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The Feynman paradox revisited

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Abstract We propose a simpler model in order to facilitate calculations of the Feynman paradox concerning the angular momentum of a static electromagnetic field. When an angular momentum is attached to the static electromagnetic field the paradox disappears. The storage of the angular momentum in the field during the assembling process is also analysed.

Laburpena *Eremu elektromagnetiko estatikoen* momentu angeluarrari buruzko Feynman-en paradoxaren kalkuluak errazteko, eredu sinplifikatu bat aurkezten da. Horrelako eremuei momentu angeluarra egokitzen bazaie, era kuantitatiboan ikusten da paradoxa desagertu egiten dela. Dispositiboaren eratze-prozesuan zehar, momentu angeluarra eremuan nola metatzen den ere aztertzen da.

1. Introduction

It is well known that, for systems of particles satisfying Newton's second and third laws, the time derivatives of the total linear and angular momenta are equal, respectively, to the net external force and torque. For isolated systems we conclude that the total linear and angular momenta are conserved. However, these conservation laws can be considered as independent principles for isolated systems, owing to their relationship with the symmetries of the system.

If the internal forces of the systems are of an electromagnetic nature, Newton's third law no longer holds, and, if the conservation principles of linear and angular momenta (and energy) must remain valid, these magnitudes must be assigned to the interaction field, giving rise to the Poynting theorems in electrodynamics and to the formalism of the energy–momentum tensor (Møller 1972).

However, in the static electromagnetic case, it seems 'intuitively' difficult to assign linear and angular momenta to fields that do not change with time. It will be helpful for pedagogical reasons to analyse some examples in order to show the coherence of the general theory. Some of them have been given by Romer (1966, 1967), Pugh and Pugh (1967) and Corinaldesi (1980). The recent experiments of Graham and Lahoz (1980) have revitalised the subject.

In this article we make a detailed study of a simplified model of the Feynman paradox (Feynman 1964).

The paradox is presented in §2. In §3 the electromagnetic angular momentum is computed; the fact that this momentum is transferred to the disc when the current falls off is easily checked. Finally, §4 is devoted to a discussion emphasising that the electromagnetic angular momentum does not satisfy the superposition principle.

2. Feynman paradox: simplified version

Let us consider a thin circular plastic disc of radius a lying on the XY plane of a reference frame, such that its only allowed motion is rotation about the $0Z$ symmetry axis. At the centre of the disc is a small hole in which a small circular ring is placed centrally so that it is free to move with respect to the disc.

Let us assume that the ring is made of a superconducting material and that a constant current of intensity I is flowing in it, giving rise to a magnetic moment $\boldsymbol{m} = m\boldsymbol{k}$, $\boldsymbol{k}$ being the unit vector along the $0Z$ axis. The ring is to be assumed small enough to be considered point-like.

A charge q is located at the point with coordinates $(a, 0, 0)$ on the edge of the disc. This system is initially static: the fields do not depend on time, and there are no forces on the charge or the ring.

Let us assume that the magnetic moment starts decreasing slowly at the small rate $\dot{m} = \mathrm{d}m/\mathrm{d}t$, owing to, for instance, a small increase of temperature; this gives rise to the appearance of a non-zero resistivity in the ring. We shall accept that this process is slow enough to neglect radiation and relativistic and retarded effects.

The electric field induced by the changing magnetic field acts on the charge, causing the disc to rotate. The vector potential $\mathbf{A}$ created by the

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ring at a point $\boldsymbol{r}$ is, according to Panofsky and Phillips (1975),

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\boldsymbol{r}}{r^3}. \qquad (1)$$

When $\boldsymbol{m}$ changes, the induced electric field $\boldsymbol{E} = -\partial\mathbf{A}/\partial t$ has a tangential component, at the point where the charge is, given by

$$E_\phi = -\frac{\mu_0}{4\pi}\frac{\dot{m}}{a^2}. \qquad (2)$$

The torque exerted by the electric force along the $0Z$ axis is

$$M_Z = -\frac{\mu_0}{4\pi}\frac{q\dot{m}}{a} \qquad (3)$$

such that the angular momentum of the disc when the magnetic moment has fallen to zero is $L_Z\boldsymbol{k}$, where

$$L_Z = \int M_Z\,\mathrm{d}t = -\frac{\mu_0 q}{4\pi a}\int_m^0 \dot{m}\,\mathrm{d}t = \frac{\mu_0 q m}{4\pi a}. \qquad (4)$$

But, if 'initially' the angular momentum of the system is zero and must be conserved, the disc must remain static. This is in brief the essence of the paradox proposed by Feynman.

3. Explanation of the paradox

It can be argued that at the beginning the angular momentum associated with the charge carriers of the superconducting ring must be taken into account. Stedman (1981) has remarked that this is the case in some actual experiments, such as that of Graham and Lahoz (1980). However, in our model this is not relevant, since it has to be compensated by an external torque on the ring in order to leave the ring at rest. There is another basis on which to reject this interpretation: we can, in fact, change the sign of the charge and reverse the velocity of the charge carriers without modifying (4), but changing the angular momentum of the carriers. Similarly, in more general terms, we can change the charge–mass ratio of the carriers. Thus, in principle, we neglect the angular momentum contribution of the charge carriers in order to give an explanation of the paradox.

The principle of angular momentum conservation can still hold if the electromagnetic field in vacuum has an associated angular momentum with respect to the origin given by

$$\boldsymbol{L}_{\mathrm{em}} = \frac{1}{c^2}\int_{R^3}(\boldsymbol{r}\times\boldsymbol{S})\,\mathrm{d}V \qquad (5)$$

where $\boldsymbol{S} = (1/\mu_0)\boldsymbol{E}\times\boldsymbol{B}$ is the Poynting vector and c is the speed of light in vacuum. At the end of the process, when $\boldsymbol{m} = 0$, $\boldsymbol{L}_{\mathrm{em}}$ is also null, since the only remaining fields are those created by the moving charge whose evolution is governed by the Lorentz equation, which excludes the self-terms of the Poynting vector (and also the remaining self-terms of the energy–momentum tensor); in order to take these into account, use has to be made of the Lorentz–Dirac equation.

Because $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, (5) can be rewritten as

$$\boldsymbol{L}_{\mathrm{em}} = \varepsilon_0\int_{R^3}\boldsymbol{r}\times[\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{A})]\,\mathrm{d}V. \qquad (6)$$

Using the vector identities

$$\boldsymbol{C}\times(\boldsymbol{\nabla}\times\boldsymbol{D}) + \boldsymbol{D}\times(\boldsymbol{\nabla}\times\boldsymbol{C}) = (\boldsymbol{\nabla}\cdot\boldsymbol{C})\boldsymbol{D} + (\boldsymbol{\nabla}\cdot\boldsymbol{D})\boldsymbol{C} + \boldsymbol{\nabla}\cdot\mathbf{T} \qquad (7)$$

$$\boldsymbol{r}\times\boldsymbol{\nabla}\cdot\mathbf{T} = \boldsymbol{\nabla}\cdot\mathbf{R} \qquad (8)$$

where

$$T_{ij} = (\boldsymbol{C}\cdot\boldsymbol{D})\delta_{ij} - (C_iD_j + C_jD_i)$$

and

$$R_{ij} = \varepsilon_j^{\,kl}x_kT_{il}$$

ε_{jkl} being the Levi–Civita completely antisymmetric tensor, and using Gauss's theorem in differential form, (6) can be expressed as

$$\boldsymbol{L}_{\mathrm{em}} = \int_{R^3}(\boldsymbol{r}\times\rho\boldsymbol{A})\,\mathrm{d}V + \varepsilon_0\int_{R^3}[(\boldsymbol{\nabla}\cdot\boldsymbol{A})\boldsymbol{r}\times\boldsymbol{E}]\,\mathrm{d}V + \varepsilon_0\int_{R^3}\boldsymbol{\nabla}\cdot\mathbf{Q}\,\mathrm{d}V. \qquad (9)$$

Here ρ is the charge density and

$$Q_{ij} = \varepsilon_j^{\,kl}x_k[(\boldsymbol{E}\cdot\boldsymbol{A})\delta_{li} - (E_lA_i + E_iA_l)]. \qquad (10)$$

The vector potential $\mathbf{A}$ given by (1) is divergenceless, since it satisfies the Coulomb gauge, and the second term in (9) vanishes. By Gauss's theorem, the third term in (9) appears as

$$\int_{R^3}\boldsymbol{\nabla}\cdot\mathbf{Q}\,\mathrm{d}V = \lim_{R\to\infty}\int_{S(R)}\mathrm{d}\boldsymbol{S}\cdot\mathbf{Q} \qquad (11)$$

where $S(R)$ is the sphere of radius R centred at the origin. It is easy to see that this integral vanishes because $\mathbf{Q}$ falls off as R^{-3} while $\mathrm{d}\boldsymbol{S}$ increases as R^2, when R goes to infinity.

In our problem $\rho = q\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a})$, $\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a})$ being Dirac's delta function, where $\boldsymbol{a}$ is the charge position vector, and then

$$\boldsymbol{L}_{\mathrm{em}} = \int_{R^3}(\boldsymbol{r}\times\rho\boldsymbol{A})\,\mathrm{d}V = q\int_{R^3}(\boldsymbol{r}\times\boldsymbol{A})\delta^{(3)}(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}V = q\boldsymbol{a}\times\boldsymbol{A}(\boldsymbol{a}) \qquad (12)$$

i.e.

$$\boldsymbol{L}_{\mathrm{em}} = \frac{\mu_0 q}{4\pi a^3}\boldsymbol{a}\times(\boldsymbol{m}\times\boldsymbol{a}) \qquad (13)$$

which coincides with (4).

In the quasi-static limit, the linear momentum associated with an arbitrary electromagnetic field can be written, according to Calkin (1966), in the form

$$\boldsymbol{P}_{\rm em} = \int_{R^3} \varepsilon_0(\boldsymbol{E}\times\boldsymbol{B})\,\mathrm{d}V = \int_{R^3} \rho\boldsymbol{A}\,\mathrm{d}V \qquad (14)$$

which according to (12) could be interpreted as being $\rho\boldsymbol{A}$, the linear momentum density in quasi-static situations. But this interpretation is wrong as far as angular momentum is concerned, because the surface integral (11) does not in general vanish. In our particular case it has vanished because, when R goes to infinity, $\boldsymbol{A}$ decreases as R^{-2}.

Furry (1969) has calculated the angular momentum of a system consisting of a small magnet of magnetic moment $\boldsymbol{m}$, placed at the origin, and a point-like charge q located at $\boldsymbol{r} = \boldsymbol{a}$. He makes use of an expansion in terms of Legendre polynomials, giving a result which reduces to (13) when $\boldsymbol{m} = m\boldsymbol{k}$ and $\boldsymbol{a} = a\boldsymbol{i}$. To arrive at (12) from (9) we have made use of a way of reasoning like that of Panofsky (1975).

In the preceding calculus, the ring has been considered strictly point-like because of the factor $\boldsymbol{r}$ in expression (5). Otherwise, in order to compute the linear momentum, the exact form of the fields in the neighbourhood of the ring must be taken into account, as has been remarked by Furry (1969).

4. Comments and discussion

One could object that, if there were present only a charge or only a magnetic dipole, the angular momentum of the corresponding electromagnetic field would be zero; and even if both parts were present, it would also be null because of the 'superposition principle'.

However, the superposition principle does not apply here, since Poynting's vector and the electromagnetic energy–momentum tensor are not linear in the total electromagnetic field, and thus this principle does not hold for the electromagnetic angular momentum. Another intuitive way of convincing oneself of this and of the result (4) is to follow the assembling process of the system, in a similar way to that developed by Calkin (1966) for the electromagnetic linear momentum.

In fact, let us assume we have initially the ring and charge infinitely far apart, so that no angular momentum exists, and we move the charge from infinity to the disc edge at a constant speed along the $0X$ axis. The charge creates a magnetic field which is null on the $0X$ axis, and in particular at the point where the ring is, so that no torque is acting on it. However, the charge at the point x will be affected by a magnetic force according to

$$\boldsymbol{F}_{\rm m} = q(\boldsymbol{v}\times\boldsymbol{B}) = \frac{\mu_0 qmv}{4\pi x^3}\,\boldsymbol{j}. \qquad (15)$$

It is interesting to observe that in the charge–ring interaction Newton's third law does not hold.

In order to move the charge at a constant speed, some external force must be applied on it, $\boldsymbol{F}_{\rm ext} = -\boldsymbol{F}_{\rm m}$, and its torque with respect to the origin will be

$$\boldsymbol{M}_{\rm ext} = \boldsymbol{r}\times\boldsymbol{F}_{\rm ext} = -\frac{\mu_0 qmv}{4\pi x^2}\,\boldsymbol{k} \qquad (16)$$

since $\boldsymbol{r} = x\boldsymbol{i}$.

When the charge has reached the point $a\boldsymbol{i}$, according to the principle that the external torque is equal to the time derivative of the total angular momentum, there is stored in the electromagnetic field an angular momentum

$$\boldsymbol{L} = \int \boldsymbol{M}_{\rm ext}\,\mathrm{d}t = -\frac{\mu_0 qm}{4\pi}\int_{-\infty}^{a}\frac{\mathrm{d}x}{x^2}\,\boldsymbol{k} = \frac{\mu_0 qm}{4\pi a}\,\boldsymbol{k} \qquad (17)$$

which coincides with (13).

It should be remarked that (17) is independent of v, justifying the assumption that the charge stops when it arrives at the disc.

Finally, the result (17) is independent of the way of assembling the system, if radiation is neglected, as can be shown by means of the formalism of conservation laws expressed in terms of the energy–momentum tensor.

References

Calkin M G 1966 *Am. J. Phys.* **34** 921

Corinaldesi E 1980 *Am. J. Phys.* **48** 83

Feynman R P 1964 *The Feynman Lectures on Physics* vol II (Reading, Mass.: Addison-Wesley) §17.4

Furry W H 1969 *Am. J. Phys.* **37** 621

Graham G M and Lahoz D G 1980 *Nature* **285** 154

Møller C 1972 *The Theory of Relativity* 2nd edn (Oxford: Clarendon) ch 5

Panofsky W H and Phillips M 1975 *Classical Electricity and Magnetism* 2nd edn (Reading, Mass.: Addison-Wesley) pp 133, 97

Pugh E M and Pugh G E 1967 *Am. J. Phys.* **35** 153

Romer R H 1966 *Am. J. Phys.* **34** 772

—— 1967 *Am. J. Phys.* **35** 445

Stedman G E 1981 *Phys. Lett.* **81A** 15