Relativity and electromagnetism: The force on a magnetic monopole

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On the occasion of the 100th anniversary of the first publication, by Oliver Heaviside, of what is now known as the Lorentz force law in electromagnetic theory, the analogous force law for magnetic monopoles is examined. Its relevance and limitations in calculating the force and torque on small current loops are discussed, and both its heuristic and practical uses are demonstrated.

I. 100 YEARS OF LORENTZ FORCE

The year 1989 marks the 100th anniversary of the first publication, by Oliver Heaviside, of the well-known formula[1] $\mathbf{F} = q\mathbf{v}\times\mathbf{B}/c$ for the mechanical force on a point charge q moving with velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$. Generalized to $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}/c)$ (in Gaussian units), it now constitutes the "fifth axiom" of electromagnetism, along with Maxwell's four equations, and is usually referred to as the *Lorentz force law*. I would like to take this occasion to make some remarks on the analogous formula (but note the minus sign!)

$$\mathbf{F} = q_m(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c) \tag{1}$$

for the force on a magnetic charge q_m (a monopole), which has been variously discussed, e.g., by Sommerfeld,[2] but to my mind not as sympathetically as it deserves to be.

Einstein, even before inventing special relativity in 1905, "was convinced that the (Lorentz) force acting on a charged body in motion through a magnetic field was nothing else but an electric field (in the body's rest frame),"[3] and so it turned out to be. This shows the intimate connection of Heaviside's formula with relativity. We shall therefore begin with an excursion into that theory.

II. RELATIVITY AND ELECTROMAGNETISM

Einstein's relativity principle asserts that any physical law valid in one inertial frame is valid in all inertial frames. (In particular, this holds for the five basic axioms of electromagnetism.) Thus, when a given experiment is observed from several different inertial frames, its outcome must be explainable in each of these by using the same totality of laws. Consider, then, the application of this principle to the following two electromagnetic experiments:[4]

(1) An electric charge moving through the $\mathbf{B}$ field of a stationary magnet in general (i.e., unless it moves tangentially to a field line) experiences a (Lorentz) force; referring our observations next to the instantaneous inertial rest frame of the charge, we see that a stationary charge must therefore experience a force when a magnet is moved in its vicinity.

(2) A stationary magnetic dipole (e.g., a compass needle) in general experiences a torque in the presence of a moving charge, since the latter creates a $\mathbf{B}$ field; transferring our observations once more to the inertial rest frame of the charge, we conclude that a magnetic dipole moving through a static electric field must experience a torque.

It is now natural to pose the problem whether these and perhaps other similar conclusions can be reached from Maxwell (i.e., standard electromagnetic) theory *without* appeal to relativity and, if so, at what cost in complication.

For problem 1 it would be tempting to use the "sixth axiom," our Eq. (1), if it were part of Maxwell theory. The magnet moving in the field of the charge is then immediately seen to experience a force, so that by Newton's law of action and reaction the charge must similarly experience a force due to the magnet. Problem 2 becomes even more immediately obvious by appeal to Eq. (1). However, whereas for problem 1 a little "trick" strictly within Maxwell theory will yield the result, we shall show that no nonrelativistic method will solve problem 2.

The "trick" for problem 1 is to use the Eulerian derivative of the vector potential $\mathbf{A}$, i.e., the time rate of change of $\mathbf{A}$ at a point moving with velocity $\mathbf{v} = dx_i/dt$,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \sum \frac{\partial \mathbf{A}}{\partial x_i}\frac{dx_i}{dt}. \tag{2}$$

Suppose a bar magnet is moved with uniform velocity $\mathbf{v}$ through an inertial frame. The vector potential $\mathbf{A}$ of its $\mathbf{B}$ field will then be constant along all comoving points, $d\mathbf{A}/dt = 0$. Thus, from the usual relation between $\mathbf{E}$ and $\mathbf{A}$ and Eq. (2),

$$\mathbf{E} = -c^{-1}\frac{\partial \mathbf{A}}{\partial t} = c^{-1}(\mathbf{v}\cdot\nabla)\mathbf{A},$$

and it is this nonzero $\mathbf{E}$ field that will set a stationary charge in motion as the magnet passes.

III. THE MONOPOLE LAW

As we have just seen, the monopole axiom (1) would be a handy tool in the solution of some electromagnetic problems. But, of course, since standard Maxwell theory denies the existence of monopoles ($\nabla\cdot\mathbf{B} = 0$), such an axiom at first sight seems superfluous. Nevertheless, the theory admits the existence of magnetic dipoles in the guise of current loops. (Here we shall think of realistic current loops, as in a copper wire, where the net charge in the rest frame of the wire is strictly zero.) Hence the theory must provide a means of calculating the torque on such a dipole when in motion. And indeed it does, since it is a well-defined problem in mechanics to evaluate the torque on a current loop as it moves through a static $\mathbf{E}$ field. However, *classical* mechanics yields unsatisfactory results. For the simplest case, when the plane of the loop is orthogonal to the direction of motion, the calculated torque is zero. For the force on each charge q, whether moving or not, is $q\mathbf{E}$, so the net force on any current element is zero. Classically this remains true even if the plane of the loop is not perpendicular to the motion, but then one might be tempted to mix in "some" relativity, allowing for the different length contractions of the positive and negative sets of charges in the wire.[5] Indeed this does give a torque, but in general too little.

Thus it is only *relativistic* mechanics that can—and indeed must—give the required torque. The actual calculation in the frame of the pure **E** field is tricky and involves not only the above-mentioned length contractions but also continuum mechanics somewhat along the lines of Tolman's lever.[6] However, if we are prepared to use full relativity, we might as well fall back on the archetypal relativistic technique, namely, transforming the far more easily obtainable[7] torque $\mathbf{m}\times\mathbf{B}$ in the rest frame of the current dipole **m** to the general frame. Even so, complications arise in the exact calculation, since the torque transforms awkwardly, being the 3×3 part of a 4×4 "world" tensor.

Far simpler than doing any of the above is the appeal to the monopole law (1), after replacing the current-loop dipole by a charge-pair dipole. Let us derive it from the basic assumption: A monopole q_m experiences a force

$$\mathbf{F}=q_m\mathbf{B} \tag{3}$$

when at rest. (This is certainly *consistent* with the torque on a stationary dipole being $\mathbf{m}\times\mathbf{B}$.) We must also assume that the measure q_m of a monopole is invariant, i.e., independent of its motion. Consider the usual two inertial frames S and S' in standard configuration (collinear x axes, parallel y and z axes, velocity of S' relative to S:v). Let a monopole q_m be at rest in S'. The background field $\mathbf{E}'$, $\mathbf{B}'$ acting on it in S' is the usual Lorentz transform of the field **E**, **B** in S, in particular,[8]

$$B_1'=B_1,\quad B_2'=\gamma(B_2+vE_3/c),$$
$$B_3'=\gamma(B_3-vE_2/c)\,, \tag{4}$$

with $\gamma=(1-v^2/c^2)^{-1/2}$. By hypothesis, the force on the monopole in S' is $\mathbf{F}'=q_m\mathbf{B}'$. Transforming this force[9] to S and using (4), we find

$$F_1=F_1'=q_mB_1'=q_mB_1,$$
$$F_2=\gamma^{-1}F_2'=\gamma^{-1}q_mB_2'=q_m(B_2+vE_3/c)\,,$$
$$F_3=\gamma^{-1}F_3'=\gamma^{-1}q_mB_3'=q_m(B_3-vE_2/c)\,. \tag{5}$$

With $\mathbf{v}=(v,0,0)$, we recognize Eqs. (5) to be equivalent to Eq. (1), the monopole law. (Of course, the Lorentz force law itself can be quite analogously derived from the basic law $\mathbf{F}=q\mathbf{E}$ in the rest frame, thus bearing out Einstein's pre-1905 "hunch" we mentioned earlier.)

IV. CONCLUSIONS

To what extent is a small ("point"-) charge-pair dipole equivalent to a small ("point"-) current-loop dipole? Both experience the same torque $\mathbf{m}\times\mathbf{B}$ when at rest in a given external field, as we have already noted. They also give *rise* to identical farfields.[10] By appeal to Newton's third law one might therefore be tempted to think that they must also *experience* the same force in a given external field **B**. However, in the framework of special relativity, Newton's third law is not valid—or even meaningful—in time-dependent fields. And, in fact, the forces are given by different expressions: $\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B})$ for the current loop,[11] and $\mathbf{F}=(\mathbf{m}\cdot\nabla)\mathbf{B}$ for the charge pair,[12] when at rest. These are equal *only* if $\nabla\times\mathbf{B}=0$, i.e. (in vacuum), if $\partial\mathbf{E}/\partial t=0$. The reason for the discrepancy is that the internal fields of the two kinds of dipole are different, and that these affect the forces experienced. A good way of seeing this is via the well-known formulas[13] giving the force either as a volume integral extended over the object in question or a surface integral extended over its surface, in terms of the total Maxwell stress tensor, to which the internal field of course contributes.

One may therefore replace small current loops with small charge pairs and apply to the latter the monopole law, *only* if one bears in mind the limitations spelled out above. When these limitations are exceeded, one must content oneself with regarding the results as approximate. (After all, $\nabla\times\mathbf{B}=c^{-1}\,\partial\mathbf{E}/\partial t$ is often small compared to $\nabla\mathbf{B}$.) The same applies to the use of Newton's third law in electromagnetism. For example, the force calculated by its use on the stationary charge in problem 1, though qualitatively correct, cannot be regarded as exact.

In sum, the monopole law furnishes a help to our intuition in predicting qualitatively the outcome of a large class of problems involving permanent magnets, current loops, and charges. It gives exact answers for the torque on small current loops and, under certain circumstances, also for the corresponding force.

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[1] Oliver Heaviside, "On the electromagnetic effects due to the motion of electrification through a dielectric," Philos. Mag. **27**, 324–339 (1889), Eq. (4); cited in Sir Edmund Whittaker, *A History of the Theories of Aether and Electricity* (1910, Harper Torchbooks, New York, 1960), Vol. I, pp. 310 and 396. In 1881 J. J. Thomson had given the force as half that amount. Note that Heaviside writes **AB** and *V***AB** for today's $\mathbf{A}\cdot\mathbf{B}$ and $\mathbf{A}\times\mathbf{B}$, respectively. This paper is further remarkable for containing a derivation of the exact field of a uniformly moving point charge [Eq. (29)] discovered by Heaviside the previous year.

[2] Arnold Sommerfeld, *Lectures on Theoretical Physics* (Academic, New York, 1952), Vol. III, p. 238.

[3] See R. S. Shankland, "Michelson–Morley experiment," Am. J. Phys. **32**, 16–35 (1964).

[4] These are, in fact, Exercises 1.5 and 1.6 in W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977), 2nd ed.

[5] A. P. French, *Special Relativity (The MIT Introductory Physics Series)* (Norton, New York, 1968), p. 260, or cf. Ref. 4, p. 104.

[6] See, for example, Wolfgang Rindler, *Introduction to Special Relativity* (Clarendon, Oxford, 1982), p. 164.

[7] John David Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed., p. 186, Eq. (5.71).

[8] See, for example, Ref. 4, p. 98, or Ref. 6, p. 120 (SI units!).

[9] See, for example, Ref. 6, p. 102.

[10] Reference 7, p. 182, Eq. (5.56).

[11] Reference 7, p. 185, Eq. (5.69).

[12] Since $\mathbf{F}=-q_m\mathbf{B}+q_m[\mathbf{B}+(d\mathbf{r}\cdot\nabla)\mathbf{B}]$.

[13] Reference 7, p. 239, Eqs. (6.121) and (6.122).