# Revisiting the Einstein field of a mass point 

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AbSTRACT. - The Einstein gravitational field of a material point at rest is derived anew - by a suitable limit process - from the field of a sphere of a homogeneous and incompressible fluid. This result supports clearly the thesis according to which the physically interesting singularities must correspond to the presence of matter in loco.

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1.     - In the last section of his fundamental memoir on the Einstein field of a sphere of a homogeneous and incompressible fluid [1], Schwarzschild gave some hints for deriving, through a limit procedure, the Einstein field of a mass point - which had been directly deduced by him in another fundamental work [2] - from the field of the fluid. In this note, following Schwarzschild's indications, I prove that it is actually possible to deduce the field of the paper cited in [2] from the field of the paper cited in [1].
2.     - I adopt here Schwarzschild's notations. In the following co-ordinates $(c=1)$ :

$$
\begin{equation*}
x_{1} \equiv r^{3} / 3, \quad x_{2} \equiv-\cos \theta, \quad x_{3} \equiv \varphi, \quad x_{4} \equiv t \tag{2.1}
\end{equation*}
$$

for the internal region of the above sphere we have (cf.[1]):

$$
\begin{equation*}
\mathrm{d} s^{2}=f_{4} \mathrm{~d} x_{4}^{2}-f_{1} \mathrm{~d} x_{1}^{2}-f_{2} \frac{\mathrm{~d} x_{2}^{2}}{1-x_{2}^{2}}-f_{2} \mathrm{~d} x_{3}^{2}\left(1-x_{2}^{2}\right) \tag{2.2}
\end{equation*}
$$

further,

$$
\begin{equation*}
f_{1} f_{2}^{2} f_{4}=1 \tag{2.3}
\end{equation*}
$$

where the $f$ 's are functions of $x_{1} \equiv x$. In the external region (cf. [1] and [2])

$$
\left\{\begin{array}{l}
\left(f_{4}\right)_{\mathrm{ext}}=1-\alpha(3 x+\rho)^{-1 / 3}  \tag{2.4}\\
\left(f_{2}\right)_{\mathrm{ext}}=(3 x+\rho)^{2 / 3} \\
\left(f_{1} f_{2}^{2} f_{4}\right)_{\mathrm{ext}}=1
\end{array}\right.
$$

The two constants of integration $\alpha$ and $\rho$ will be determined from the mass and the radius of our fluid sphere, see eqs. (33) and (34) of [1]. Schwarzschild found the functions $f(x)$ by solving Einstein equations with the following matter tensor:

$$
\left\{\begin{array}{l}
T_{1}^{1}=T_{2}^{2}=T_{3}^{3}=-p \quad ; \quad T_{4}^{4}=\rho_{0} ; \quad T_{\mu}^{v}=0, \text { for } \mu \neq v  \tag{2.5}\\
T:=T_{\alpha}^{\alpha}=\rho_{0}-3 p
\end{array}\right.
$$

where $\rho_{0}$ is the invariant and constant density of the fluid, and $p=p(x)$ is the pressure [3], which must be equal to zero on the surface of the sphere (the suffix $a$ denotes, here and in the sequel, the value of a quantity on the spherical surface):

$$
\begin{equation*}
p_{a}=p\left(x_{a}\right)=0 \tag{2.6}
\end{equation*}
$$

For our aim it is not necessary to report all the results obtained by Schwarzschild in [1]. We can limit ourselves to some points.

Let us put, with our Author,

$$
\begin{equation*}
f_{2}=\eta^{2 / 3}, \quad f_{4}=\zeta \eta^{-1 / 3}, \quad f_{1}=\frac{1}{\zeta \eta} \tag{2.7}
\end{equation*}
$$

at $x=0$ we must have $\eta(0)=0$, as it is easily seen. Then, externally to the sphere:

$$
\begin{equation*}
\eta_{\mathrm{ext}}=3 x+\rho, \quad \zeta_{\mathrm{ext}}=\eta^{1 / 3}-\alpha \tag{2.8}
\end{equation*}
$$

A basic clause in Schwarzschild's treatment is that the pressure must be positive and finite also at the centre of the sphere. On the contrary, to obtain the field of a

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mass point through a convenient "mathematical contraction" of our sphere, we must assume that at $x=0$ both $p(x)$ and $\rho_{0}$ are infinite (and positive).

Since (see eqs. (10) and (22) of [1])

$$
\begin{equation*}
\left(\rho_{0}+p\right) \sqrt{f_{4}}=\text { constant }=\rho_{0} \sqrt{\left(f_{4}\right)_{a}}, \tag{2.9}
\end{equation*}
$$

we must determine $f_{4}(x)$ in such a way that $f_{4}(0)=0$. Now, Schwarzschild proves (see p. 430 of [1]) that for very small $\eta$ the function $f_{4}$ is given by

$$
\begin{equation*}
f_{4}=\frac{\lambda}{\eta^{1 / 3}}\left[K+\frac{\kappa \rho_{0} \sqrt{\left(f_{4}\right)_{a}}}{7} \frac{\eta^{7 / 6}}{\lambda^{3 / 2}}\right]^{2}, \tag{2.10}
\end{equation*}
$$

where: $(8 \pi)^{-1} \kappa \equiv$ the constant $G$ of universal gravitation; $\lambda$ is an integration constant; $K$ is another constant (depending, in particular, on $\lambda$ ), defined by formula (27) of [1]. If $K=0$, we have clearly $f_{4}(0)=0$, as desired. Thus

$$
\begin{equation*}
f_{4}=\left(\frac{\kappa \rho_{0}}{7 \lambda}\right)^{2}\left(f_{4}\right)_{a} \eta^{2} \tag{2.11}
\end{equation*}
$$

$\lambda$ is evidently given by

$$
\begin{equation*}
\lambda= \pm \frac{\kappa}{7} \rho_{0} \eta_{a} \tag{2.12}
\end{equation*}
$$

In conclusion,

$$
\begin{equation*}
f_{4}(x)=\eta_{a}^{-2}\left(f_{4}\right)_{a} \eta^{2}(x), \tag{2.13}
\end{equation*}
$$

where

$$
\left(f_{4}\right)_{a}=1-\alpha\left(3 x_{a}+\rho\right)^{-1 / 3} .
$$

If the radius $r_{a}$ of the sphere goes to zero we get

$$
\begin{equation*}
\lim _{x_{a} \rightarrow 0}\left(f_{4}\right)_{a}=1-\alpha \rho^{-1 / 3}, \tag{2.13"}
\end{equation*}
$$

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in order that this limit is equal to zero we must put $\rho=\alpha^{3}$ : but this was just the value chosen by Schwarzschild in [2], owing to a physical analogy with Newton's theory.

Next,

$$
\begin{equation*}
\left(f_{2}\right)_{a}=\eta_{a}^{2 / 3}=\left(3 x_{a}+\alpha^{3}\right)^{2 / 3}, \tag{2.14}
\end{equation*}
$$

$$
\begin{equation*}
\left(f_{1}\right)_{a}=\frac{1}{\left(f_{2}\right)_{a}^{2}\left(f_{4}\right)_{a}}=\frac{1}{\left(3 x_{a}+\alpha^{3}\right)\left[\left(3 x_{a}+\alpha^{3}\right)^{1 / 3}-\alpha\right]} \tag{2.15}
\end{equation*}
$$

thus, in the limit $x_{a} \rightarrow 0$ we obtain actually the values at the origin $O$ of the coordinates for the components of the field of a mass point [2]. And the expression of $\mathrm{d} s^{2}$ in the customary polar co-ordinates $\rho, \theta, \varphi$ is:

$$
\begin{equation*}
\mathrm{d} s^{2}=\left(1-\frac{\alpha}{R}\right) \mathrm{d} t^{2}-\frac{\mathrm{d} R^{2}}{1-\alpha / R}-R^{2}\left(\mathrm{~d} \theta^{2}+\sin ^{2} \theta \mathrm{~d} \varphi^{2}\right) \tag{2.16}
\end{equation*}
$$

where

$$
R \equiv\left(r^{3}+\alpha^{3}\right)^{1 / 3}
$$

and $\alpha \equiv \kappa M /(4 \pi)$ - if $M$ is the mass of the gravitating point - , as it can be easily proved, e.g., by computing the motion of a test particle at a great distance from $O$.

We have found anew the space-time interval corresponding to a material point in the form of Schwarzschild's paper [2]. And we have found a strong argument in favour of the original choice $\rho=\alpha^{3}$.
3. - In the literature of mathematical character on general relativity we find very many papers concerning the various kinds of singular space-times. We have, typically, quasi-regular singularities and curvature singularities, with their subspecies. From the physical standpoint, however, their interest is rather poor: as our deduction of eqs. $(2.13) \div\left(2.16^{\prime}\right)$ has evidenced, the physically significant

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singularities do correspond to a real presence of matter in loco, i.e. to a mass tensor different from zero. This was perfectly clear to Karl Schwarzschild - and to Albert Einstein. (In Newton's theory, the gravitational potential $U=G M / r$ is solution of Laplace-Poisson equation $\nabla^{2} U(\mathbf{r})=-4 \pi G M \delta(\mathbf{r})$.)

A final remark. If you calculate the collapsing process of a spherical cloud of "dust" by relying on formulae (2.16)-(2.16') for the external region, you will obtain a very transparent and "Galilean" result, owing to the fact that the $\mathrm{d} s^{2}$ given by (2.16)-(2.16') is quite regular for $r=\kappa M /(4 \pi)$.

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## References

[1] Schwarzschild K., Berl. Ber., Phys.-Math. Kl., (1916) 424. (An English translation can be requested to A.L., e-mail: sloinger@tin.it). The problem was subsequently re-investigated by Weyl, see Weyl H., Raum-Zeit-Materie, Siebente Auflage (Springer-Verlag, Berlin, etc.) 1988, p. 262.
[2] Schwarzschild K., Berl. Ber., Phys.-Math. Kl., (1916) 189. (An English translation by S.Antoci and A.Loinger at http://xxx.lanl.gov/abs/physics/9905030 (12 May 1999)).
[3] Bauer investigated the problem of a sphere of a fluid characterized by a linear equation of state, see Bauer H., Wiener Ber., Math.-Naturw. Kl., Abt. IIa, (1918) 2141.

