The twin "paradox" and the conventionality of simultaneity

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A new approach to understanding the twin paradox, based on the conventionality of simultaneity, is presented and illustrated. The canonical version of the twin paradox is discussed with reference to its historical origins and the standard explanations given for the differential aging of the twins. It is shown that these are merely specific examples of an infinite class of possible accounts, none of which is privileged. The bounds of this class are given a novel geometrical interpretation. Nonstandard versions of the twin paradox are discussed, and the conventionality of the simultaneity approach is generalized. The role of accelerated reference frames in explaining the twins' aging is also critically examined. The application of the conventionality of simultaneity to the twin paradox hopefully provides a way to settle the often discussed issue of the twins' differential aging.

I. INTRODUCTION

The "twin paradox" has been the subject of a great deal of interest and discussion since the introduction of special relativity by Einstein in 1905. Henri Arzelies has pointed out that while Einstein had suggested the kernel of the paradox, it was Paul Langevin in 1911 who first posed the problem in its current more or less standard form.[1] Describing much of the subsequent scholarly discourse concerning the paradox, Arzelies complains that "the same arguments are always advanced, and the same replies given."[2] A review of recent work dealing with the twins in special relativity seems to bear this out. This paper attempts to provide an approach to the twin paradox, suggested by one of us,[3] which will make superfluous much of the standard discussion that Arzelies finds so exasperating.

In Langevin's 1911 paper on space and time he discussed at length the implications of special relativity and presented what came to be called the twin paradox as an "exemple concret," of its implications.[4] Langevin described a scenario in which a traveler leaves the Earth for a distant star at a speed close to the speed of light and returns in the same manner having aged only 2 years, while on Earth 2 centuries have elapsed. This is essentially the standard account of the paradox to which is often added the idea that the traveler has a twin who stays on the Earth, so that at the end of the trip the twins will have aged differently.

Langevin did not include his calculations, but the differential aging can be demonstrated by calculating the proper time, τ, along the two paths through Minkowski space-time, as shown in Fig. 1, where the outward and inward coordinate speeds are the same. These are labeled as path 1, from the origin of the Earth's frame to time $t=2T$ along the vertical axis and path 2, from the same origin to the turning point e and back again, and they correspond to the earthbound and the traveling twin, respectively. To obtain proper times one can integrate along each path using the fact that, for constant speeds,

$$d\tau=\gamma^{-1}\,dt, \tag{1}$$

where

$$\gamma=1/(1-v^2/c^2)^{1/2}; \tag{2}$$

t is time measured in the inertial frame of the Earth, v is the speed of departing and returning, and c is the speed of light. Equating this calculation of proper time with the time measured by ideal physical clocks is what has been referred to as the "clock hypothesis."[5] We will assume that this hypothesis holds, although there is some discussion as to when clocks do actually measure proper time.[6] Since γ^{-1} is always less than or equal to 1, the proper time measured along path 2 will always be less than that along path 1. That is to say,

$$\tau_1=2T>\tau_2=\gamma^{-1}(2T). \tag{3}$$

The differential aging suggested by Langevin comes directly from the fact that proper time is a path dependent quantity in special relativity.

From these straightforward calculations it is not clear where there is a "paradox" in the story of the twins. One way to make the twin paradox seem paradoxical or at least unexpected would be to note that the Lorentz transformation predicts reciprocal dilatation of moving clocks, according to which each clock is calculated to move slower than the other, and to contrast this to the nonreciprocal dilatation predicted for the round-trip journey.[7] Wesley Salmon refers to this symmetrical time dilatation as the "clock paradox" as opposed to the asymmetrical dilatation which takes place in the twin paradox.[8] Perhaps confusingly "clock paradox" can also refer to the attempt "To avoid ⋯ the [biological] issue of whether a traveler's aging is in accord with the standard clock that he carries.[9] Because proper times are path dependent quantities, the time dilatation which produces the "clock paradox" fails to produce a truly paradoxical version of the twins' story. With notable exceptions including Herbert Dingle,[10] most commentators agree that the proper times on the two paths of the twin paradox are unambiguously different, and that as such there is formally speaking no paradox. Instead, most of the significant discussion of the twins has focused on the asymmetry between the two paths and on trying to explain where and when the differential aging actually occurs. It is these sorts of arguments which we will address.

II. ASYMMETRIES

Langevin himself was the first to emphasize the fundamental lack of symmetry between the path through space-

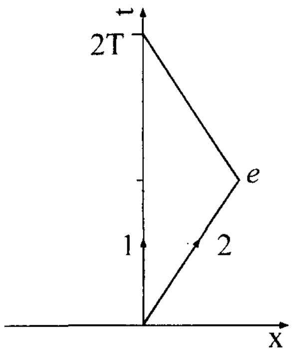

Fig. 1. The standard version of the twin paradox on Minkowski space-time. The labeled arrows designate the paths of the two twins.

time of his traveler and that of a stationary observer on the Earth.[11] In his 1911 paper, he called attention to two asymmetries that have been the basis for many if not most of the standard explanations of the differential aging advanced since then. The first of these is in the difference between the experience of the traveler and that of the Earth observer if they try to keep track of each other's progress using radio signals. The second fact that Langevin used to support the "dissymetrie" between the two paths was that of the acceleration that the traveler must undergo in order to return to Earth. Many arguments have been advanced since Langevin's paper up through the 1990s which have followed the lines suggested by these two asymmetries. These arguments can be grouped into those that focus on the effect of different standards of simultaneity in different frames and those that designate the acceleration as the main reason for the differential aging.

The family of explanations of how the twins age differentially based on the relativity of simultaneity effectively includes several different but related approaches. Among these explanations are the radio signal approach first suggested by Langevin and Lord Halsbury's "three brothers" approach. These both explicitly or implicitly remove from consideration the role of the acceleration. Each then tells a story about how during the course of the journey the proper times measured by earthbound and traveling clocks change with respect to one another.

David Bohm gives a detailed version of the radio signal approach. The different experiences he describes of traveler and earthbound observer while maintaining radio contact follows Langevin's qualitative discussion closely (see Fig. 2). From the relativistic Doppler shift equations, Bohm notes that from the point of view of the earthbound observer he or she will receive "first of all a set of slower pulses and later [after time q], another set of faster ones,"[12] where $q=T(1+v/c)$ is the time that the first signal is received after the traveling twin turns around.

Conversely Bohm concludes that "If the rocket observer were watching the fixed observer he would then see the life of the latter slowed down at first and later speeded up."[13] The change between slow and fast would in this case occur at the time $p=T(1-v/c)$ when a signal from Earth reaches the traveler at the turnaround point e (see Fig. 3). Bohm concludes that for the traveling twin "the effect of the speeding up more than balanced that of the slowing down. He would not therefore be surprised to find on meeting with his twin that the latter had experienced more of life than he had."[14] Bohm's account of the relative lapse of proper time

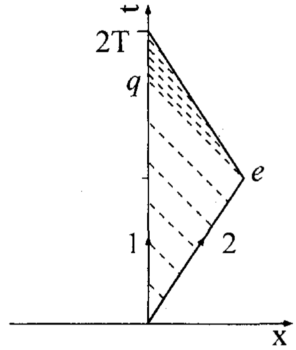

Fig. 2. The dashed lines represent radio signals sent at constant intervals from the frame of the traveling twin, path 2, back to Earth. The point q represents the moment that the signal sent at the turning point, e is received by the earthbound twin on path 1.

for each observer does not give the acceleration any special treatment and describes a situation where each observer sees the other going more slowly than him or herself at first and going faster after a certain moment in time, p or q.

The radio communication solution exemplified by Bohm is similar to the "three brothers" approach suggested by Lord Halsbury. This is a situation where instead of turning the corner at e the traveling twin's clock is synchronized with a third clock carried by a third sibling already moving at the opposite velocity toward the Earth; the time measured by both clocks will together give us the proper time along the whole of path 2.[15] This is intended to remove any question of the effect of acceleration on the motion.[16] The difference in measurements of proper times on the two paths, according to those who have adopted this approach is (as in Bohm's discussion) based on the relativity of simultaneity. Each inertial frame, stationary, departing, and returning, has lines of simultaneity, horizontal, and parallel to the lines re and se respectively defined by the Einstein convention, as shown in Fig. 4. Therefore, on outgoing and returning legs both traveling and stationary clocks seem to be going faster than each other, but the change of inertial frames at e constitutes a change of lines of simultaneity which results in a jump ahead between the times r and s as measured on the moving clocks with respect to the stationary clocks. The "missing time" between r and s becomes then the reason for the differential aging.

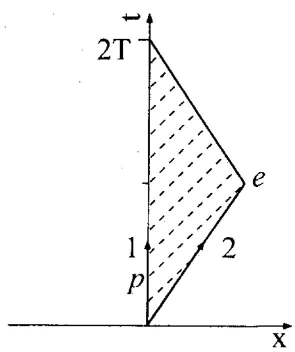

Fig. 3. The dashed lines represent radio signals sent at constant intervals from the frame of the earthbound twin, path 1, to the traveling twin. The point p represents the moment of transmission of the signal from the Earth to the turning point e on path 2.

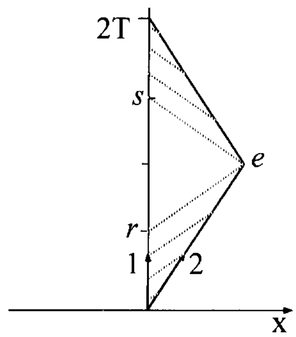

Fig. 4. The dotted lines represent lines of simultaneity from the point of view of a traveler on path 2.

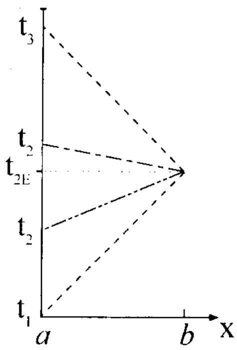

Fig. 5. Establishing synchrony between spatially separated points. The dashed lines represent radio or light signals. The dotted line represents the line of simultaneity associated with the Einstein convention. The mixed dashed and dotted lines represent other possible lines of simultaneity.

In addition to those who have used simultaneity considerations to account for the twins age difference, some have taken Langevin's second asymmetry, the role of the direction-reversing acceleration, as essential to a complete explanation of the paradox. Many seem to feel that the introduction of general relativity and a gravitational field at the point of acceleration is the best way to explain this second asymmetry. Bohm expresses this view and notes that "two clocks running at places of different gravitational potential will have different rates."[17] However, since we are dealing with flat space-time, we regard the reference to general relativity in this context as decidedly misleading.[18] We shall return to the issue of acceleration in Sec. VI below.

III. THE CONVENTIONALITY OF SIMULTANEITY

The result of both simultaneity based and acceleration based explanations of the twin paradox has been a situation in which discussion centers on trying to say at what event the traveler loses time against the Earth. However, there is another approach to the differential aging problem that promises to lay all of this discussion aside. One of us has recently suggested the application of the conventionality of simultaneity, introduced by Reichenbach and Grünbaum, to the twins problem.[19] This approach to simultaneity denies that there is a fact of the matter in designating one standard of simultaneity within the bounds of the light cone, even relative to a given reference frame. Applied to the twins, this undermines much of the discussion of their specific relative aging.

As philosopher Michael Friedman puts it, the conventionality of simultaneity implies that only proper time has "objective status in special relativity."[20] More specifically, this approach refers to Grünbaum's concept of "topological simultaneity.[21] This is simply the assertion that in using light signals to synchronize two spatially separated clocks, at points a and b shown in Fig. 5, one need not divide the difference of transmission, t_1, and reception, t_3, of a signal by two as originally described by Einstein. Doing so gives Einstein's convention of simultaneity, represented by t_{2E}, which is equivalent to assuming the constancy of the one-way speed of light. Instead, one could choose any time, t_2, measured at position a between t_1 and t_3 to be simultaneous with the time of reception of the signal at position b. Another way of saying this is that the interval from t_1 to t_3 is topologically simultaneous with the time of reception recorded at b. John Winnie has investigated some consequences of this approach to establishing simultaneity, using Reichenbach's notation:[22]

$$t_2=t_1+\epsilon(t_3-t_1), \tag{4}$$

such that $0<\epsilon<1$. When $\epsilon=1/2$, this is equivalent to the Einstein convention. Winnie points out that any simultaneity criterion, such that $0<\epsilon<1$, may in fact be applied without affecting the differential aging in what is equivalent to a Halsbury type phrasing of the twin paradox. This is what one would expect if the choice of ϵ is truly one of convention. However, Winnie also concludes that the standard time dilatation in special relativity described by the phrase "moving clocks run slow" is in some ways an artifact of the Einstein convention. Winnie calculates specific criteria, i.e., values of ϵ, according to which clocks can be seen to run synchronously in rest and moving frames. To do this and remove any one-way time dilatation, he shows that one must choose different values of ϵ for when the clocks being synchronized are receding, ϵ_r, or approaching, ϵ_a, with respect to the rest frame, and that these values are additive inverses of each other, such that

$$\epsilon_r+\epsilon_a=1. \tag{5}$$

What Winnie's work suggests for the twin paradox is that while round-trip differential aging is not dependent on convention, the one-way description of relative clock rates is. This is exactly the position the authors support, that any choice of simultaneity criterion, ϵ, will give the overall difference in age for the twins, but that each different choice will represent an equally acceptable story about the relative rates of clocks along each portion of the journey. If this is the case then any of the discussions of where or when during his or her journey the traveler gains on the earthbound twin become equally conventional and thus entirely uninteresting.

The conventionality thesis and the concomitant issue of whether it is possible, in a noncircular fashion, to make a factual measurement of the one-way speed of light, has attracted considerable discussion in both the philosophical and physics literature. For a spirited defence of the thesis reference may be made to the work of Salmon[23] and of Erlichson.[24] Objections to the conventionality thesis take three main forms. First, the loss of simplicity arising from the anisotropic effects associated with using nonstandard synchrony. Thus Robert Brehme writes, "It can be done, but it is so artificial as to jar our sense of fitness."[25] But, the conventionality thesis is an issue, not about simplicity, but about what is factual and what is conventional in the foundations of special relativity.

A more promising criticism is to introduce methods of establishing distant synchrony which do not depend on a prior choice of the ϵ parameter. Much discussed in this connection is the method of slow clock transport due to Bridgman.[26] Introducing a notion of self-measured velocity, (i.e., what is now usually referred to as proper velocity) Bridgman showed that in the limit as the self-measured velocity tended to zero, slow clock transport agrees with the Einstein convention, without presupposing it. A similar line of argument has been developed by Brehme,[27] who uses clocks moving in opposite directions with the same proper speed to establish distant synchrony in agreement with the Einstein convention.

Third, a more sophisticated line of argument can be traced back to the work of Robb.[28] The essential idea here is to note that standard Einstein synchrony is equivalent to Minkowski orthogonality to the time axis of the reference frame, and then to demonstrate that Minkowski orthogonality is definable from the causal structure of Minkowski space-time, i.e., the light cone structure, without any assumptions about the one-way speed of light. Nevertheless, the conventionality thesis can still be defended on the grounds that any method that establishes standard synchrony in a moving frame will automatically define nonstandard synchrony in a stationary frame, so the conventional element is restored in specifying simultaneity in the stationary frame, viz., the choice of whether to import into *that* frame the standard synchrony defined in any of the moving frames.

In this paper, we shall proceed on the assumption that the conventionality thesis is correct, and refer the reader for a comprehensive review of this issue to the work cited in Ref. 3.

IV. CONVENTIONALITY OF SIMULTANEITY AND THE TWINS' AGING

In order to apply the conventionality of simultaneity to the problem of relative rates of clocks in the twin paradox, we consider a situation in which the traveling twin continuously sends and receives signals from the Earth and uses these to set upper, u, and lower, l, bounds on possible values of his or her clock that may be judged simultaneous with a given reading of the clock of the earthbound twin (see Fig. 6).[29] We subsequently plot the proper time along each path against one another which produces a parallelogram, as shown in Fig. 7, the upper and lower boundaries of which are the bounds on possible times measured by the traveler for a given instant on the earthbound clock. That is to say that for each instant of proper time measured along path 1, τ_1, there exists a range of proper time values along path 2, τ_2, which would all be equally good choices to be considered simultaneous with that particular value of τ_1. It is important to note that the diagonal of the parallelogram lies below the line of slope 1 such that the differential aging by the end of the journey is undisputed. This parallelogram, OPQR, also allows one to see that it would be possible for either clock to run more quickly than the other over any particular interval within its bounds. The essence of the conventionality of simultaneity approach to the twin paradox can be made apparent by remarking that any nondecreasing curve inside the parallelogram would be equally acceptable as a way of describing the relative rates of the two clocks.

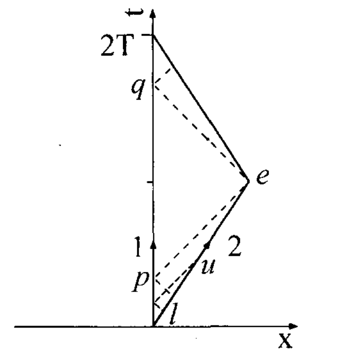

Fig. 6. The dashed lines represent radio or light signals. The bounds on possible simultaneous points are given by the proper times measured on path 2 at each point of transmission, l, for the lower bound, and reception, u, for the upper bound.

With this approach to the twin paradox, it is easy to see that any discussion about where during the journey the differential aging takes place is unnecessary. In fact, many of the standard explanations can be plotted onto the parallelogram. Two of those already discussed involve the use of the Einstein convention from the point of view of the traveler.[30] In the first, one could simply halve the difference between the traveler's sending and receiving times over the whole journey to establish the progress of the twins' clocks relative to one another. Graphically this would mean taking the average between the upper and lower boundaries of the parallelogram, as depicted by the dashed line segments in Fig. 7. Adopting this method, the traveler's clock seems to run more quickly than the Earth's up to $\tau_1 = T(1 - \nu/c)$, the time in the Earth's frame that the first signal reaches the turnaround point. Then the traveler's clock seems to lose ground against the Earth's until $\tau_1 = T(1 + \nu/c)$, the time that the Earth receives its first signal after the turnaround. Thereafter, the traveler again ages more quickly, but the overall effect is such that his or her total age is less than that recorded on Earth.

The Halsbury "three brothers" approach to explaining of the differential aging can also be represented as a curve inside the parallelogram, shown by the dashed line segments in Fig. 8. To get this curve, the Einstein convention for simultaneity for a frame receding with velocity, ν, is used until the traveler reaches his or her halfway point, half of $2T^*$, im-

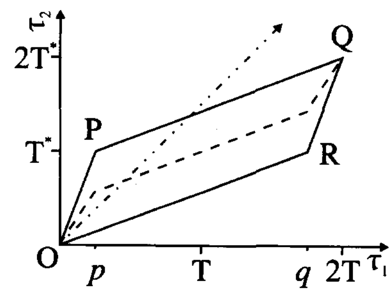

Fig. 7. A parallelogram of possibly simultaneous points. The dashed line represents a possible convention of simultaneity. $T^* = T\gamma^{-1}$. The mixed dashed and dotted line represents the line of slope=1 on which $\tau_1 = \tau_2$.

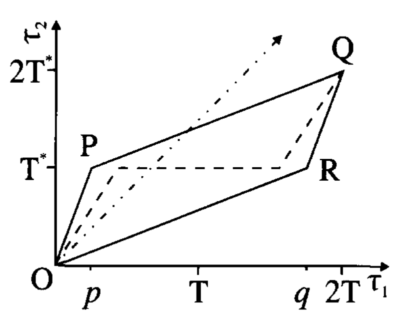

Fig. 8. The dashed line segments represent a possible choice of simultaneity conventions corresponding to the Halsbury "three brothers" approach to the twin paradox.

plying that the traveler is aging more quickly. On the second half of $2T^*$, the same convention for an approaching frame is used, and the traveler ages more quickly again. The overall youth of the traveler is due then to the "missing time" from the traveler's journey which is represented by the horizontal section in which the Earth ages instantaneously from his or her point of view. Conversely, from the Earth, the traveling twin's clock seems to stand still during this period.

Infinitely many other stories may also be told which fit into the bounds of convention set by the parallelogram. As we have seen, the simultaneity criterion, ϵ, can be chosen so as to eliminate one-way time dilatation if $\epsilon=\epsilon_r$, for receding clocks, or ϵ_a, for approaching ones. The result of choosing these criteria is represented by the dashed line segments in Fig. 9. During the first half of the traveling twin's journey his or her clock runs in synchrony with the clock on earth, the dashed line segment runs along the line of slope one. These lines are also parallel during the second half of the twin's journey where the clocks run at the same rates again. The overall differential aging is caused by the horizontal dashed segment over which, from the Earth, the traveler's clock stands still. Interestingly, the additive inverse relationship of Eq. (5) can be seen readily on the parallelogram. Modifying Reichenbach's notation in Eq. (4),

$$\epsilon=(t_2-t_1)/(t_3-t_1). \tag{6}$$

From the traveler's point of view, t_3 and t_1 are given by the adjacent sides of the upper and lower boundary of the parallelogram, respectively, while t_2 is the chosen simultaneous moment represented by the dashed line segments. This implies that on the first part of the traveler's journey

$$\epsilon_r=B/(A+B), \tag{7}$$

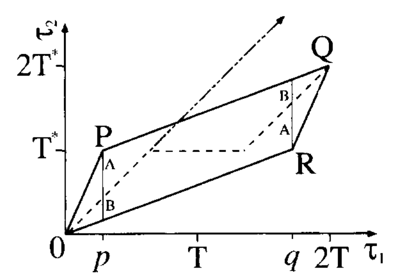

Fig. 9. The dashed line segments represent a possible choice of simultaneity conventions.

where A and B are the magnitudes labeled on Fig. 9. On the last part of the journey,

$$\epsilon_a=A/(A+B). \tag{8}$$

This gives the result that

$$\epsilon_r+\epsilon_a=(A+B)/(A+B)=1, \tag{9}$$

as expected.

The boundaries of the parallelogram can also be seen to represent the approach to explaining the twin paradox, exemplified above in Bohm's discussion, which uses Doppler shifted radio signals. Looking more closely at the boundaries of the parallelogram, we can see that the lower boundary has the slope $[(1-\nu/c)/(1+\nu/c)]^{1/2}$ and the upper boundary has the slope $[(1+\nu/c)/(1-\nu/c)]^{1/2}$ where ν is taken to be the outgoing velocity of the traveler and $-\nu$ the returning velocity. This is not surprising as these slopes are the relative rates of the measurement of proper times in frames moving with respect to one another. This relationship can be seen in the relativistic Doppler shift equation according to which

$$\tau'=[(1+\nu/c)/(1-\nu/c)]^{1/2}\tau, \tag{10}$$

where τ' is the period of radiation received in a frame moving with velocity ν and τ is the period of the radiation in the rest frame, in the situation where the radiation is propagating in the same direction as ν. Noting that $\tau_2=\Sigma\tau'$ and $\tau_1=\Sigma\tau$ over their respective paths, and that the Doppler shift equation describes the periods of radiation on the upper bound, and the multiplicative inverse describes the lower bound, the slopes of the sides of the parallelogram can be easily confirmed.

Looking to the story Bohm tells of the twins' relative progress, we can see that he is actually describing the two boundaries of the parallelogram which come directly from the Doppler shifted signals he sets out to discuss. Bohm first discusses the appearance of signals coming from the traveler as seen on the Earth. Looking at the parallelogram, we can see Bohm's explanation[31] by looking at the lower boundary ORQ in Fig. 9. Taking this approach, the Earth observer sees the traveling twin aging more slowly up to the time $p=T(1+\nu/c)$, represented by segment OR of the parallelogram. Subsequently, he or she sees the traveler aging more quickly than earthbound clocks, segment RQ.

From the other point of view, Bohm expects that the moving twin will see the Earth's clock running slower than the moving clock up until the time $q=T(1-\nu/c)$, and subsequently he or she will see the Earth's clock running more quickly than the moving clock.[32] This is exactly the story that is represented by the upper boundary, OPQ, of the parallelogram in Fig. 9. Bohm explains the differential aging by pointing out that the speeding up of the Earth's clock witnessed by the traveler after time q "more than balanced" the slower relative rate prior to q.[33] The use of the parallelogram makes it obvious that the Doppler shifted signal approach to the paradox is concerned with the outer bounds of an infinite number of acceptable stories about the twins relative rates of aging.

V. THE TWINS ON NONSTANDARD PATHS: THE CONVENTIONALITY APPROACH GENERALIZED

Another approach to explaining the differential aging of the twins without reference to the point of acceleration has

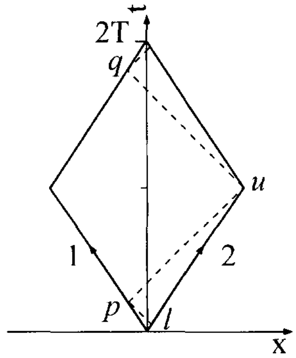

Fig. 10. A one-dimensional "Hafele, Keating" scenario. The labeled arrows designate the paths of the two twins. Symmetrical paths do not take into account the Earth's rotation, there is no differential aging. The dashed lines represent radio or light signals. The point q represents the moment that the signal sent at the turning point on path 2 is received by the twin on path 1. The point p represents the moment of transmission of the signal from the twin on path 1 to the turning point on path 2.

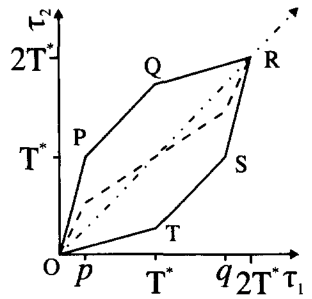

Fig. 11. A symmetrical hexagon of possibly simultaneous points. The dashed line represents a possible convention of simultaneity: $T^* = T\gamma^{-1}$.

been to put the two paths onto cylindrical space-time. By cylindrical space-time, we mean here a two-dimensional universe in the shape of an infinitely long cylinder with time running "up" the cylinder and space running "around" it. In this way the stationary twin is considered to travel up the cylinder, the time axis parallel to the axis of rotation of the cylinder, and the traveling twin to depart and return by simply going around the cylinder at a constant velocity. The calculation of the proper times on the cylinder has been done recently by more than one individual.[34] At first it might seem that one might be able to get a real paradox out of this situation without the obvious asymmetry in the two paths, that is, the asymmetry provided by the acceleration and change in direction on the traveler's path. This turns out not to be possible because of the structure of simultaneity relations in cylindrical space-time.[35] This should not be surprising as some asymmetry must exist between the two paths in order to calculate different proper times. This scenario has also been examined by Redhead,[36] who has shown how the parallelogram construction can be adapted to spell out the conventionality limits on synchronizing distant clocks in cylindrical space-time.

One place to observe the requirement of an asymmetry to get differential aging is in the Hafele–Keating experiment. In this experiment, differential aging was observed on two atomic clocks traveling on jets at the same speed around the Earth in opposite directions.[37] The rotation of the Earth provided the asymmetry that was necessary to produce the difference in proper times. These two paths without the rotation of the Earth can be compared schematically to the two paths going around in opposite directions on cylindrical space-time. The addition of the rotation gives us a scenario which looks roughly like the twin paradox in cylindrical space-time.

Using flat noncylindrical coordinates, we can set up an idealized one-dimensional Hafele, Keating situation with two symmetrical paths for each twin corresponding to the case when the Earth's rotation is not considered (see Fig. 10). It turns out that using the Einstein convention of simultaneity implies a specific story about the relative rates of clocks even when there is no overall differential aging. Taking the same approach as above, we can consider that the twin on path 2 checks the progress of the other by sending and receiving signals. With this information, the twin on path 2 again sets upper and lower bounds for acceptable values of its own proper time, τ_2, for each given value of the other twin's proper time, τ_1. The result this time is a symmetrical hexagon with its diagonal along the line of slope 1, so that there is no overall differential aging (see Fig. 11). The slopes of the sides of the hexagon can once again be explained using the Doppler shift equation, Eq. (10), according to which, as previously stated, the lower boundary has the slope $[(1-v_r/c)/(1+v_r c)]^{1/2}$ and the upper boundary has the slope $[(1+v_r/c)/(1-v_r/c)]^{1/2}$, where this time v_r is the relative velocity between the twins each moving with velocity v. Using relativistic velocity addition,

$$v_r = 2v/(1+v^2/c^2), \tag{11}$$

from $\tau_1=0$ to p, where $p=[T-T(2v/c)/(1+v/c)]\gamma^{-1}$, which corresponds to segments OP and OT in Fig. 11. Plugging into the Doppler shift equation gives slopes for these segments of $(1+v/c)/(1-v/c)$ and $(1-v/c)/(1+v/c)$, respectively. The relative velocity, v_r, is zero from $\tau_1=p$ to q, where $q=[T+T(2v/c)/(1+v/c)]\gamma^{-1}$, which implies that segments PQ and TS are both of slope 1. The other boundaries are similarly calculated and should be apparent from symmetry.

The dashed line segments inside the hexagon represent the story given if the Einstein convention is used in the sense that the average of the upper and lower bounds is taken everywhere. As we can see, use of this convention implies that the clocks on different paths are seen to move faster or slower than each other at different moments during the journey even when there is no overall differential aging. In fact, no single simultaneity criterion, ϵ, will pick out the diagonal which seems to make the most sense as a description of the clocks in this situation. The arbitrary nature of the implications of any single criterion supports the conventionality approach of setting the boundaries and not discussing paths within them.

We can also create a one-dimensional Hafele–Keating experiment in which there is differential aging by adding a velocity in one direction (see Fig. 12). In this situation the magnitude of the velocities on path 1 is less than that of the velocities on path 2, and the overall aging of the twin on path 1 will be greater. A hexagon can also be drawn to incorporate the possible values of one proper time versus the other, as shown in Fig. 13. The slopes of the sides, as in the previous hexagon, can be given using the relativistic Doppler shift equation, Eq. (10) and the relative velocities of the two twins using relativistic addition of velocities. As one would expect, substitution of the same velocity for paths 1 and 2 gives us

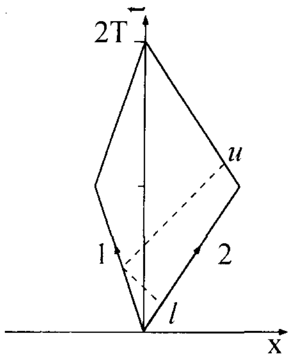

Fig. 12. A one-dimensional "Hafele, Keating" scenario. The labeled arrows designate the paths of the two twins. Nonsymmetrical paths correspond to the case when the Earth's rotation is considered, differential aging takes place.

back the symmetric hexagon, and substitution of zero velocity for one of the paths gives us the parallelogram from the standard twin paradox.

Some general features of this approach to depicting the relative progress of clocks between two paths in Minkowski space-time can be observed. First of all, it is possible to construct a region of possibly simultaneous points for any two paths. The Doppler shift equation relating periods of signals can be used to sum over all periods to get the upper and lower bounds on this region as long as relative velocity along each path is constant over each individual period. Much simpler methods can be used to calculate these bounds if the paths are straight and accelerations are instantaneous. In this situation, one can see from the examples done so far that the number of vertices, V, on the boundary of the simultaneity region is given by

$$V=2(n+1), \tag{12}$$

where n is the number of instantaneous points of acceleration on the twins' paths excluding the accelerations at separation and return.

The conventionality of simultaneity approach to the twin paradox also clarifies some implications of a method for estimating distance suggested by Clive Kilmister. Kilmister has suggested that the traveler could keep track of his or her distance from the origin of the rest frame using the same signals used above to discuss simultaneity.[38] By this method, described as a "radar" method by Hermann Bondi, the traveling twin could estimate distance from the Earth by estimating the time it takes for a signal to make the trip using the

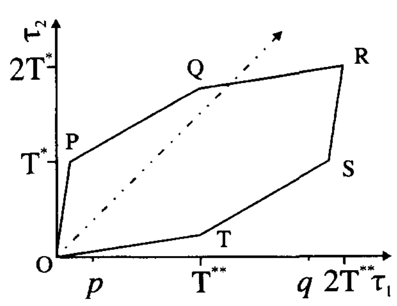

Fig. 13. A nonsymmetrical hexagon of possibly simultaneous points. $T^{**}=T\gamma_1^{-1}$ and $T^{*}=T\gamma_2^{-1}$, where time is dilated by different amounts, γ_1 and γ_2 on paths 1 and 2, respectively.

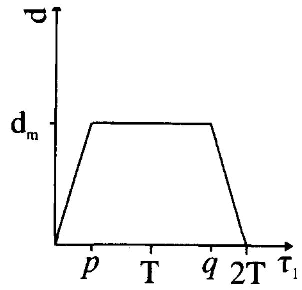

Fig. 14. Distance estimate by the radar method vs proper time on path 1. Estimate levels off at d_m.

Einstein convention and multiplying by the speed of light.[39] This is equivalent to saying that the distance, d, for a specific value of τ_1, the proper time measured on path 1, is given by

$$d=c(t_u-t_l)/2, \tag{13}$$

where t_u and t_l are the upper and lower bounds on the value of proper time on path 2, for that value of τ_1. From the parallelogram and hexagons, we have already constructed, we can get an idea of how the quantity $(t_u-t_l)/2$ varies at different proper times τ_1. Multiplying by c, we can get the distance estimate of Eq. (13). For the standard twin paradox situation, the radar method implies that the distance between twins is constant at a maximum distance, d_m, near the change of direction (see Fig. 14). In the symmetrical one-dimensional Hafele–Keating situation, the radar method also gives an artificially low estimate of distance which implies that the relative velocity between the twins is lessened near the turning points. The distortion of these distance estimates is a result of the use of the radar method, and the specific simultaneity criterion it assumes, and these strange results are artifacts of its adoption. This example demonstrates, this time from the point of view of relative distance instead of relative aging, the arbitrary results of choosing a single simultaneity criterion.

VI. THE ROLE OF ACCELERATION CRITICIZED

Finally, the conventionality approach to the twins' differential aging can also be used to illustrate that discussions which try to pin the age difference to the direction-reversing acceleration are misconceived. Recently Boughn[40] has stressed this point by considering the case of two *identically* accelerated twins who nevertheless age *differently* as a result.

In a comment by Desloge and Philpott,[41] they have described in more detail the paths in Minkowski space-time that Boughn's scenario requires, if the journeys of the twins are to start and finish in spatial coincidence. We shall now apply our analysis to a version similar to that which they describe, as illustrated in Fig. 15. In this case the twins are separated symmetrically, given the same acceleration into a new frame at a point e in time, and brought back together symmetrically with respect to their new frame. At the end, the twin on path 2 has a greater total elapsed proper time.

Using the conventionality of simultaneity approach on this version of the twin paradox, one could, in principle, draw a region of possible simultaneous points that would have 14 vertices according to our previous general observations [see Eq. (12)]. Any path within this region would be an acceptable account of the differential aging. One need not construct the entire region to see that assigning the difference in age to

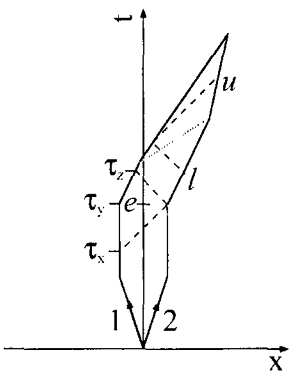

Fig. 15. An "identically accelerated twins" version of the twin paradox. The labeled arrows designate the paths of the two twins. The dashed lines represent radio or light signals. τ_x, τ_y, τ_z, are points in proper time measured on path 1 near the moment of acceleration.

the point of acceleration is only one of these accounts. We can see this in a rough diagram of the shape of this region around the point of acceleration (see Fig. 16). The dashed line segments designate a story that allows the differential aging to take place at the point of acceleration. However, it is obvious that many other nondecreasing curves could fit within the appropriate bounds.

It remains true, of course, that without acceleration, then in Minkowski space-time at any rate, it is impossible to have twice intersecting trajectories so as to formulate the twin paradox with the twins starting and finishing in spatial coincidence. So, in this sense, acceleration is an essential ingredient in understanding the twin paradox. It may be noted however that even this role for acceleration can be eliminated in formulations of the twin paradox in curved space-time, where the twins can fall freely along space-time geodesics between successive meetings.[42]

VII. CONCLUSIONS

Having discussed a new approach to looking at the twin paradox with regard to where and when the differential aging occurs, it is possible to conclude that the conventionality of simultaneity and in particular the concept of topological simultaneity provides a means to put an end to this question. One can conclude that any explanation of relative aging that stays within the bounds set by the light cone is equally valid. In addition, discussion based on the application of particular simultaneity criteria, i.e. ϵ values, either to establish simultaneity or estimate distance is also uninteresting as it discusses only one of an infinite number of conventional ways to approach the problem. Perhaps the method discussed in this paper, the conventionality of simultaneity applied to depicting the relative progress of two travelers in Minkowski space-time, will settle the issue of the twin paradox, one which has been almost continuously discussed since Langevin's 1911 paper.

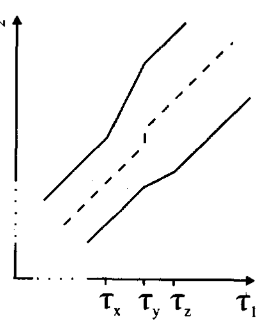

Fig. 16. An approximate detail of the region of possibly simultaneous points near the moment of acceleration. The dashed line segments represent a possible choice of simultaneity convention.

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[2]Reference 1, p. 189.
[3]Michael Redhead, "The Conventionality of Simultaneity," in *Philosophical Problems of Internal and External Worlds: Essays on the Philosophy of Adolf Grünbaum*, edited by J. Earman, A. I. Janis, G. J. Massey, and N. Rescher (University of Pittsburgh, Pittsburgh, 1993), pp. 103–128.
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[8]W. C. Salmon, *Space, Time, and Motion: A Philosophical Introduction* (Dickenson, Encino, 1975), p. 95.
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[10]For a detailed discussion of Herbert Dingle's work on the twin paradox, see H. Chang, "A Misunderstood Rebellion: The Twin Paradox Controversy and Herbert Dingle's Vision of Science," Studies in History and Philosophy of Science **24**, 741–790 (1993).
[11]See Ref. 4, pp. 51 and 52.
[12]D. Bohm, *The Special Theory of Relativity* (W. A. Benjamin, New York, 1965), pp. 168–170.
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[15]For discussion of this version of the paradox, see Ref. 8, pp. 96–97; H. Bondi, "The Spacetraveller's Youth," Discovery **18**, 505–510 (1957).
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[29]The initial stages of the analysis here follow that included in Ref. 3.
[30]Both of these examples were also part of the analysis in Ref. 3.
[31]Reference 12, pp. 168–169.

[32]Reference 12, pp. 170–171.
[33]Reference 12, p. 171.
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[35]See Ref. 34, p. 825 of Dray (1990); p. 26 of Low (1990).
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[37]J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Predicted Relativistic Time Gains," Science **177**, 166–168 (1972); "Around the World Atomic Clocks: Observed Relativistic Time Gains," Science **177**, 166–168 (1972).
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[41]E. A. Desloge and R. J. Philpott, "Comment on 'The case of the identically accelerated twins,' by S. P. Boughn," Am. J. Phys. **59**, 280–281 (1991).
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Elementary derivation of Kepler's laws

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A simple derivation of all three so-called Kepler laws is presented in which the orbits, bound and unbound, follow directly and immediately from conservation of energy and angular momentum. The intent is to make this crowning achievement of Newtonian mechanics easily accessible to students in introductory physics courses. The method is also extended to simplify the derivation of the Rutherford scattering law. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

The so-called Kepler laws of planetary motion have been of central interest for Newtonian mechanics ever since the appearance of Newton's *Principia*.[1] They are discussed in most introductory textbooks of physics[2,3] and continue to be a subject of lively interest in the pages of the *American Journal of Physics*.[4] This interest is not surprising because the understanding of planetary motion has been one of the oldest challenges in many human cultures and continues to excite the sense of wonder among young scientists today.

The purpose of the present article is to give a new elementary derivation of all three of the Kepler laws intended to make their physics accessible to first year university students taking introductory mechanics. I have used this derivation in my own introductory classes for more than a decade and find that it, and the many associated problems, are a highlight of the introduction which I give to physics. In contrast, most first-year textbooks give a description of Kepler's laws but apparently regard their derivation as too difficult. Perhaps the derivation given here can then fill an important gap.

The elementary proof, given in the next section follows directly, in a few easy steps, from conservation of energy and angular momentum which, in turn, follow from $F=ma$ and the central nature of the universal gravitational force, $F=GmM/r^2$. These conservation laws, on which we build, are usually covered thoroughly, and often even elegantly, in first year textbooks.

In succeeding sections, beyond the proof, we provide further discussion of bound elliptic orbits and extend the treatment to the unbound Kepler orbits and to the Rutherford scattering law.

II. ELEMENTARY PROOF OF KEPLER'S LAWS

A. Kepler's first law (the law of orbits): All planets move in elliptical orbits having the Sun at one focus

For a planet of mass m in a bound orbit (negative total energy E), around the Sun of mass M, we have the constant total energy, E

$$E \equiv mv^2/2 - GM\ m/r, \tag{1}$$

where r is the distance of the planet from the Sun and v its velocity. $(-E/m)$ is a positive constant of the motion. Because the force is central we also have conserved angular momentum, l

$$l \equiv mvh, \tag{2}$$

where $h(\equiv r \sin \phi$, with ϕ the angle between $\mathbf{v}$ and $\mathbf{r}$) is the perpendicular distance from the planet's instantaneous velocity vector to the Sun (see Fig. 1). From the definition of h we have $h \leq r$. (l/m) is also a positive constant of the motion. Using Eq. (2) in Eq. (1), we obtain

$$\frac{[(l/m)^2/2(-E/m)]}{h^2} - \frac{[GM/(-E/m)]}{r} = -1. \tag{3}$$

The relationship (3) between r and h, both taken from a common center of force defines an ellipse. In the next section we show that for an ellipse of semimajor axis a and semiminor axis b we have

$$\frac{b^2}{h^2} - \frac{2a}{r} = -1 \quad (h \leq r). \tag{4}$$

