Revisiting the Einstein field of a mass point

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ABSTRACT. — The Einstein gravitational field of a material point at rest is derived anew – by a suitable limit process – from the field of a sphere of a homogeneous and incompressible fluid. This result supports clearly the thesis according to which the physically interesting singularities must correspond to the presence of matter *in loco*.

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1. – In the last section of his fundamental memoir on the Einstein field of a sphere of a homogeneous and incompressible fluid [1], Schwarzschild gave some hints for deriving, through a limit procedure, the Einstein field of a mass point – which had been directly deduced by him in another fundamental work [2] – from the field of the fluid. In this note, following Schwarzschild's indications, I prove that it is actually possible to deduce the field of the paper cited in [2] from the field of the paper cited in [1].

2. – I adopt here Schwarzschild's notations. In the following co-ordinates (c=1):

(2.1)
$$x_1 \equiv r^3/3, \quad x_2 \equiv -\cos\theta, \quad x_3 \equiv \phi, \quad x_4 \equiv t,$$

for the *internal* region of the above sphere we have (cf.[1]):

(2.2)
$$ds^{2} = f_{4} dx_{4}^{2} - f_{1} dx_{1}^{2} - f_{2} \frac{dx_{2}^{2}}{1 - x_{2}^{2}} - f_{2} dx_{3}^{2} (1 - x_{2}^{2}) ;$$

further,

(2.3)
$$f_1 f_2^2 f_4 = 1$$
,

where the *f*'s are functions of $x_1 \equiv x$. In the external region (cf. [1] and [2])

(2.4)
$$\begin{cases} (f_4)_{\text{ext}} = 1 - \alpha (3x + \rho)^{-1/3} \\ (f_2)_{\text{ext}} = (3x + \rho)^{2/3} \\ (f_1 f_2^2 f_4)_{\text{ext}} = 1 \end{cases}$$

The two constants of integration α and ρ will be determined from the mass and the radius of our fluid sphere, see eqs. (33) and (34) of [1]. Schwarzschild found the functions *f*(*x*) by solving Einstein equations with the following matter tensor:

,

(2.5)
$$\begin{cases} T_1^1 = T_2^2 = T_3^3 = -p ; & T_4^4 = \rho_0; & T_\mu^\nu = 0, \text{ for } \mu \neq \nu ; \\ T := T_\alpha^\alpha = \rho_0 - 3p , \end{cases}$$

where ρ_0 is the invariant and constant density of the fluid, and p=p(x) is the pressure [3], which must be equal to zero on the surface of the sphere (the suffix *a* denotes, here and in the sequel, the value of a quantity on the spherical surface):

(2.6)
$$p_a = p(x_a) = 0$$
.

For our aim it is not necessary to report all the results obtained by Schwarzschild in [1]. We can limit ourselves to some points.

Let us put, with our Author,

(2.7)
$$f_2 = \eta^{2/3}, \quad f_4 = \zeta \eta^{-1/3}, \quad f_1 = \frac{1}{\zeta \eta};$$

at x = 0 we must have $\eta(0) = 0$, as it is easily seen. Then, externally to the sphere:

(2.8)
$$\eta_{\text{ext}} = 3x + \rho, \qquad \zeta_{\text{ext}} = \eta^{1/3} - \alpha ,$$

A basic clause in Schwarzschild's treatment is that the pressure must be positive and finite also at the centre of the sphere. On the contrary, to obtain the field of a mass point through a convenient "mathematical contraction" of our sphere, we must assume that at x = 0 both p(x) and ρ_0 are infinite (and positive).

Since (see eqs. (10) and (22) of [1])

(2.9)
$$(\rho_0 + p) \quad \sqrt{f_4} = \text{constant} = \rho_0 \quad \sqrt{(f_4)_a}$$

we must determine $f_4(x)$ in such a way that $f_4(0) = 0$. Now, Schwarzschild proves (see p.430 of [1]) that for *very small* η the function f_4 is given by

,

(2.10)
$$f_4 = \frac{\lambda}{\eta^{1/3}} \left[K + \frac{\kappa \rho_0 \sqrt{(f_4)_a}}{7} \frac{\eta^{7/6}}{\lambda^{3/2}} \right]^2 ,$$

where: $(8\pi)^{-1}\kappa \equiv$ the constant *G* of universal gravitation; λ is an integration constant; *K* is another constant (depending, in particular, on λ), defined by formula (27) of [1]. *If K*=0, we have clearly $f_4(0) = 0$, as desired. Thus

,

(2.11)
$$f_4 = \left(\frac{\kappa \rho_0}{7\lambda}\right)^2 (f_4)_a \eta^2 \quad ;$$

 λ is evidently given by

(2.12)
$$\lambda = \pm \frac{\kappa}{7} \rho_0 \eta_a$$

In conclusion,

(2.13)
$$f_4(x) = \eta^{-2}{}_a (f_4)_a \eta^2(x) ,$$

where

(2.13')
$$(f_4)_a = 1 - \alpha (3x_a + \rho)^{-1/3}$$

If the radius r_a of the sphere goes to zero we get

(2.13")
$$\lim_{x_a \to 0} (f_4)_a = 1 - \alpha \rho^{-1/3} ,$$

in order that this limit is equal to zero we must put $\rho = \alpha^3$: but this was just the value chosen by Schwarzschild in [2], owing to a physical analogy with Newton's theory.

Next,

(2.14)
$$(f_2)_a = \eta_a^{2/3} = (3x_a + \alpha^3)^{2/3}$$

(2.15)
$$(f_1)_a = \frac{1}{(f_2)_a^2 (f_4)_a} = \frac{1}{(3x_a + \alpha^3)[(3x_a + \alpha^3)^{1/3} - \alpha]} ;$$

thus, in the limit $x_a \rightarrow 0$ we obtain actually the values at the origin *O* of the coordinates for the components of the field of a mass point [2]. And the expression of ds^2 in the customary polar co-ordinates ρ , θ , ϕ is:

(2.16)
$$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \alpha/R} - R^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) ,$$

where

(2.16')
$$R \equiv (r^3 + \alpha^3)^{1/3}$$

and $\alpha \equiv \kappa M/(4\pi)$ – if *M* is the mass of the gravitating point –, as it can be easily proved, e.g., by computing the motion of a test particle at a great distance from *O*.

We have found *anew* the space-time interval corresponding to a material point *in the form of Schwarzschild's paper* [2]. And we have found *a strong argument in favour of the original choice* $\rho = \alpha^3$.

3. – In the literature of mathematical character on general relativity we find very many papers concerning the various kinds of singular space-times. We have, typically, quasi-regular singularities and curvature singularities, with their subspecies. From the physical standpoint, however, their interest is rather poor: as our deduction of eqs. $(2.13) \div (2.16')$ has evidenced, the physically significant

singularities do correspond to a real presence of matter *in loco*, i.e. to a mass tensor different from zero. This was perfectly clear to Karl Schwarzschild – and to Albert Einstein. (In Newton's theory, the gravitational potential U=GM/r is solution of Laplace-Poisson equation $\nabla^2 U(\mathbf{r}) = -4\pi \ GM \ \delta(\mathbf{r})$.)

A final remark. If you calculate the collapsing process of a spherical cloud of "dust" by relying on formulae (2.16)–(2.16') for the external region, you will obtain a very transparent and "Galilean" result, owing to the fact that the ds^2 given by (2.16)–(2.16') is quite regular for $r = \kappa M/(4\pi)$.

Zum Andenken an Karl Schwarzschild (1873-1916).

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