The simplest, and the full derivation of Magnetism as a Relativistic side effect of ElectroStatics

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Abstract

The notion of Magnetism as relativistic side effect of Electro Statics can be derived from the work of Lienard & Wiechert around 1900, and the correct form of the Lorentz transformations established a few years later. 100 years later we are now teaching this concept to undergrade students with the help of a popular derivation following Purcell who presented this derivation in his 1963 textbook.

This derivation however is questionable. Mainly so because the test-charge used to measure the force outside the wire-with-current, has a velocity alongside the wire which is, unrealistically, taken to be always the same as the charge velocity inside the wire itself. So, when the test-charge doubles it speed, the current I through the wire is also doubled and the magnetic force is quadrupled. This however, makes it impossible to determine if the contributions to the higher magnetic force come from either the higher current I, the higher speed v, or both.

The electrons in a real live wire drift with a wide range of different velocities which together produce the current I. We will discuss our derivation, which only needs the current I through the wire and the speed v of the test-charge. Surprisingly, this derivation turns out to be even simpler as Purcell's. (for the case of the charge moving parallel to the wire).

We will also derive the case where the charge is moving perpendicular instead of parallel to the wire. We discuss the required charge-carrier-density in a current carrying wire in order to be *electrically neutral* in the rest-frame. To be self consistent we will derive the relativistic EM Potential and the relativistic Electrostatic Field for a point particle from the classical EM wave equations in a way which is both short and simple.

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1 The correct derivation based on non-simultaneity.

We'll now give the correct derivation for the case when the test-charge moves parallel with the wire. This derivation is actually even simpler than Purcell's one. We want to prove that Special Relativity alters the apparent *net charge density* q_L of the wire from neutral to a non-zero value.

As stated, we in fact do not need to deal with the speed of the electrons in the wire at all, just with the current I which the wire is carrying. We start of with the two standard textbook formulas involved. Here Q is the test-charge which is moving alongside the wire with a speed of v and y is the distance between the test-charge and the wire.

$$F_{mag.} = \frac{vI}{2\pi\epsilon_o c^2 y}\,Q \qquad \equiv \qquad F_{elec.} = \frac{q_L}{2\pi\epsilon_o y}\,Q \qquad \Rightarrow \qquad q_L \;=\; I\frac{v}{c^2} \tag{1}$$

The first formula gives the magnetic force on the test-charge moving in parallel with the wire. The second formula gives the electric force on the test-charge from the wire when the wire has a non-zero charge-density q_L. By equating the two formulas we get an expression for the net charge density. That is: The test-charge moving with a speed v along a neutral wire with current I should see a *non-neutral* wire with a charge density of $q_L = Iv/c^2$. With this expression in mind we now turn to the standard Lorentz transformations:

$$t \;=\; \gamma\left(t' - \frac{v}{c^2}x'\right), \qquad\qquad x \;=\; \gamma\,(x' - vt') \tag{2}$$

All we need here is the factor v/c^2 in the expression for the time t which describes the non-simultaneity. The magnetic force is proportional to this factor as shown above. We can even ignore the factor γ in the formula for t, We can use v/c^2 instead of $\gamma\ v/c^2$. The difference at low speeds can be neglected for our purpose. The typical average drift velocity of electrons in domestic electro motors is in the order of a millimeter per second.

We need an understanding of what is happening as a result of non-simultaneity: An observer in the moving test-charge frame, who moves along with the wire (say in the same direction as the electrons in the wire) will, due to non-simultaneity, 'see' into the *future* of the wire downstreams of the electrons, and into the *past* at the side where the electrons enter the wire. 'Seeing' is of course the wrong word. We just redefine simultaneity different in a different reference frame. We however have to adopt our calculations as if these events in the past and future are simultaneous to our time when we are observing from the test charge's rest-frame.

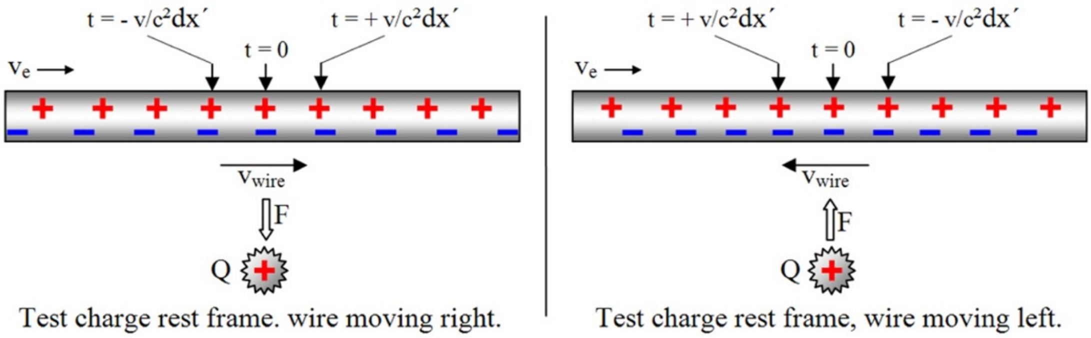

Figure 1: The correct derivation based on non-simultaneity.

The *future* at the downstream side of the electrons means that they did stream further out of the wire there. At the other hand, the *past*, at the side where the electrons enter the wire, means that less electrons have streamed into the wire there. The overall result is thus that we must calculate with less electrons in the wire per unit of length as positive ions. More electrons have streamed out while less of them have streamed in. Now, how much does the charge density change? Well,

$$q_L \; dx \;=\; I \; dt \;=\; I \frac{v}{c^2} \; dx \tag{3}$$

$q_L \; dx$ is the total charge in the piece of wire dx, this corresponds with the current I going on for a time dt. The time difference dt due to non-simultaneity between the two ends of a piece of wire dx is $v/c^2 dx$ as we did see from the standard Lorentz transformation.

It means that the current-out-of-the-wire has gone on longer for a time of $dt = v/c^2 \; dx$ compared to the current-into-the-wire at the other side. This then simply gives us the change of the charge in the piece of wire dx and thus gives us the charge density q_L. We see that the expression for q_L is exactly the one we found which was needed.

2 Neutral wire condition from the relativistic charge field.

After the simple derivation given above we want to look at the full derivation. We start with the electrostatic field of a point charge which depends on the speed of the charge, That is, the shape of the electrostatic field itself changes as a result of Special Relativity. This was first shown by Lienard & Wiechert in 1900, before Einstein's paper of 1905. They derived it from classical physics, presuming that the speed of propagation is c. Indeed, Lorentz contraction and also time dilation arise from classical physics. We will derive the relativistic electrostatic field in the final section of this paper in a both simple and short way directly from the classical wave equation of the EM potentials.

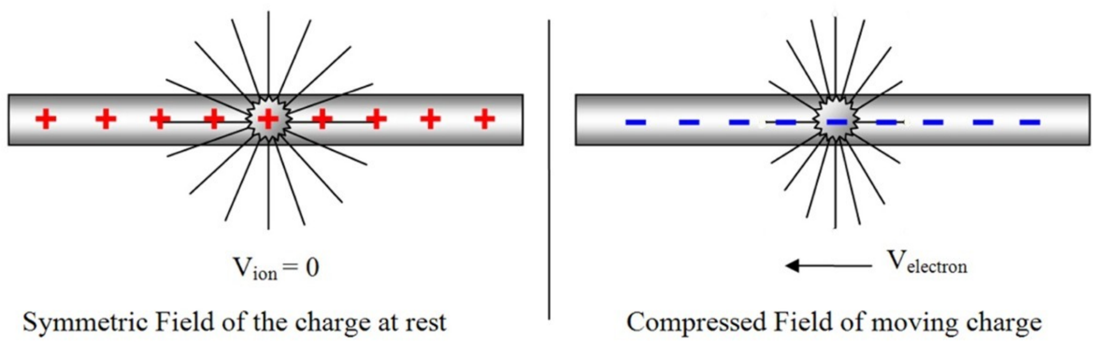

Figure 2: The Electrostatic Field of the Ion at rest and the moving electron

Here we will start with the formula in the form as given by Jackson (11.154). We will use the remainder of this section to show one thing: In order for a current carrying wire to be neutral in its rest frame it is required that the density of the moving electrons (density per unit of length) is simply equal to that of the density of the positive ions. So even if the field of the individual moving electrons is different, the end result is the same.

This is because two effects cancel each other. The field becomes stronger in the y direction by a factor γ but weaker in the x direction (along the wire) by a factor γ^2. The two effects cancel if we integrate over the wire. This does mean that the presumption that the electron density is equal to the ion density for a neutral current carrying wire was indeed correct. We'll do the math starting with the formula as given by Jackson:

$$\mathbf{E} \;=\; \frac{q\mathbf{r}}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \qquad\qquad \left(\lim_{v\to 0}\ \mathbf{E} \;=\; \frac{q\mathbf{r}}{4\pi\epsilon_o r^3}\right) \tag{4}$$

Unlike Jackson we maintain the factor $4\pi\epsilon_o$. This is just a question of using more familiar units. $\mathbf{E}$ and $\mathbf{r}$ are vectors, pointing in the same direction and $\beta = v/c$ as usual. If we set $\sin\psi = 0$ we have the field in the x-direction along the wire, which is weaker by a factor γ^2. Setting $\sin\psi = 1$ gives us the field in the y-direction, stronger by a factor γ. We can separate this formula in its individual E_x, E_y components:

$$E_x \;=\; \frac{qx}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}, \qquad E_y \;=\; \frac{qy}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \tag{5}$$

We've can get rid of the $\sin\psi$ factor here by substituting $\sin\psi = y/\sqrt{x^2+y^2}$. We start using β_x for β here just to emphasize that the speed is in the x-direction.

$$E_x \;=\; \frac{q(1-\beta_x^2)x}{4\pi\epsilon_o(x^2+(1-\beta_x^2)y^2)^{3/2}}, \qquad E_y \;=\; \frac{q(1-\beta_x^2)y}{4\pi\epsilon_o(x^2+(1-\beta_x^2)y^2)^{3/2}} \tag{6}$$

Now we have the formula in a way which is appropriate to integrate over x, along the (infinitely long) wire. Thanks to the fact the we have symbolic integrators nowadays we immediately get:

$$\int E_x dx \;=\; \frac{-(1-\beta_x^2)q_L}{4\pi\epsilon_o y\sqrt{x^2+(1-\beta_x^2)y^2}}, \quad \int E_y dx \;=\; \frac{q_L x}{4\pi\epsilon_o y\sqrt{x^2+(1-\beta_x^2)y^2}} \tag{7}$$

Well, this one wasn't that difficult, so you might have recognized it. Note that we have changed q into q_L, the charge-*density* per unit of length. Since the wire is infinitely long we integrate from $-\infty$ to $+\infty$.

$$\int_{-\infty}^{+\infty} E_x dx \;=\; 0, \qquad \int_{-\infty}^{+\infty} E_y dx \;=\; \frac{q_L}{2\pi\epsilon_o y} \tag{8}$$

We see that the field of the moving electrons is the same as the field for the electrons at rest. The result is independent of the speed of the electrons. We arrived at this result by assuming that the number of electrons per unit of length stays the same, regardless of if they do move or not move.

3 The test charge moving perpendicular to the current.

Here we derive the Magnetic force on a test-charge moving perpendicular to the wire. This force should be parallel to the wire itself. This derivation is a little bit more involved, so we start with the formula for the relativistic electrostatic field for a point charge from Jackson (11.154)

$$\mathbf{E} \;=\; \frac{q\mathbf{r}}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \tag{9}$$

We repeat: $\mathbf{E}$ and $\mathbf{r}$ are vectors, pointing in the same direction and $\beta = v/c$ as usual. If we set $\sin\psi = 0$ we have the field along the direction of motion of the point charge, which is weaker by a factor γ^2. Setting $\sin\psi = 1$ gives us the field perpendicular to the direction of motion of the charge, being stronger by a factor γ. Again we start by separating this formula in its individual E_x, E_y components:

$$E_x = \frac{qx}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}}, \qquad E_y = \frac{qy}{4\pi\epsilon_o r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \tag{10}$$

In this case we have a $\beta = v/c$ which can have components both in the x and y-direction. We shall consider the test-charge at rest. This means that the positive ions in the wire move either to us or away from us. Their speed has an y-component only. The electrons in the wire have the same y-component and, in adition to that, a component in the x-direction because they move through the wire.

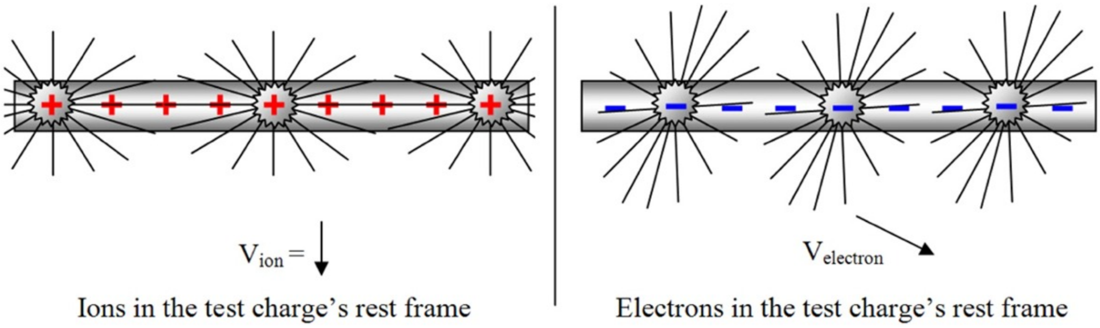

Figure 3: The charge moving perpendicular with the current.

The image shows how the electron fields are rotated. There will be a net force pulling a positive test charge from left to right. In order to integrate we want to get rid of the $\sin\psi$ term first. We can do this with the following identity:

$$|\ \vec{r}\times\vec{\beta}\ | \ = \ r\beta\sin\psi \tag{11}$$

$$(\vec{r}\times\vec{\beta})\cdot(\vec{r}\times\vec{\beta}) \ = \ r^2\beta^2\sin^2\psi \ = \ (\beta_y x - \beta_x y)^2 \tag{12}$$

We get the absolute value of the cross-product by doing a dot-product with itself. It turns out that the squared value is exactly what we need to get rid of the $\sin^2\psi$ terms:

$$E_x = \frac{q(1-\beta_x^2-\beta_y^2)x}{4\pi\epsilon_o(x^2+y^2-(\beta_y x-\beta_x y)^2)^{3/2}}, \qquad E_y = \frac{q(1-\beta_x^2-\beta_y^2)y}{4\pi\epsilon_o(x^2+y^2-(\beta_y x-\beta_x y)^2)^{3/2}} \tag{13}$$

These expressions show all the x values explicitly. This is what we need to integrate the fields from all the charges over the wire. (in the x-direction)

Symbolic integration software really helps with these ones. We obtain the following as the results:

$$\int E_x dx \ = \ \frac{-Q_L\left(\beta_x\beta_y x \ + \ (1-\beta_x^2)y\right)}{4\pi\epsilon_o y\sqrt{(1-\beta_y^2)x^2+2\beta_x\beta_y xy+(1-\beta_x^2)y^2}} \tag{14}$$

$$\int E_y dx \ = \ \frac{Q_L\left((1-\beta_y^2)x \ + \ \beta_x\beta_y y\right)}{4\pi\epsilon_o y\sqrt{(1-\beta_y^2)x^2+2\beta_x\beta_y xy+(1-\beta_x^2)y^2}} \tag{15}$$

Note that we have changed q into Q_L, the *total* charge-density per unit of length for each type of charge carrier, either the electrons or the positive ions. This is the *total* charge and not the *net difference* of the charges, for which we have used q_L. Now, since the wire is infinitely long we integrate from $-\infty$ to $+\infty$.

$$\int_{-\infty}^{+\infty} E_x dx \;=\; \frac{-\beta_x\beta_y\; Q_L}{2\pi\epsilon_o\sqrt{1-\beta_y^2}\; y}, \qquad \int_{-\infty}^{+\infty} E_y dx \;=\; \frac{\sqrt{1-\beta_y^2}\; Q_L}{2\pi\epsilon_o y} \tag{16}$$

We must take these values for both the electrons and positive ions. Only the electrons have a non-zero E_x value since the ions have no β_x, which is the speed at which the electrons move trough the wire. We see that the E_y fields for both charge types cancel each other. We use the following to make a few substitutions:

$$I \;= v_x\; Q_L \;=\; c\beta_x\; Q_L, \qquad\qquad v_y \;=\; c\beta_y \tag{17}$$

Now, in this case, we have the right formula for the current I through the wire, because v_x is the speed of the electrons relative to the positive ions and Q_L is the charge-density of the electrons. Substituting and ignoring the γ factor we get the required result. The magnetic force on a charge moving perpendicular to a current carrying wire:

$$F_{mag} = \frac{v_y I}{2\pi\epsilon_o c^2 y}\; Q \qquad \equiv \qquad F_{elec} = \frac{v_y v_x\; Q_L}{2\pi\epsilon_o c^2 y}\; Q \qquad\qquad \Rightarrow \qquad I = v_x Q_L, \quad v_y = c\beta_y \tag{18}$$

4 The relativistic Potentials of the point charge.

To be self consistent and complete we will derive here the relativistic EM potentials of a moving point charge. In the next section we will use this to get the relativistic electrostatic field formula which we have used several times in this document. We start with the classical wave function which naturally generates the Lorentz contraction of the potentials. Yes. classical physics generates these relativistic effects.

$$\text{Classical Wave equation:} \qquad \frac{\partial^2\psi}{\partial t^2} \;=\; v^2\frac{\partial^2\psi}{\partial x^2} \tag{19}$$

The derivatives in time and space are equal, except for a constant which comes from the characteristic speed of the medium. This simply means that the equation is satisfied by any arbitrary function which shifts along with a speed v (or $-v$). We can expand this equation to three dimensions, for instance for the electric potential field V:

$$\text{Electric Potential:} \qquad \frac{\partial^2 V}{\partial t^2} \;=\; c^2\frac{\partial^2 V}{\partial x^2} + c^2\frac{\partial^2 V}{\partial y^2} + c^2\frac{\partial^2 V}{\partial z^2} \tag{20}$$

Where c is the speed of light. The same expression holds for the three components of the magnetic vector potential. Again these equations are satisfied by any arbitrary function which shifts along with the characteristic speed c.

In our world however we also see things which are stationary or move at other speeds as the speed of light. If we go to three (or more) space dimensions then such solutions become possible. A stable solution which shifts along with an arbitrary speed v in the x direction will satisfy both (19) with a speed of v and (25). We can use this to eliminate the time dependency:

$$\left(1-\frac{v^2}{c^2}\right)c^2\frac{\partial^2 V}{\partial x^2} \ + \ c^2\frac{\partial^2 V}{\partial y^2} \ + \ c^2\frac{\partial^2 V}{\partial z^2} \ = \ 0 \tag{21}$$

This shows that the solutions are Lorentz contracted in the direction of v by a factor γ, The first order derivatives are higher by a factor γ and the second order by a factor γ^2. Velocities higher then c are not possible. The solution for $v = 0$ is:

$$\frac{\partial^2 V}{\partial x^2} \ + \ \frac{\partial^2 V}{\partial y^2} \ + \ \frac{\partial^2 V}{\partial z^2} \ = \ 0, \qquad \Rightarrow \qquad V \ = \ \frac{1}{r} \tag{22}$$

Which is the electro static potential. The equation is satisfied at all points except for $r = 0$ where we have a singularity. This singularity is now associated with the classical (point)charge. Without it there would be no solutions at sub-luminal speeds. The total solution is an arbitrary superposition of $1/r$ functions. This includes the Quantum Mechanical fields where the charge is spread out over the wave function. Now going to the EM potentials. For a non-relativistic speed (along the x-axis) these are the familiar:

$$V \ = \ \frac{q}{4\pi\epsilon_o r}, \qquad A_x \ = \ \beta_x V, \qquad A_y = A_z = 0 \tag{23}$$

These now simply become contracted in the x-direction by a factor γ, so we multiply any occurrence of x by γ. Since the size of the whole decreases we multiply the overall result also by a factor γ, to maintain the same result if we integrate over all of space:

$$V \ = \ \frac{q\ \gamma}{4\pi\epsilon_o\sqrt{\gamma^2 x^2 + y^2 + z^2}}, \qquad A_x \ = \ \beta_x V, \qquad A_y = A_z = 0 \tag{24}$$

We should mention here, although without further proof, a property of fundamental importance of the wave function in the form of the d'Alembert operator:

$$\text{d'Alembert operator:} \qquad \Box\,\psi \ = \ \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} - c^2\frac{\partial^2}{\partial y^2} - c^2\frac{\partial^2}{\partial z^2}\right)\psi \tag{25}$$

This operator performs a *deconvolution* of the field ψ with the *light cone*. What this means is that, if ψ is for instance the potential caused by numerous moving and accelerating (radiating) charges on arbitrary curved paths, that, this operation will give you back a function which is everywhere zero except on all the paths of the charges. Using this operator on the magnetic vector potentials will give you back the velocities which the charges had on every location of their paths.

5 The relativistic Electrostatic Field of the point charge.

Finally we want to derive Jackson's formula (11.154) for the relativistic electric field $\vec{E}$ here. This is straight forward by using Maxwell's laws to obtain the field from the potentials V and A_x, A_y and A_z.

$$E_x \ = \ -\frac{\partial V}{\partial x} \ - \ \frac{\partial A_x}{\partial t}, \qquad E_y \ = \ -\frac{\partial V}{\partial y} \ - \ \frac{\partial A_y}{\partial t}, \tag{26}$$

The magnetic vector potentials A_x and A_y have a simple relation with the potential V:

$$A_x \;=\; \beta_x V, \qquad\qquad A_y \;=\; \beta_y V = 0 \tag{27}$$

We can change a derivative in t to derivative in x by simply multiplying it with $-\beta_x$ since our solution shifts in the x-direction with a speed v_x, so we have:

$$\frac{\partial A_x}{\partial t} \;=\; -\beta_x \frac{\partial A_x}{\partial x} \;=\; -\beta_x^2 \frac{\partial V}{\partial x}, \qquad \text{thus:} \tag{28}$$

$$E_x \;=\; -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} \;=\; -(1-\beta_x^2)\frac{\partial V}{\partial x} \;=\; (1-\beta_x^2)\frac{q\,\gamma^3 x}{4\pi\epsilon_o(\gamma^2x^2+y^2)^{3/2}} \tag{29}$$

The E_y case is simple since $A_y = 0$. We now get for the components of the electric field:

$$E_x \;=\; \frac{q\;\gamma\;x}{4\pi\epsilon_o(\gamma^2x^2+y^2)^{3/2}}, \qquad\qquad E_y \;=\; \frac{q\;\gamma\;z}{4\pi\epsilon_o(\gamma^2x^2+y^2)^{3/2}} \tag{30}$$

Which we can write in vector form:

$$\mathbf{E} \;=\; \frac{q\;\gamma\;\mathbf{r}}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \qquad\qquad \left(\lim_{v\to 0}\mathbf{E} \;=\; \frac{q\mathbf{r}}{4\pi\epsilon_o r^3}\right) \tag{31}$$

Finally by the substitution of $\sin^2\psi \; = (y^2+z^2)/(x^2+y^2+z^2)$ we get Jackson's formula (11.154)

$$\mathbf{E} \;=\; \frac{q\mathbf{r}}{4\pi\epsilon_o\; r^3\gamma^2(1-\beta^2\sin^2\psi)^{3/2}} \tag{32}$$