve: how can the twins tell which twin took the trip?

Accelerometers will not suffice to determine acceleration, on account of the principle of equivalence. If the acceleration of the traveling twin is produced by means of the gravitational fields of massive objects, then no accelerometer will register the acceleration.

Gravitational field gradients will not provide the needed information, since objects of small gravitational potential can produce large gradients when located close to the gradient detector.

Examination of the twins after the trip to determine relative aging likewise will not work: if the Earth-bound twin spends his time near a massive object, he could in principle age less than the traveling twin.

The surprising fact is, the determination of which twin took the trip can be made only by observation of "the fixed stars," i.e., all the mass, or at least all of the "important" mass[20] in the universe. The trip is in principle defined in terms of motion relative to the fixed stars, and they must be consulted in order to determine whether a trip occurred. No measurement in a closed laboratory will suffice.[21] Mach's principle, adopted by Einstein[22] and later elaborated,[20] suggests that if there were no fixed stars then there would be no fundamental distinction between the Earth and ship frames of reference, and that if the fixed stars were somehow to be accelerated the effect would be the same as if the observer were accelerated. Thus when we state that the space ship undergoes an acceleration, and the Earth does not, in order to resolve the twin paradox, there is a hidden assumption concerning the distribution of mass in the universe. The implicit "absolute" acceleration means not "felt" acceleration but acceleration relative to the fixed stars.


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[13]Similarly, Einstein proceeded from the concept that time slows down near massive objects to the idea that light moves more slowly there; see footnote 4.

[14]In R. Perrin, Ref. 6, there is a misprint in the corresponding formula; the equation immediately preceding his Eq. (44) should read:

$$t_{III} = s/c = (v/c^2)(L + 2d + c^2/g).$$

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