Visualization of Objects Traveling with Relativistic Velocities

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Abstract: When first introducing students to special relativity, students are often misleadingly told that Lorentz contraction "describes the appearance" of rapidly moving objects. This paper discusses the difference between an object's physical location and its apparent location. Transformations which incorporate effects such as aberration of light and Lorentz contraction are formulated and derived to illustrate how these phenomena affect an object's appearance. The basic structure of a C program to visualize these transformations is also briefly described and several interesting results are discussed. These results should be particularly interesting for those unfamiliar with the nature of light aberration.

1 Introduction, Overview and Motivation

Measurement and observation have played a large role in the evolution of physics. Ever since Galileo measured the acceleration of balls as they rolled down an inclined plane, physicists have utilized a gamut of devices to measure and record the behavior of the world around them. From these observations, physicists have been able to deduce patterns and regularities in nature, and with the help of the scientific method some of these patterns have become "physical laws." One of the fundamental building blocks of the scientific method is the notion that data recorded by one person can be systematically transformed to agree with data of the same event recorded by another person.[1] This transformation between two *reference frames* is known in physics as a *change of coordinates.*[2]

[1] Without a *systematic* way of transforming measurements from one coordinate system to another it would not make sense to talk about a "*universal* law of physics." The notion of transforming data is essential in the success of the scientific method.

[2] Usually, it is *inertial* reference frames which are used in a change of coordinates.

The most basic change of coordinates transforms the location and time of an event recorded by one observer into the location and time of the same event as recorded by a second observer moving at a constant velocity with respect to the first observer. Up until the start of the 20th century, the change of coordinates that governed these type of situations was thought to be a very basic and intuitive transformation known as the Galilean transformation[3]. However, Albert Einstein's Theory of Special Relativity,[4] introduced in 1905, challenged this belief in a simple change of coordinates. The basis for this theory was the following two postulates:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light in a vacuum has the same value, c, in all inertial reference frames.

With these two simple postulates, Einstein argued that the classical relationship between space and time, the relationship which is expressed in the Galilean Transformation, is incorrect. In place of the Galilean Transformation, Einstein used a less intuitive transformation in which time and space become intermixed when one changes coordinate systems. This transformation, known as the Lorentz transformation[5], was derived by Henry Lorentz in order to preserve the form of the wave equation and with Einstein's interpretation it has become accepted as the correct transformation in a universe without gravitational forces.

Most students first learn about the Lorentz transformation in a modern physics course. In this first introduction, students often study the non-intuitive phenomena, such as length contraction and time dilation, which arise in this transformation. The classic example used when discussing length contraction is a meter stick traveling past an observer at some relativistic velocity.[6] Unfortunately, in such a discussion one often refers to the "appearance" of the contracted meter stick and students can easily be misled into thinking that the Lorentz contraction actually describes how an object moving at a constant relativistic speed appears to a fixed observer. However, neither the Lorentz contraction nor the Lorentz transformation alone is enough to describe the *appearance* of events or objects in an inertial reference frame.[7]

[3]See section 3.1 for the explicit form of the Galilean transformation.

[4]Albert Einstein, Zur Elektrodynamik bewegter Körper, Annalen der Physik, 17, 1905.

[5]See section 3.3 for the explicit form of the general Lorentz transformation.

[6]Relativistic velocities are those velocities which are comparable in magnitude to the speed of light. For velocities significantly smaller in magnitude than the speed of light the Galilean Transformation serves as good approximation for the Lorentz Transformation.

[7] See for example G. D. Scott and M. R. Viner (The Geometrical Appearance of Large Objects moving at Relativistic Speeds, *Am. J. Physics.* , vol. 33 p. 534 (2), 1965) and the references contained therein.

Describing the location and time of an event is not the same as describing the appearance of the event.[8] The appearance of an event is determined by which rays of light from this event arrive at the point of observation when the observation is made. The Lorentz transformation only describes how the spacetime coordinates transform between inertial reference frames. In this paper, points described in this manner are referred to as **physical points** while points which describe the *observed* location of objects and events in the fixed frame are called **apparent points**.

One way to think of the physical points of an object is to to imagine viewing the object with a special type of light that has an infinite speed.[9] In this case, there is no difference between the physical points of the object and its apparent points. To get a feeling for the *apparent points* of an object one needs to consider the effects of light aberration[10] which, unlike the relativistic effect of length contraction, tends to stretch out the an object's appearance. The next section includes a discussion of light aberration. In section 3, several transformations are derived to help illustrate how effects such as Lorentz contraction and aberration affect the apparent position of a point. Section 4 contains an outline for a C program which implements these transformations and calculates the apparent image of objects moving at a relativistic speed. Section 5 presents several images generated by the program and concluding remarks are given in section 6.

2 The Phenomenon of Light Aberration

Light propagates at a finite speed in both relativistic and (most) classical models of physics. This fact leads to a distortion of a moving object's image known as light aberration. In everyday experience, most objects people observe travel at speeds so miniscule in comparison to that of light that the effects of aberration are never noticed. However, one can gain an understanding of the nature of aberration by performing a thought experiment in which a high speed camera is set up to take a picture of an object passing by at a relativistic speed. One can determine the photograph's image by determining all of the light rays that arrive

[8] Up until 1959 most physicists incorrectly believed that the Lorentz contraction could *in general* be observed and even photographed (reference 7).

[9] Of course, if one imagines the speed of light to be infinite then the Lorentz transformation, in the limit, is the same as the Galilean transformation.

[10] Ironically, even though aberration also occurs in classical physics, it is often not understood as well as counter-intuitive non-classical phenomena such as length contraction and time dilation.

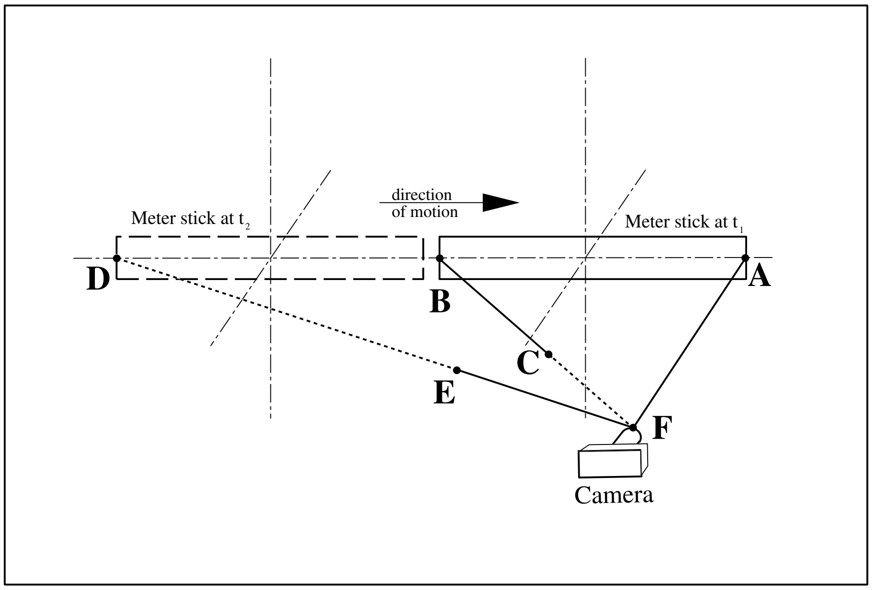

Figure 1: Aberration of a Meter Stick with Orientation Parallel to Direction of Motion

at the camera the instant a picture is taken. Let us perform this thought experiment now by considering a camera photographing a fast moving meter stick.

First, let the meter stick fly by with an orientation parallel to the direction of motion (see figure 1). Assuming the meter stick emits light continuously, all its points will be visible at the camera when the shutter clicks (event F). For example, at some time t_1, a photon will be emitted from the *front* of the meter stick and will travel toward the camera, arriving at the exact instant of the shutter click F. Call the emission of this photon event A. In a similar manner, at some time before the shutter click F, a photon will be emitted from the *back* of the meter stick such that it travels toward the camera, also arriving just in time to get caught on film. One's intuition might say that this photon would also have to be emitted at t_1. Event B denotes the emission of such a photon destined for the camera. But notice that the distance the photon from A had to travel is shorter than the distance the photon emitted at B would have to travel to reach the camera. If a photon from B arrived at the camera at the same instant as a photon from A then the B photon would have had to travel faster than the A photon. However, even classically, photons emitted from the same object could not travel with the necessary speeds. Thus, when the photon from A arrives at F, the photon from B will only have traveled part of the distance to the camera. This photon will be visible at C, but will remain invisible to the camera. In order for a photon from the back end of the meter stick to arrive in time it will have to leave the meter stick *before* the photon

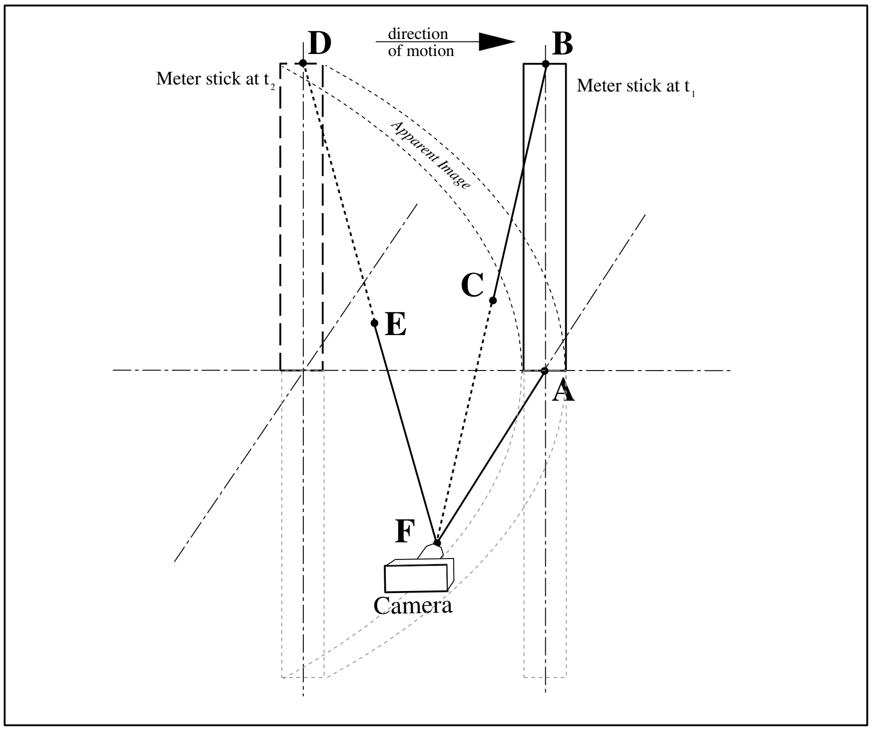

Figure 2: Aberration of a Meter Stick with Orientation Perpendicular to Direction of Motion

from A. This will allow the photon to *race ahead* [11], of the meter stick, thus making up for the additional distance it must travel. Such a photon would be emitted at some time $t_2 < t_1$. In figure 1 the emission of this photon is labeled as event D. Event E shows the position of the photon at t_1 when the photon for A is emitted. Note that since D occurs before A, the light observed from the end of the meter stick will be *older* than the light observed from the front of the meter stick. The image of this rapidly moving meter stick will thus contain light of mixed ages with the older light illuminating the stretched and backwardly distorted parts of the image. The apparent image, recorded by our camera, will be a meter stick stretched from the location of A to the location of D.

Now, before packing up our camera, let us take one more picture with the meter stick rotated such that its orientation is perpendicular to the direction of motion (see figure 2). The events and arguments for arrival times proceed exactly as above. But note how this

[11] This assumes the object is traveling at less than the speed of light. Using classical physics one could consider the appearance of objects traveling faster than the speed of light. However, this type of phenomena is not realizable and since it does not enhance one's understanding of the final "correct" transformation (the Lorentz transformation with aberration effects) it is not considered in this paper.

image demonstrates the non-linear nature of aberration. In figure 2, the "front" and "back" points on meter stick are replaced by a point at eye level and a second point directly above the first. One can see from this figure that the straight edge of the meter stick will appear to bend in the direction opposite the direction of motion. Notice that by a symmetry argument points below eye level will also appear to bend away. In general, the appearance of an object will have all distant points warped or "pushed back" against its direction of motion in a non-linear fashion with distant points getting "pushed back" farther than nearby points.

To further develop the notion of aberration, and to determine how the stretching of aberration and the compression of the Lorentz contraction interact, let us move to a more mathematical treatment.

3 Derivation of Point Transformation

In this section, the equations used to describe the apparent positions for arbitrary points of an inertial reference frame are derived. When considering how to obtain the apparent position of a point in an inertial reference frame, there are four different models one can use. Two models follow directly from the Galilean or Lorentz transformations and the other two models take these transformations and include aberration[12] effects. Of these four models, the Lorentz transformation with aberration is the only transformation which produces an image that would correspond to the "real" appearance of an object moving at a relativistic velocity. However, the other three transformations are instructive insofar as they provide a perspective to help understand the various effects that contribute to the overall observed appearance as described by the Lorentz transformation with aberration. The following notation is used in all four of the derivations:

- Let $\mathcal{O}$ be the coordinate system of the viewing reference frame. This coordinate system remains fixed with respect to the observer. Any physical points which are described by this coordinate system are done so using unprimed 3-tuples such as (x, y, z). Also,

[12]In the Lorentz Transformation with aberration there is only one model. However, in the classical realm, when considering aberration one can use several different models such as a particle-light model or an aether-light model. For the purpose of illustrating aberration in a manner similar to that in the Lorentz Transformation with aberration, the classical model of aberration used in this paper is the aether-light model in the special case where the observer is in the stationary frame of the aether. Thus, in this model light emitted from a point will travel toward the observer at the speed of light (c) regardless of the velocity of the object.

whenever reference to the time of some event observed in this reference frame is made an unprimed time variable such as t is used.

- Let $\mathcal{O}'$ be an inertial reference frame moving with respect to viewing reference frame. This inertial reference frame corresponds to a coordinate system in which the object described remains stationary. Let $\mathbf{v} = (v_x, v_y, v_z) \neq (0,0,0)$ be the velocity of the origin of this reference frame with respect to the origin of the fixed frame $\mathcal{O}$.[13] To distinguish between physical points described by the $\mathcal{O}$ reference frame and physical points described by the $\mathcal{O}'$ coordinate system, primed 3-tuples such as (x', y', z') will be used to denote those physical points described in the $\mathcal{O}'$ reference frame. In the same manner, a primed time variable such as t' will be used to denote any time measured in the $\mathcal{O}'$ frame.

- To relate the $\mathcal{O}$ and $\mathcal{O}'$ reference frames, let the origin of $\mathcal{O}'$ be located at (x_0, y_0, z_0) in $\mathcal{O}$ at $t' = t = 0$.

- The apparent locations of all points are calculated so that light emitted from these apparent points will all arrive at the origin of the fixed frame $\mathcal{O}$ at the same time. Let T denote this arrival time.

Using the above notation we now proceed with the derivation of the four equations.

3.1 Galilean Transformation

In the first model, the transformation of the object from its fixed frame to the fixed frame of observation is done using a Galilean transformation. At time t, a point located in an object's reference frame at

$$(x', y', z')$$

will appear in the fixed frame at

$$(x(t), y(t), z(t)) = (x', y', z') + (x_0, y_0, z_0) + t \cdot (v_x, v_y, v_z)$$

In this model the effects of aberration are ignored so the above physical position is the same as the apparent position of the point. Thus, the transformation which yields an apparent

[13] The requirement that $\mathbf{v} \neq (0,0,0)$ is not strictly necessary but it allows some nice simplification of results. Furthermore, note that stationary objects experience no aberration and thus are already completely described in the $\mathcal{O}$ coordinate system as both physical objects and apparent objects.

position of points viewed from the origin of the $\mathcal{O}$ fixed frame at time T is simply

$$(x(T), y(T), z(T)) = ((x' + x_0 + T \cdot v_x), (y' + y_0 + T \cdot v_y), (z' + z_0 + T \cdot v_z))$$

3.2 Galilean Transformation with Aberration Effects

As in the previous section, the transformation of physical points from the object's fixed frame, $\mathcal{O}'$, to the fixed frame of the observation, $\mathcal{O}$, is done with the Galilean transformation. Thus at time t a point located in the object's reference frame, $\mathcal{O}'$, at

$$(x', y', z')$$

will have the following physical location in $\mathcal{O}$

$$(x(t), y(t), z(t)) = ((x' + x_0 + t \cdot v_x), (y' + y_0 + t \cdot v_y), (z' + z_0 + t \cdot v_z))$$

in $\mathcal{O}$. In this model, the apparent location of a point is determined with aberration effects. Furthermore, to obtain aberration effects similar to those which will occur in the Lorentz Transformation with aberration we used an aether model of light in which the observer is in the rest frame of the aether. In this model, light from the object will travel from the physical point to the observer at the speed of light (c) regardless of the objects velocity.

The appropriate time, t_E, for light to be emitted from the point so that it will arrive at the origin of $\mathcal{O}$ at $t_A = T$ needs to be determined. The light emitted from the physical point, $(x(t), y(t), z(t))$, is a distance of $\sqrt{x(t)^2 + y(t)^2 + z(t)^2}$ from the point of observation. In the aether fixed frame model, by dividing this by the speed of light (c) yields the transit time. Adding transit time to the emission time gives the time at which the emitted light will arrive at the origin

$$t_A = t_E + \frac{\sqrt{x(t_E)^2 + y(t_E)^2 + z(t_E)^2}}{c}$$

Setting $t_A = T$ and solving the resulting equation for t_E gives:

$$t_E = \frac{B - \sqrt{B^2 - AC}}{A}$$

where

$$\begin{aligned} A &= c^2 - v_x^2 - v_y^2 - v_z^2 \\ B &= T \cdot c^2 + v_x(x_0 + x') + v_y(y_0 + y') + v_z(z_0 + z') \\ C &= T^2 \cdot c^2 - (x_0 + x')^2 - (y_0 + y')^2 - (z_0 + z')^2 \end{aligned}$$

Thus, the apparent location of the point at time T at the origin of $\mathcal{O}$ is given by

$$(x, y, z) = ((x' + x_0 + t_E \cdot v_x), (y' + y_0 + t_E \cdot v_y), (z' + z_0 + t_E \cdot v_z))$$

3.3 Lorentz Transformation

In this model, the transformation of the object from its fixed frame to the fixed frame of observation is done using a Lorentz transformation. The general form of a Lorentz transformation can be expressed as a linear transformation from one set of spacetime coordinates to another. Let $\gamma_v = \frac{1}{\sqrt{1-(v_x^2+v_y^2+v_z^2)/c^2}}$. Then the general Lorentz transformation takes the form[14]

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma_v & \frac{\gamma_v v_x}{c^2} & \frac{\gamma_v v_y}{c^2} & \frac{\gamma_v v_z}{c^2} \\ \gamma_v v_x & 1+\frac{\gamma_v^2 v_x^2}{(1+\gamma_v)c^2} & \frac{\gamma_v^2 v_x v_y}{(1+\gamma_v)c^2} & \frac{\gamma_v^2 v_x v_z}{(1+\gamma_v)c^2} \\ \gamma_v v_y & \frac{\gamma_v^2 v_y v_x}{(1+\gamma_v)c^2} & 1+\frac{\gamma_v^2 v_y^2}{(1+\gamma_v)c^2} & \frac{\gamma_v^2 v_y v_z}{(1+\gamma_v)c^2} \\ \gamma_v v_z & \frac{\gamma_v^2 v_z v_x}{(1+\gamma_v)c^2} & \frac{\gamma_v^2 v_z v_y}{(1+\gamma_v)c^2} & 1+\frac{\gamma_v^2 v_z^2}{(1+\gamma_v)c^2} \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

Applying this general form to the notation of our model, a point located in an object's reference frame $\mathcal{O}'$, at

$$(x', y', z')$$

will transform into the fixed frame point

$$(x(t), y(t), z(t)) = ((X(t') + x_0), (Y(t') + y_0), (Z(t') + z_0))$$

where

$$t = t'\gamma_v + x'\frac{\gamma_v v_x}{c^2} + y'\frac{\gamma_v v_y}{c^2} + z'\frac{\gamma_v v_z}{c^2}$$

and

$$\begin{aligned} X(t') &= x' + \gamma_v v_x \left(t' + \tfrac{\gamma_v}{(1+\gamma_v)c^2}(v_x x' + v_y y' + v_z z')\right) \\ Y(t') &= y' + \gamma_v v_y \left(t' + \tfrac{\gamma_v}{(1+\gamma_v)c^2}(v_x x' + v_y y' + v_z z')\right) \\ Z(t') &= z' + \gamma_v v_z \left(t' + \tfrac{\gamma_v}{(1+\gamma_v)c^2}(v_x x' + v_y y' + v_z z')\right) \end{aligned}$$

Solving X,Y, and Z in terms of t and simplifying gives

$$\begin{aligned} X(t) &= x' + v_x \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right) \\ Y(t) &= y' + v_y \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right) \\ Z(t) &= z' + v_z \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right) \end{aligned}$$

Once again, in this model the effects of aberration are ignored so the above physical position is the same as the apparent position of the point. Thus, the transformation which yields the

[14]Ungar, Abraham A., The abstract Lorentz transformation group, Am. J. Phys. **60** (9), September 1992, p.821.

apparent position of points viewed from the origin of the $\mathcal{O}$ fixed frame at time T is

$$
\begin{aligned}
x &= x_0 + x' + v_x \left(T - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)\\
y &= y_0 + y' + v_y \left(T - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)\\
z &= z_0 + z' + v_z \left(T - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)
\end{aligned}
$$

3.4 Lorentz Transformation with Aberration Effects

The final model considered uses the Lorentz transformation but also incorporates light aberration. The transformation of the physical point from $\mathcal{O}'$ to $\mathcal{O}$ is first done using the Lorentz transformation so that the point

$$(x', y', z')$$

is transformed into the physical point

$$(x(t), y(t), z(t))$$

where

$$
\begin{aligned}
x(t) &= x_0 + x' + v_x \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)\\
y(t) &= y_0 + y' + v_y \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)\\
z(t) &= z_0 + z' + v_z \left(t - \tfrac{1}{v_x^2+v_y^2+v_z^2}\left(\tfrac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')\right)
\end{aligned}
$$

As with the previous aberration model, the emission time t_E, needs to be determined such that light emitted from the physical point at this time will arrive at the origin of $\mathcal{O}$ at $t_A = T$. This gives rise to the equation

$$T = t_A = t_E + \frac{\sqrt{x(t_E)^2 + y(t_E)^2 + z(t_E)^2}}{c}$$

Plugging in the expressions for $x(t)$, $y(t)$, $z(t)$ and solving yields

$$t_E = \frac{B - \sqrt{B^2 - AC}}{A}$$

where

$$
\begin{aligned}
A &= c^2 - v_x^2 - v_y^2 - v_z^2\\
B &= T \cdot c^2 + \tfrac{v_x(x_0+x') + v_y(y_0+y') + v_z(z_0+z')}{\gamma_v}\\
C &= T^2 \cdot c^2 - (x_0 + x')^2 - (y_0 + y')^2 - (z_0 + z')^2 + \tfrac{(v_x x' + v_y y' + v_z z')^2}{c^2}
\end{aligned}
$$

Finally, letting

$$t_V = t_E + \frac{1}{v_x^2 + v_y^2 + v_z^2}\left(\frac{1}{\gamma_v} - 1\right)(v_x x' + v_y y' + v_z z')$$

the apparent location of the point at time T at the origin of $\mathcal{O}$ is given by

$$(x, y, z) = ((x' + x_0 + t_V \cdot v_x), (y' + y_0 + t_V \cdot v_y), (z' + z_0 + t_V \cdot v_z))$$

A derivation of this transformation for the special case of motion in the ẑ direction can be found in reference 7.

4 Outline of Program Operation

Using the transformations derived in the previous section and the graphical capabilities of modern computers, a program can be written to take the description of an object, calculate the effects of a transformation and display the resulting image. The following is a brief outline of such a program written in C using the OpenGL libraries.[15] The C code of this program was written with all device dependent procedures collected into separate modules, allowing one to create suitable replacements with avaliable graphical routines and recompile the program to work on any desired platform. Figure 3 is a flow chart that shows the various modules and their interactions. Modules which call functions defined elsewhere have arrows pointing to where these functions are defined.

Execution of the program begins at the `main` function found in the **application module**. It is here that general platform-independent types of initialization are done. Execution is transfered to the **viewer module** via the `init_viewer` initialization function. This is where device dependent initialization of graphic windows and other code specifically used in setting up the viewing window is located.

In the OpenGL libraries, several functions to easily handle program execution and interrupt processing are avaliable. One such function is `auxMainLoop`, which is used to initialize any function as the core loop. OpenGL then runs this function, branching off to handle any interrupt events when they occur. In the particular program used to generate images for this paper, the `auxMainLoop` was used to initialize the function `display` as the core loop. The function `display` is contained in the viewer module and is described below. OpenGL also allows interrupt events to be generated by a key press. This allows one to effectively map

[15]The OpenGL libraries are based on the graphical routines found on Silicon Graphics computers.

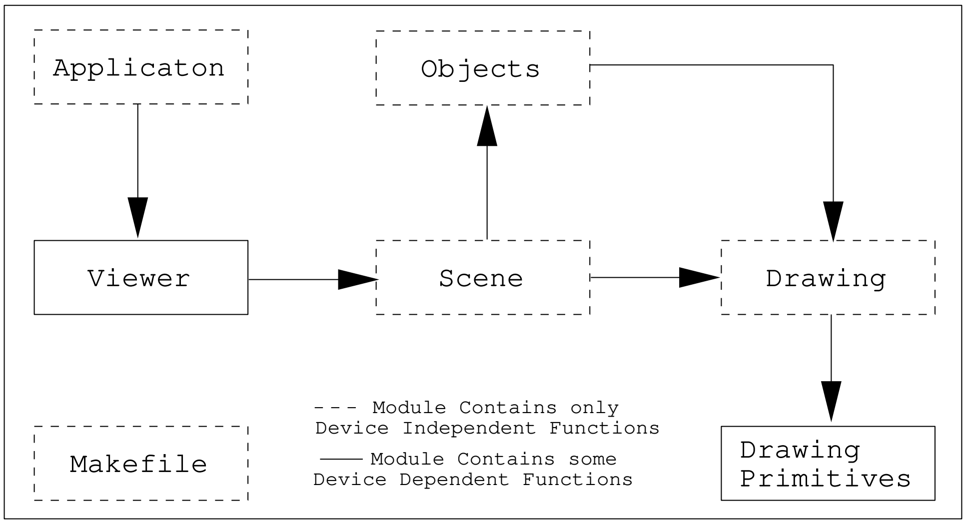

Figure 3: Program Flow Chart

any given key directly to a function so that by pressing it the core loop is interrupted and the appropriate function is executed. In this manner, the viewer module of the program mapped the arrow keys to the functions `panUp`, `panDown`, `panLeft` and `panRight` which increment or decrement local polar and azimuthal variables. By altering these variables, the user can center the camera on any desired location. The user can also increment and decrement the viewing time by pressing the t and r keys which were mapped to `fwdTime` and `revTime` respectively. All of these direct key functions were defined within the viewer module.

The function `display`, defined and used in the viewer module, sets up the viewer to draw the transformed objects. It does this by first rotating the viewing coordinate system into the standard fixed coordinate system of $\mathcal{O}$, using the internal polar and azimuthal variables. Then `display` draws a green 10 meter square located 2 meters below the point of observation. This square is in the x-y plane and serves to give a "ground" orientation. Additionally two posts are placed in the corners of the square. A blue post is located to the left and originates from the x-y plane at $(5, 5)$ and a red post is located to the right at $(5, -5)$ in the x-z plane. Both posts are 5 meters tall. After providing these landmarks for orientation `display` transfers control to the **scene module** via the function `drawScene(view_time)`.

When `drawScene(view_time)` is called, any objects that were specified in this function when the application was compiled are drawn.[16] Specification of objects was done by using

[16]This is not the ideal data structure for storing object information. However, the inconvenience of hard coding objects and recompiling the application every time a change is made is offset by the simplification

either "object" functions defined in the **objects module** (such as `drawCircle` or `drawGrid`), or the `drawLine` function found in the **drawing module**. The *appearance* of the object is specified by using the functions `setEffectFlag`, `setOrigin`, `setVelocity`, `setColor` and `setTime`. These functions determine which transformation to use, where to set the origin of the $\mathcal{O}'$ coordinate system, what relative velocity to use between the $\mathcal{O}'$ and $\mathcal{O}$ coordinate systems, and finally, what color to use when drawing the lines of the object. To give a clearer understanding of how `drawScene` is used a sample of the C code used to define a simple scene is given at the end of this section.

The **drawing module** contains the most essential aspects of the program. It is here that the transformations for an object's points are calculated. It is also here that all of the primitive line drawing commands are generated. The module has several external functions which allow the internal variables and flags to be set. These internal variables and flags in turn determine the viewing transformation, the origin of the $\mathcal{O}'$ coordinate system, the relative velocity of the object, the color used to draw the lines of the object and the arrival time to use in the viewing transformation. The drawing module also contains the `drawLine` function from which all objects are built. This function determines the curvature of each line by calling a recursive line-drawing function contained inside the drawing module. This recursive function calculates the endpoints and midpoint of each line and recursively draws lines connecting the midpoint and the two endpoints with the base case of the recursion halting when either a maximum recursion depth is exceeded or a small enough error between the expected and observed warped midpoints is observed. Once the base case is reached, the device dependent line drawing function, located in the **drawing primitives module**, is used.[17] When the program reaches this base case for all of the lines which make up the scene, the entire scene will have been drawn.

Before the results of this program are discussed, let us consider a simple example to demonstrate how a scene is specified. The sample scene which is used is a "meter stick". Figure 4 shows the actual `drawScene` code which describes a red[18] "meter stick" moving to the right at $0.9c$. It is passing one meter in front of the observer and is oriented parallel to the direction of motion. The "meter stick" itself is composed of ten 0.10 meter squares

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that this allows in program complexity. Future versions of this program may incorporate a more flexible data structure.

[17]Since the drawing module contains only device independent functions, the device dependent drawing functions are separated into the drawing primitives module.

[18]Here red is only used as the line color. Doppler shifting effects, which do affect the observed color of an object, are not incorporated into this program.

```
void drawScene(double view_time)   {
   double origin[3], velocity[3], frontLeftLowerCorner1[3], dx_dy_dz[3];
   int size[3];
   /* --------- optional parameters which can be explicitly set -------- */
   setTime(view_time);
   velocity[0] = 0.0;
   velocity[1] = -0.9 * c;
   velocity[2] = 0.0;
   setVelocity(velocity);
   setEffectFlag(2);             /* EFFECTS 0 = stationary frame         */
                                 /*         1 = Galilean, no aberration  */
                                 /*         2 = Galilean with aberration */
                                 /*         3 = Lorentz, no aberration   */
                                 /*         4 = Lorentz with aberration  */
   origin[0] = 1.0;  origin[1] = 0.0;  origin[2] = 0.0;
   setOrigin(origin);
   /* --------- parameters specific to a "meter stick" ----------------- */
   frontLeftLowerCorner1[0] = 0.0;
   frontLeftLowerCorner1[1] = -0.5;
   frontLeftLowerCorner1[2] = -0.05;
   dx_dy_dz[0] = 0.1;
   dx_dy_dz[1] = 0.1;
   dx_dy_dz[2] = 0.1;
   size[0] = 0;
   size[1] = 10;
   size[2] = 1;
   setColor(4);      /* COLORS  0 = black, 1 = white,         */
                     /*         2 = blue, 3 = green, 4 = red  */
   /* ------------------- draw the "meter stick" ---------------------- */
   drawGrid(frontLeftLowerCorner1, dx_dy_dz, size);
}
```

Figure 4: An example of the `drawScene` function used to describe a meter stick moving to the right at $0.9c$. To change the scene, one simply changes the code in this function and recompiles the program.

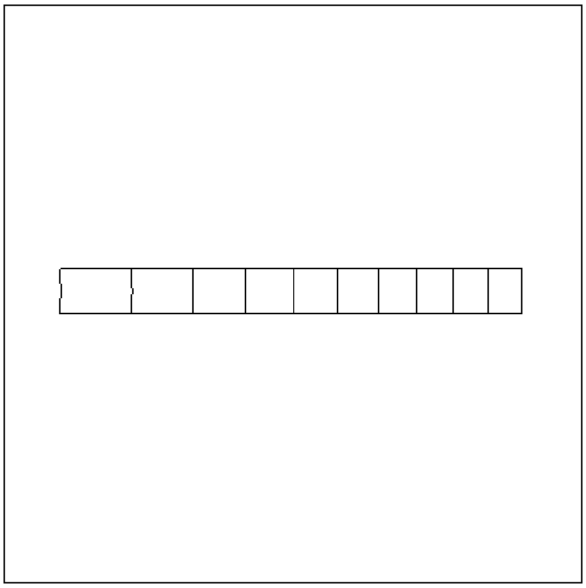

Figure 5: Image of the meter stick described in `drawScene` function of Figure 4.

stacked side by side. The Galilean Transformation with aberration is used as the viewing transformation. The viewing time is given by the variable `view_time`, which (after compiling the program) was incremented[19] to 3.7×10^{-9} seconds and the image of a distorted meter stick was observed (see figure 5).

5 Results

In this section, several images generated by various transformations are explored in an effort to provide the reader with a more intuitive understanding of the nature of light aberration and its interplay with relativistic length contraction.

As a starting point, we return to the examples used in section 2. With the help of the computer, we explore the appearance of a relativistic meter stick. The `drawScene` function shown in figure 4 was used to create several images of a meter stick traveling to the right at $0.8c$ with the transformations derived in section 3. The viewing time was adjusted on each image so that the apparent center of the meter stick was directly ahead of the observer.

[19] Since the physical origin of the meter stick in this example is directly in front of the observer the measured time can be thought of as the number of seconds which have elapsed since the physical center point of the "meter stick" passes directly in front of the observer.

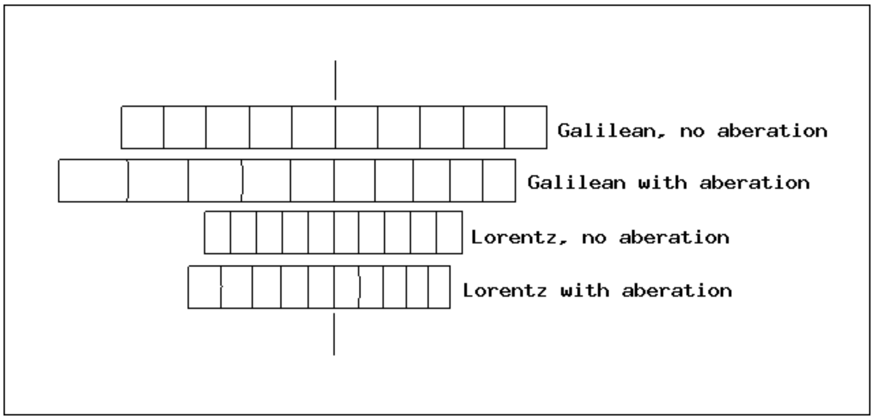

Figure 6: The above images were obtained by considering the appearance of a meter stick traveling to the right at $0.8c$ and passing one meter in front of the observer. For each image the "appearance" is calculated using a different transformation and the observation time of each image is set so that the apparent location of the meter stick's center appears directly ahead.

These separate images were then used to create[20] figure 6. Simultaneously displaying the results of these four transformations, one can see how the inclusion of aberration tends to stretch out the left end of the meter stick (which is approaching the observer) and contract the right end (which is moving away from the observer). Comparing the image obtained from the Galilean transformation with aberration and the image obtained from the Lorentz transformation with aberration, notice how the Lorentz contraction significantly reduces the observed aberration.

Next, consider the appearance of the meter stick as it travels perpendicular to the direction of motion (see figure 7). Notice both the images generated using the Galilean and Lorentz transformations with no aberration appear "normal," while the same transformations with aberration appear "warped," showing the non-linear nature of aberration. Also notice that while the the contraction of the Lorentz transformation can be observed, it does not alter the dimensions of the object enough to affect the overall distance between the observer and the points on the object. Thus, unlike the images in figure 6, the images produced by the Galilean transformation with aberration and the Lorentz transformation with aberration are very similar.

Now, for a more complicated scene, we will parallel an example used in the introductory

[20]This figure was created as a means of verifying that the program contained a correct implementation of the various transformations. The figure duplicates the results obtained in reference 7

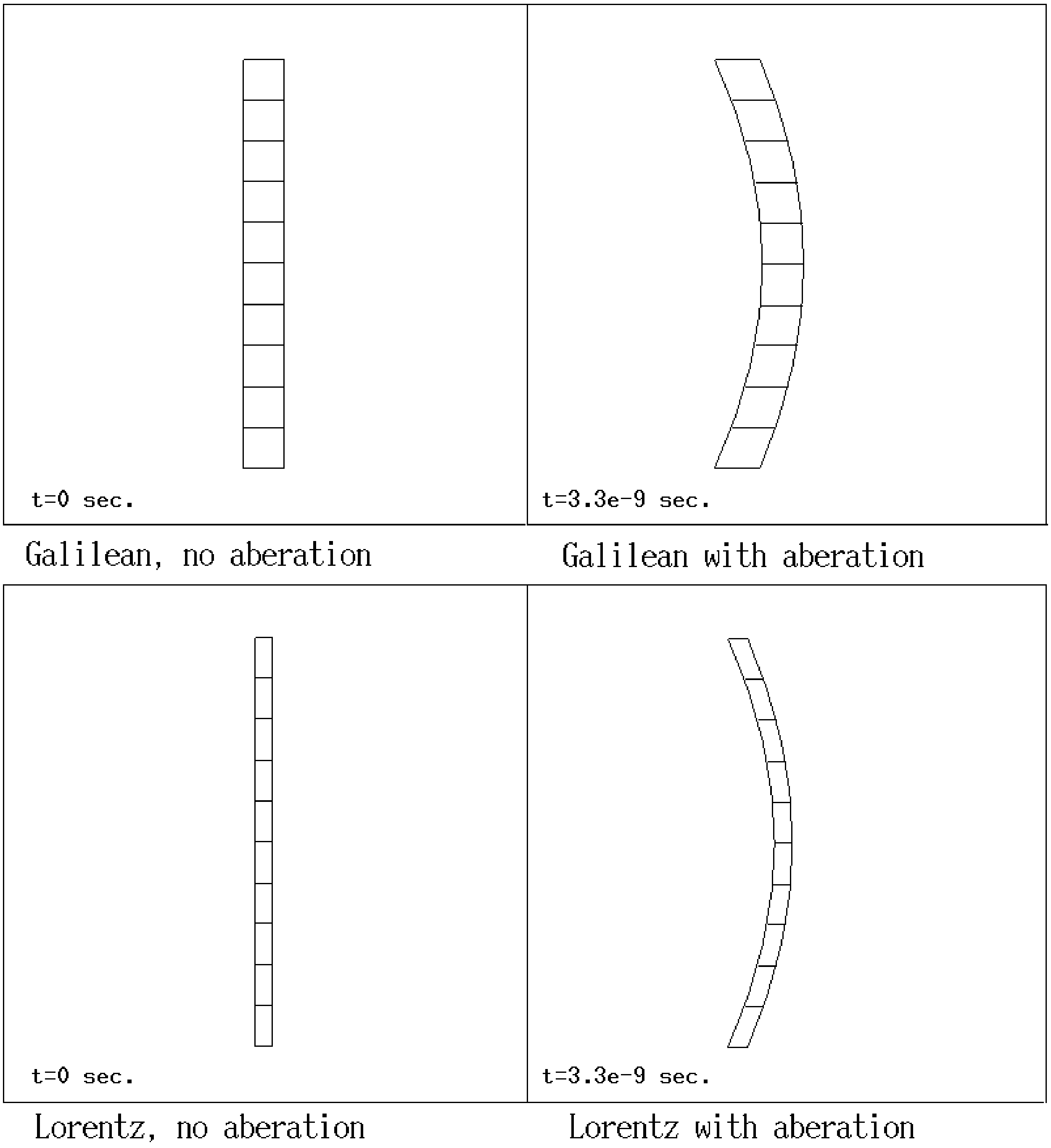

Figure 7: The above images were obtained by considering the appearance of a meter stick traveling to the right at $0.9c$ and passing one meter in front of the observer. For each image the "appearance" is calculated using a different transformation and the observation times are show. The observation times are chosen such that that the apparent position of the center of the meter stick appears directly in front of the observer.

physics video series The Mechanical Universe.[21] In this series, Dr. David L. Goodstein often uses computer animation to help illustrate the physical concepts under consideration. However, in his lecture # 24 entitled "The Lorentz Transformation, " Dr. Goodstein mistakenly uses length contraction *without* aberration to describe how a stationary observer would see a moving observer. In this video, "Albert" (A. Einstein) sees an image of "Henry" (H. A. Lorentz) standing a flatbed train car with meter stick in hand. As Henry passes by, Albert sees everything compressed in the direction of motion by the normal length contraction factor $\sqrt{1-\beta^2}$. To correctly determine what Albert will observe and as another example to provide insight on the aberration of light, let us consider a stick figure standing on a flatbed train car holding a meter stick. Again, we look at the result of all four transformations as the train car passes to the right at $0.8c$.

The top image of figure 8 shows how "Henry" would have appeared to Galileo and other early physicists. However some of these physicists may have realized that this image lacked the necessary consideration of light transit. Perhaps these physicists dreamed of a distorted "Henry" flying by as shown in the bottom image of figure 8. Modern physicists would realize that all of these apparitions rely on the Galilean transformations and would visualize a compressed "Henry" as shown in the top image of figure 9. Those that realize the need to consider aberration would also contemplate distorted images of "Henry" as shown at the bottom of figure 9. Notice that the effects of the Lorentz contraction are significant enough so that the Galilean transformation with aberration (bottom of figure 8) and the Lorentz transformation with aberration (bottom of figure 9) do have different appearances.

In considering the "correct" appearance of this passing train car, one interesting thing to note is the shifting nature of the distortion (see figures 10 and 11). When the observed image is some distance away and moving toward the observer the distortion is rather large (upper left image in figures 10 and 11). However, as the image of the object gets closer and begins to recede from the observer, the image begins to look more and more like a pure Lorentz transformation without aberration (see bottom image in figures 10 and 11).

Our visualization program is in no way limited to viewing two dimensional objects. However, without the interactive computer environment, many of the interesting three dimensional examples cannot be sufficiently captured by static two dimensional black and white images. In spite of this encumbrance, we include a series of examples to demonstrate the appearance of a cubic grid. To generate the cube we simply modified the the code shown in

[21] P. Buffa, The Mechanical Universe and Beyond, Intellimation, Santa Barbara, CA, 1985.

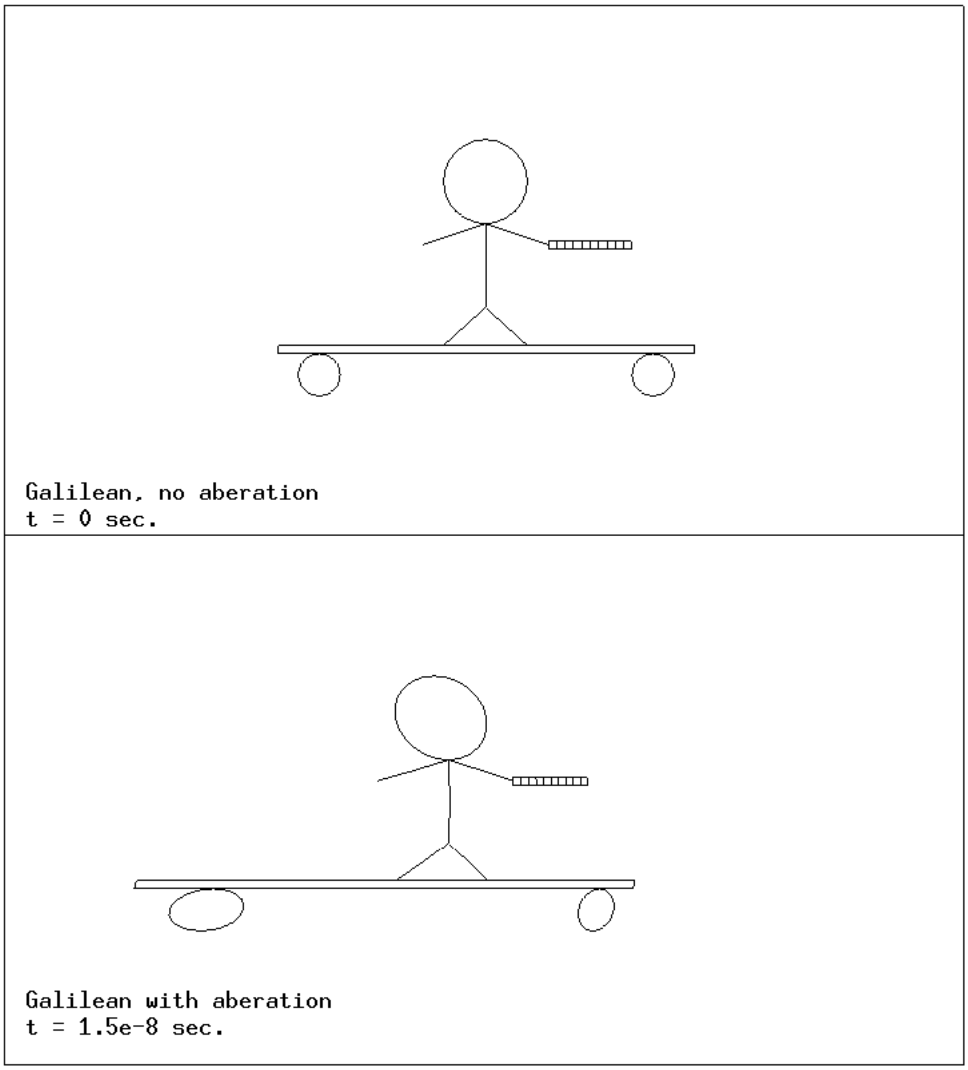

Figure 8: The above images show a stick figure on a train car holding a meter stick while traveling at $0.8c$ to the right. The viewing transformation for the above image is the Galilean Transformation while the lower image is generated using the Galilean Transformation with aberration. The physical center point of the train is 5 meters in front of the observer at $t = 0$.

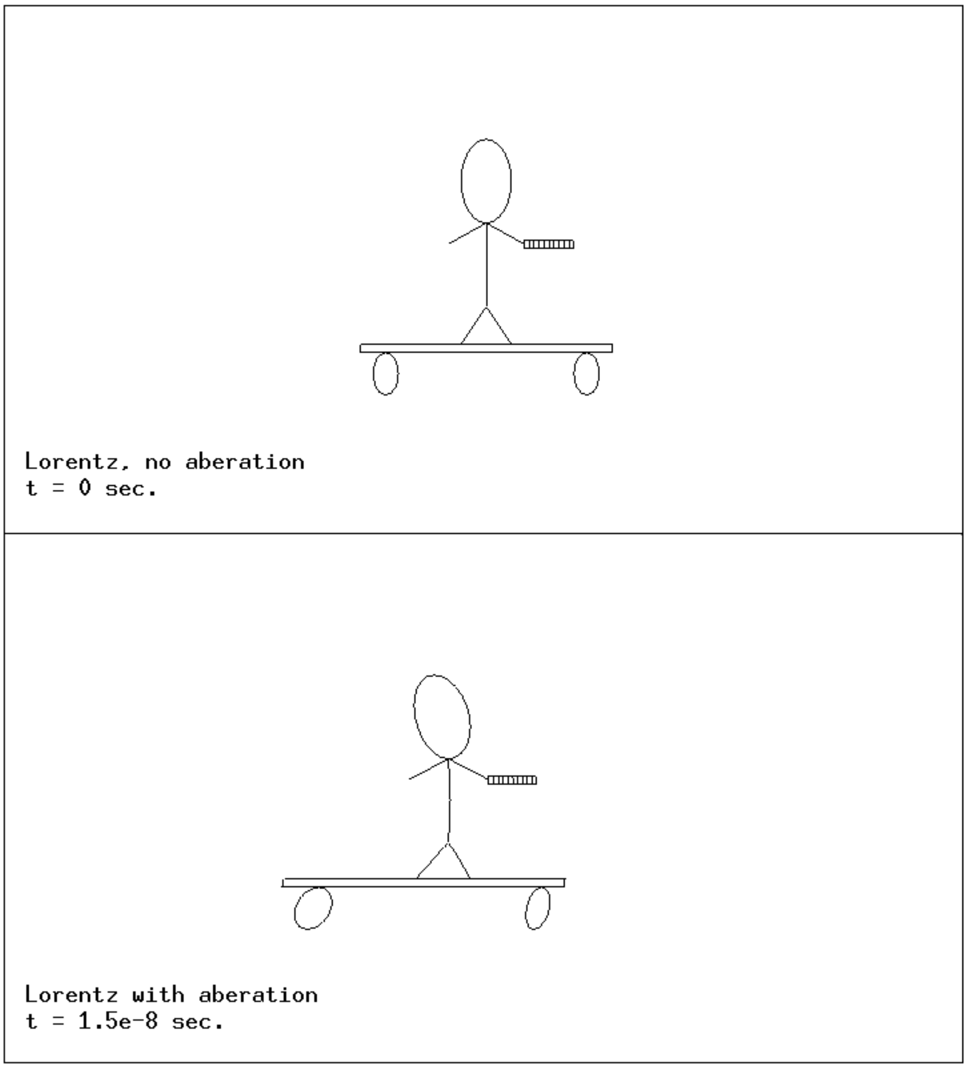

Figure 9: The above images show a stick figure on a train car holding a meter stick while traveling at $0.8c$ to the right. The viewing transformation for the above image is the Lorentz Transformation while the lower image is generated using the Lorentz Transformation with aberration. The physical center point of the train is 5 meters in front of the observer at $t = 0$.

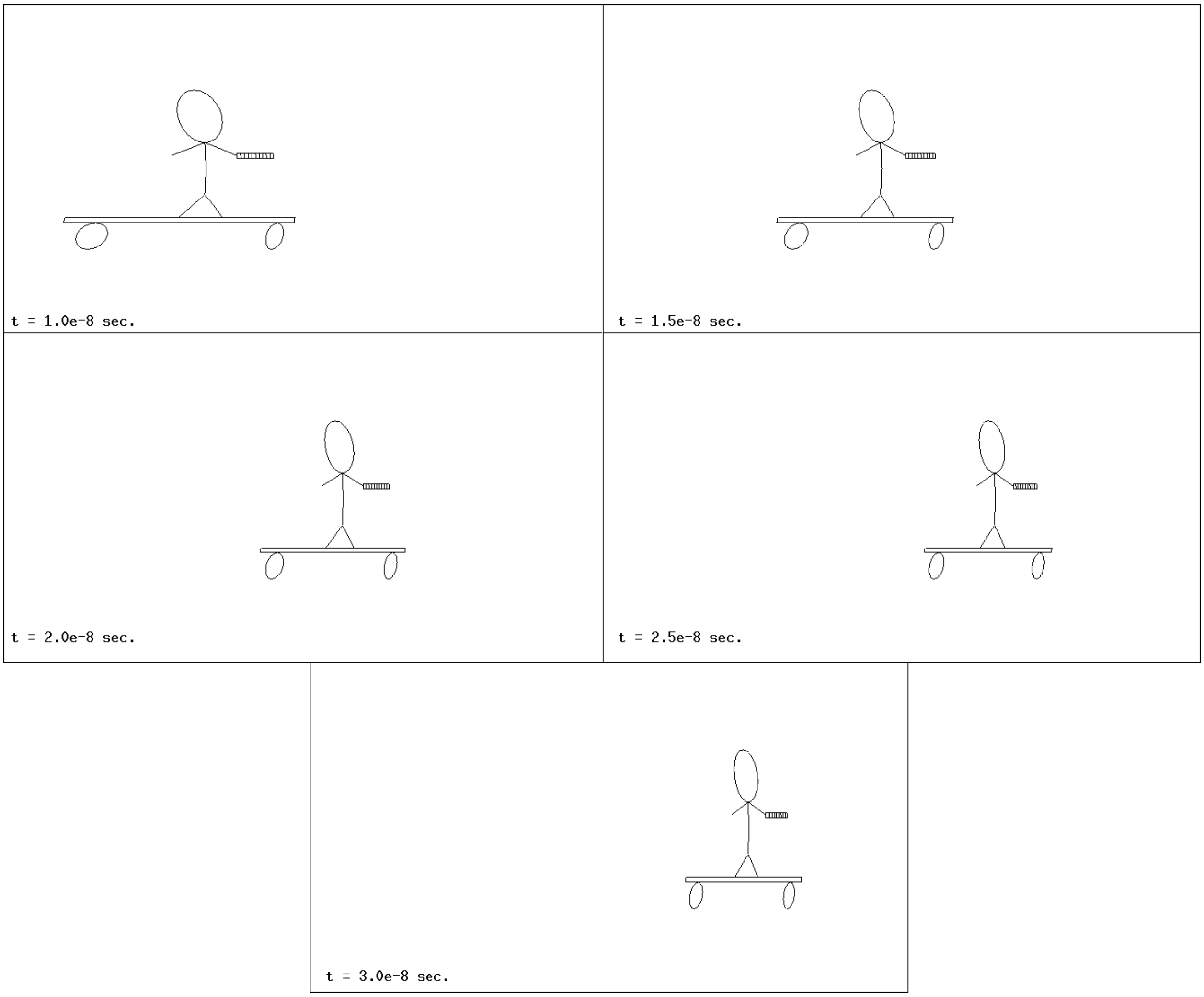

Figure 10: The above images show a stick figure on a train car holding a meter stick while traveling at $0.8c$ to the right. The viewing transformation for all images was the Lorentz Transformation with aberration, and the physical center point of the train was 5 meters in front of the observer at $t = 0$.

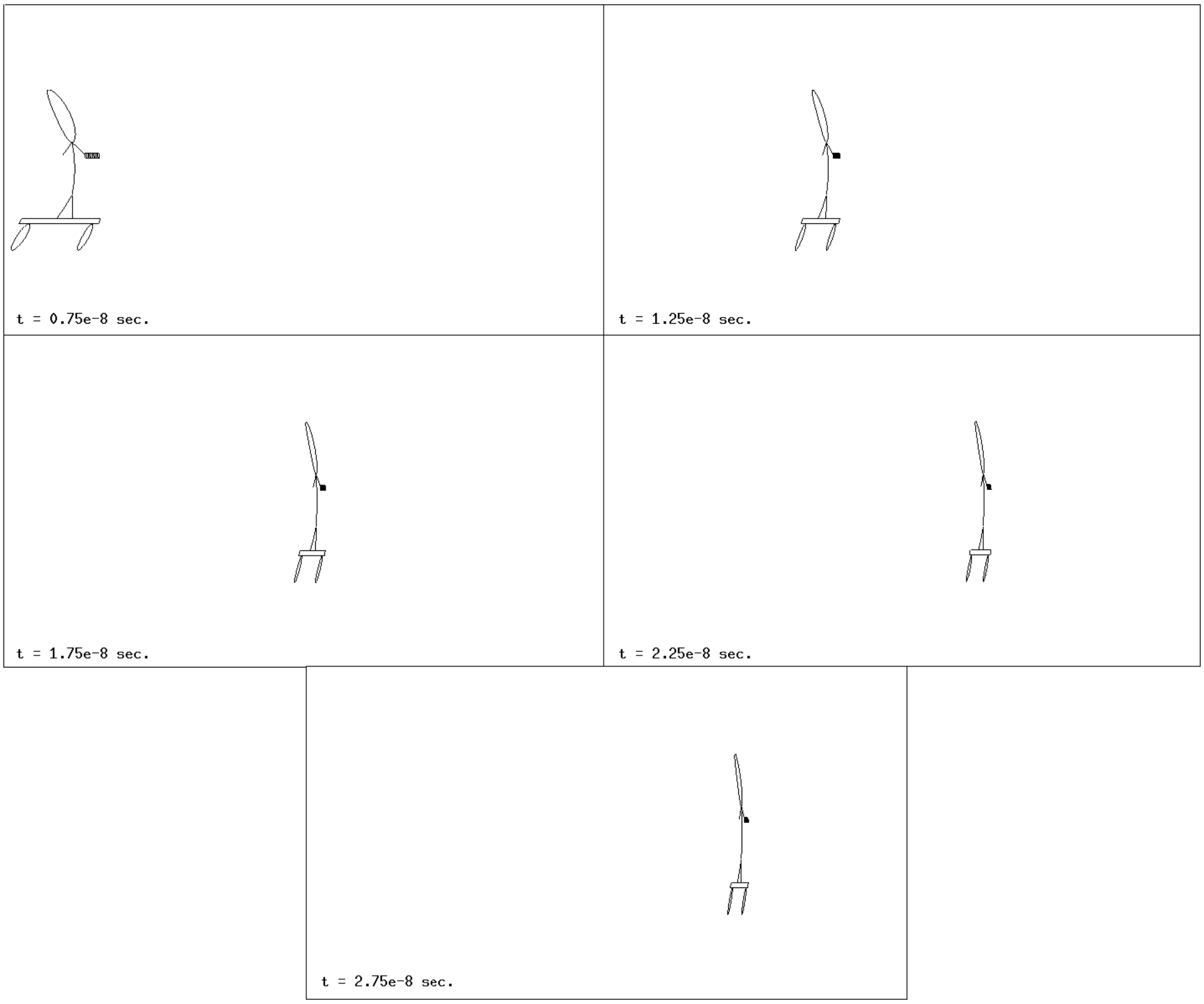

Figure 11: The above images show a stick figure on a train car holding a meter stick while traveling at $0.995c$ to the right. The viewing transformation for all images was the Lorentz Transformation with aberration, and the physical center point of the train was 5 meters in front of the observer at $t = 0$.

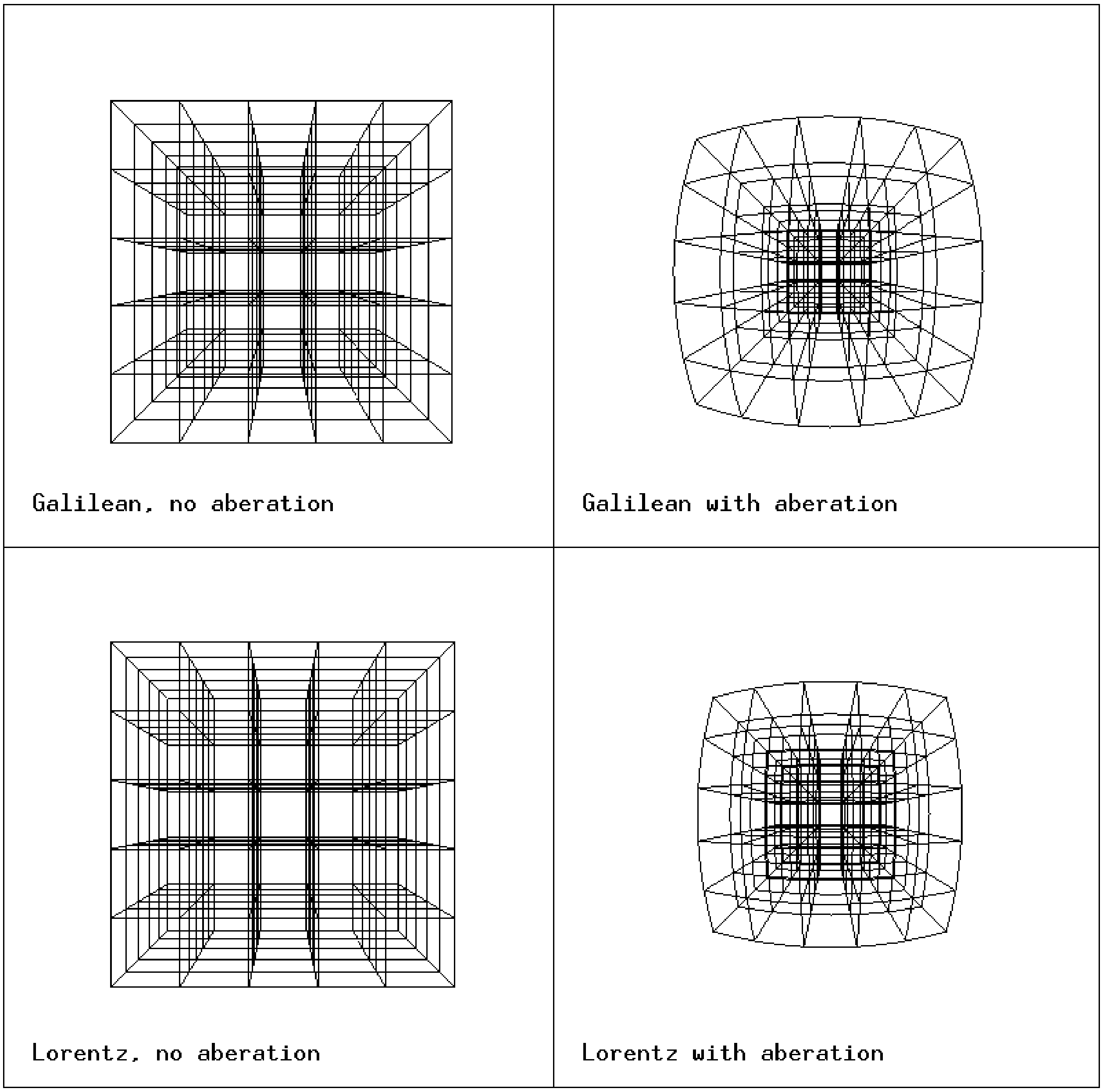

Figure 12: The above images show the various viewing transformations for a grid traveling at $0.8c$ toward the observer.

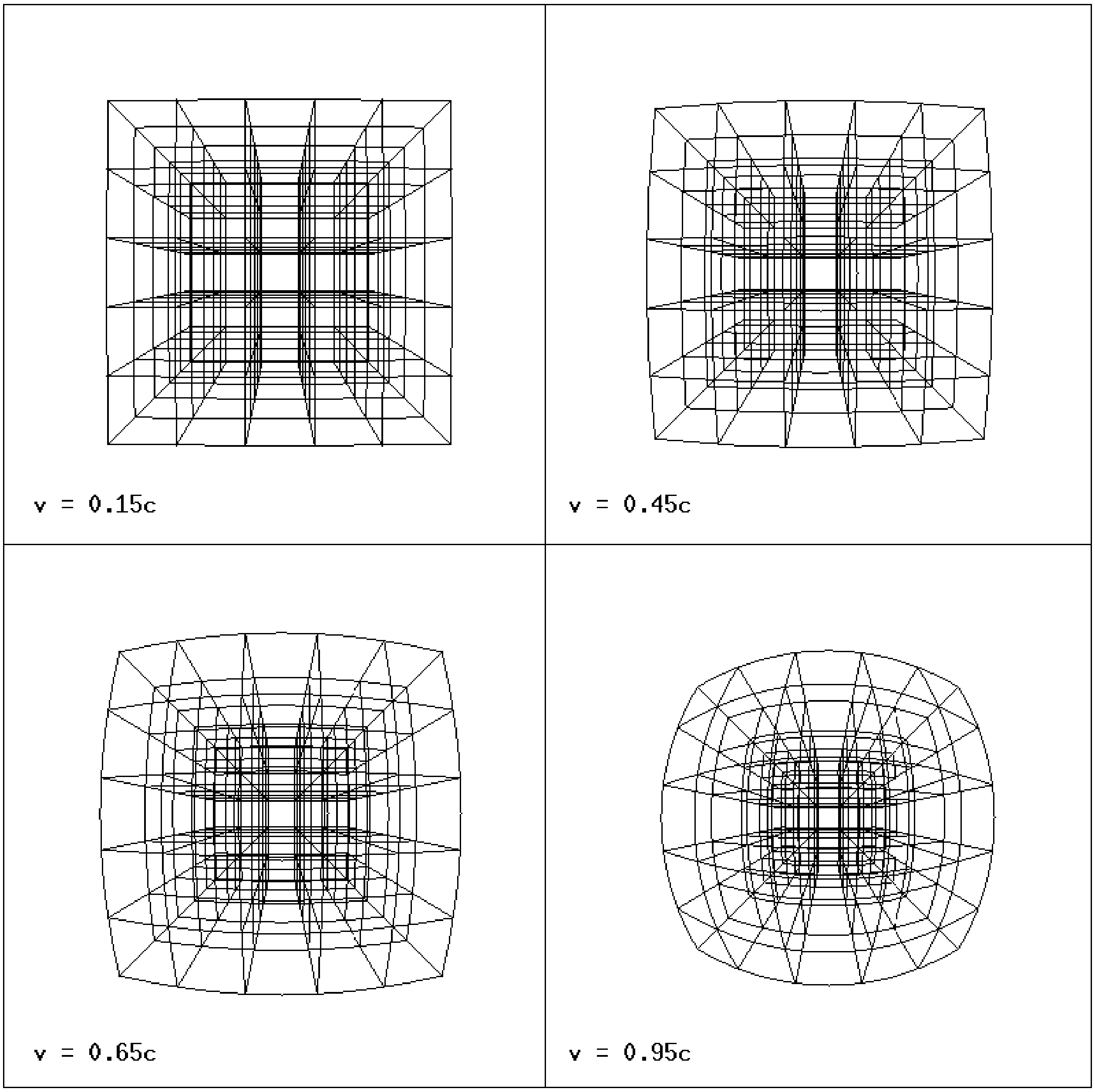

Figure 13: The above images shows a several grids traveling toward the observer at various velocities. Transformations of physical descriptions to apparent images are done using the Lorentz transformation with aberration.

figure 5 by changing the direction of the `velocity` and increased the `size` array. The grid is one cubic meter and is built up from smaller 0.20 meter cubes. We consider the appearance of this grid as it moves toward the observer at $0.8c$ (see figure 12). Notice that the aberration effects for objects moving toward the observer, while arising from the same transformation, appear to be significantly different. Part of this effect is due to the nature of matrix projection which allows the computer to simulate a three dimensional environment. However, when working interactively with the program the relics of the projection matrix become less noticeable and the aberration effects which are observed are still strikingly different from those in the previous examples.

Finally, we conclude by showing how the "real" image (i.e. the image obtained using Lorentz contraction and aberration) of the grid would appear[22] when traveling toward the observer at various fractions of the speed of light (figure 13). Notice that the "bubbling" type distortion increases as the velocity of the object increases. Not shown but also interesting are images of the object as it gets closer and passes by the observer. In these pictures the "bubbling" distortion greatly increases to the point that viewing fixed images is no longer insightful in understanding the effect.

6 Conclusion

The phenomena of light aberration is both important and interesting. As was demonstrated, light aberration leads to a non-linear distortion of in the appearance of objects moving at relativistic velocities. Furthermore, we have seen that for different situations, the battle between aberration's stretching and Lorentz's contracting can be "won" by either side. In some simple cases one can use logical reasoning to determine all of the pertinent effects and get an intuitive understanding of a moving object's physical appearance. However, using visualization tools such as the computer program discussed here, one can readily create and explore new situations.

There are many aspects of this project which could be explored further. Additional work can easily be done of this visualization program to make it more accessible to students. Implementing a more efficient method of scene specification and porting the code to the PC platform are two obvious directions. Other features which could be developed are solid

[22] Images similar to Figure 13 are also presented by Ivars Peterson in Space-Time Odyssey, *Science News*, vol. 137 no. 15, p. 232 (3), 1990.

modeling and implementation of a realistic color scheme that includes Doppler shift effects. Finally, including more of the classical aberration models such as a general aether model and a particle model would be interesting.

Hopefully, the results presented in this paper have provided the reader with a deeper intuitive understanding of the appearance of objects moving at relativistic velocities.

References

[1] P. Buffa, The Mechanical Universe and Beyond, Intellimation, Santa Barbara, CA, 1985.

[2] A. Einstein, Zur Elektrodynamik bewegter Körper, Annalen der Physik, 17, 1905.

[3] I. Peterson, Space-Time Odyssey, *Science News*, vol. 137 no. 15, p. 232 (3), 1990.

[4] G. D. Scott and M. R. Viner, The Geometrical Appearanceof Large Objects moving at Relativistic Speeds, *Am. J. Physics.* , vol. 33 p. 534 (2), 1965.

[5] A. Ungar, The abstract Lorentz transformation group, Am. J. Phys. **60** (9), September 1992, p.821.