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Mach's principle and space–time structure

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Abstract

Mach's principle, that inertial forces should be generated by the motion of a body relative to the bulk of matter in the universe, is shown to be related to the structure imposed on space–time by dynamical theories. General relativity theory and Mach's principle are both shown to be well supported by observations. Since Mach's principle is not contained in general relativity this leads to a discussion of attempts to derive Machian theories. The most promising of these appears to be a selection rule for solutions of the general relativistic field equations, in which the space–time metric structure is generated by the matter content of the universe only in a well-defined way.

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1. Introduction

{The} investigator must feel the need of . . . knowledge of the *immediate* connections, say, of the masses of the universe. There will hover before him as an ideal an insight into the principles of the whole matter, from which accelerated and inertial motions result in the *same* way.

This rarely quoted passage from Mach (1883) is a clear, if distant, presentiment of a general theory of relativity. Mach's sought-after equivalence of accelerated and inertial motion can be related to Einstein's use of the principle of covariance in the development of general relativity. According to this, physical laws are required to take the same form in all systems of coordinates and, by implication, for all observers. Equally, Mach's insight touches on Einstein's other cornerstone of general relativity, the principle of equivalence. This, in one of its many forms, states that inertial forces generated by 'absolute' acceleration cannot be distinguished from gravitational forces in an 'inertial' frame of reference. We shall discuss this more fully in §2, where we shall find it follows that inertial forces can be considered as an aspect of gravity, and therefore dependent on the masses in the universe. Indeed, that inertial forces should be generated entirely by motion relative to matter has become the commonest abbreviation of this group of ideas that Einstein referred to as Mach's principle, and one that we too shall adopt.

Einstein's general theory is not only a theory of gravity, but also an invitation to consider dynamics from a geometrical point of view. For Newtonian dynamics such an approach was initiated by Cartan (1923, 1924) and continued more recently in the work of Trautman (1964, 1966). A complete understanding of the geometry of dynamical theories, and in particular of the relation of the equivalence principle to the geometry of space–time, has been achieved only relatively recently (Penrose 1968, Ehlers 1973, Misner *et al* 1973). It is in this context that Mach's principle finds its clearest expression. This is taken up in §2, which will serve as an introduction to the geometrisation of dynamics in general, and general relativity in particular. I shall illustrate the place of Mach's ideas in the geometry of three representative dynamical theories, those associated with Aristotle, Newton and Einstein. We shall see that Mach's principle can be taken to imply that certain geometrical structures must be determined by the distribution of matter and energy throughout space–time. In this way Mach's principle appears as an aspect of dynamical theories in general and not merely tied to general relativity.

How then does general relativity theory fare in this regard? The first attempt to demonstrate the explicit inertial effect of distant matter was made by Einstein himself (Einstein 1955). This attempt to show the change in mass of a body brought about by the presence of other bodies in the general theory failed because it turned out to be no more than an effect of the arbitrary choice of coordinates (Brans 1962). In general relativity, inertial mass is an intrinsic local and invariant property of bodies, a point to which we shall return (§5). Lens and Thirring (1918) reinterpreted the problem in terms of the induction of Coriolis and centrifugal forces and this aspect of Mach has developed a long and continuing history (for reviews see Heller (1975) and Raine and Heller (1981)). The answer to Mach's question as to how Newton's bucket 'experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick' (Mach 1883, p284) was answered by Brill and Cohen (1966) in the case that the rotating system becomes the whole universe. The result is the complete dragging of inertial frames by the rotating matter in accordance with Mach's principle (Orwig 1978).

Einstein remarked that distant matter should not just influence inertia, but should completely determine it. In a simple, non-relativistic, vector model of gravity Sciama (1953, 1969) showed how a $1/r$ acceleration-dependent inertial induction force coupling to matter with the same strength as the $1/r^2$ Newtonian force could provide the inertial forces required for Mach's symmetry between accelerated and non-accelerated motion. In a sense this leads to the integral formulation of general relativity in §6. Nevertheless, it is now well-known that general relativity does not in general encompass Mach's principle because it admits universes in which the principle does not hold (Ostvath and Schücking 1962). This can be seen clearly not only in explicit examples but in the structure of general relativity itself.

To examine this problem, in §3 we consider general relativity in its converse role as the dynamics of geometry. We shall find that Einstein's field equations provide a Hamiltonian system for the evolution of the metric on space-like hypersurfaces in space–time. The existence of free degrees of freedom for the field, the specification of which is independent of the distribution of non-gravitational energy, can be interpreted as the origin of the failure of Mach's principle in general relativity, since an aspect of the geometry becomes unconstrained by matter. An alternative view is that this elucidation of the plan of relativity provides us with a way of specifying the distribution of gravitational energy, which too is to be allowed to contribute to inertial forces (Wheeler 1964a). Isenberg (1974) in particular has suggested that space–times which arise from suitably posed initial conditions should be called Machian. This is taken up in §5.

Section 4 is devoted to the observational status of Mach's principle. This involves firstly the status of general relativity, for which I give a brief review of recent reviews. The main point here is the considerable strengthening of the observational basis for general relativity, which increasingly points to a role for Mach's principle as, at best, a selection rule within that theory. Such a role was again first proposed by Einstein with the suggestion that requiring space to be closed would eliminate the unconstrained element in the geometry and thereby implement Mach's principle (§5). Secondly, Mach's principle, embracing as it does the whole universe, is confronted by cosmological evidence. Thus I consider in §4 the extent to which the observed universe can be seen to satisfy Mach's principle.

Attempts to express Mach's principle other than through general relativity are considered in §5. Any successful alternative must violate one or more of the assumptions leading to general relativity, while remaining compatible with the observations. It is clear from the preceding discussions that these represent severe constraints. Several attempts have been made to generate inertial masses, rather than inertial forces, by interaction with the universe, and in such cases inertial mass clearly ceases to be a local invariant property. A favourite red-herring in this context is the arbitrariness of the choice of fundamental units. In the most viable of these theories (the Brans–Dicke theory or theories closely related to this) it is a strong version of the equivalence principle that is violated. But there is increasing observational evidence against this theory, and it is not in any case at all clear that it is any more successful than general relativity in incorporating Mach's principle. Perhaps the least compelling aspect of general relativity is the field equations. It is not, after all, necessary that these should agree formally with the Newtonian limit; only an agreement with observation is mandatory (for all but the most extreme Machians). Thus alternative generalisations of Newtonian theory may be possible. Finally the most ambitious recent attempt at a reformulation is that of Barbour (1974) who seeks to construct a dynamics using the space–time structure appropriate to a purely relative concept of motion.

In §6 I return to the implementation of Mach's principle as a selection rule on solutions of the field equations of general relativity through the integral equation approach pioneered by Al'tschuler (1967) and Lynden-Bell (1967). An integral formulation of the field equations provides a separation of the gravitational potential (and hence inertial forces) into a part due to material sources and a part due to arbitrary boundary conditions. Mach's principle can then be imposed as a selection rule in the sense that in physically acceptable space–times satisfying Einstein's equations the contribution from the boundary terms must vanish when the contributions from all the material sources are included. In principle this provides a well-defined meaning to the condition that inertial forces should be generated entirely by motion relative to matter, but there are technical difficulties associated with this approach which are reviewed. I then outline some new work in which I attempt to re-express the field equations in an integral form that admits only Machian solutions.

It should be clear that this review is intended to set the ancient and venerable principle of Mach in the context of recent developments in gravity theories. I aim, in particular, to make these developments accessible to interested readers with the minimum of technicalities. In the final section (§7) I discuss what one might call the psychological power of Einstein's presentation of Mach's ideas and the reasons for the continuing interest in them. Appendix 5 contains a summary of notation and conventions.

2. The geometry of dynamics

Dynamical theory is concerned with a comparison between the motion of bodies subject to forces and force-free motions. To begin dynamics then, it is necessary to specify the force-free motions of particles, and this is the task of a 'first law'. How such motions should relate to the matter content of the universe we might call the *Mach problem.*

It turns out that a first law of dynamics is a statement about the geometrical structure of space–time. The geometry of space–time, in turn, is investigated by observation of the motion of bodies and the behaviour of clocks in much the same way that one thinks of the geometry of space as related to the results of measurements with rigid rods. In this way the Mach problem becomes a question of the origin, or determination, of an aspect of geometry.

To illustrate these ideas in a sufficiently broad context I start with an account of what is essentially Aristotle's theory of dynamics, and go on to consider the Newtonian and relativistic theories. It will be seen that the last can be introduced in a fairly straightforward way given a suitable formulation of the Newtonian picture. A fuller interpretation of the evolution of ideas concerning dynamics and space–time geometry is given in Raine and Heller (1981).

2.1. Aristotelian dynamics

The first law of Aristotelian dynamics is:

A body subject to no forces retains its state of absolute rest.

That this is observed to be wrong is beside the point. We shall see that it leads to a coherent dynamics of relevance to the subsequent discussion. More immediately it leads to a structure for the Aristotelian space–time manifold $\mathscr{A}$. One can think of this manifold heuristically as the set of space–time events $\{a\}$, with a concept of smoothness in passing from one event to another but without, as yet, any quantitative measure of distances between points of $\mathscr{A}$. (For a more extensive discussion see, for example, Raine and Heller (1981) or Davies (1977); for mathematical details see, for example, Hicks (1965), Kobayashi and Nomizu (1963) or any book on differential geometry.) As to the geometry of $\mathscr{A}$ imposed by the dynamics we see that bodies at rest define 'the same place at different times' and hence an absolute meaning for location. This provides a projection π_Σ of any event in $\mathscr{A}$ to a three-dimensional manifold, Σ, of spatial locations (figure 1).

Further, we may assume that clocks associated with bodies at rest run at the same rate, so we can give a meaning to 'the same time at different places'. This provides an absolute simultaneity which associates a time $t=\pi_T(a) \in T$ to any event $a \in \mathscr{A}$. Thus Aristotelian space–time is a product manifold, $\mathscr{A}=\Sigma \times T$, of Euclidean space and time. Equivalently, the first law asserts the existence of a global privileged vector field on $\mathscr{A}$, which provides a foliation of the space–time into a continuous sequence of three-dimensional hypersurfaces orthogonal to the vector field.

Aristotle's second law, interpreted in modern terms†, gives the velocities, v, induced by 'forces', f, acting on a body of mass m as

$$f=mv \tag{2.1}$$

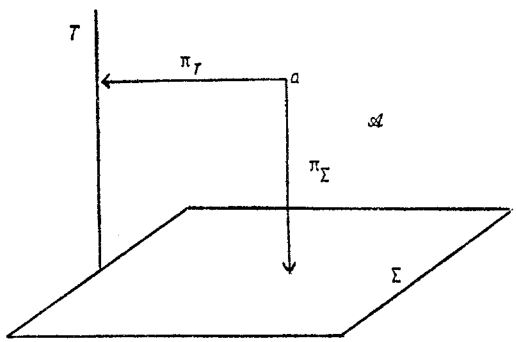

Figure 1. The projections, π_Σ, π_T of Aristotelian space–time, $\mathscr{A}$, into space, Σ, and time T.

and is, of course, consistent with the space–time structure just described, since f is zero if and only if v is zero. The velocities in the law are absolute velocities, measured by observers at absolute rest. To an observer with velocity u the law becomes

$$f+mu=mv' \tag{2.2}$$

where $v=v'-u$. We see that what, by analogy with Newtonian theory, we should call inertial 'forces', mu, have to be introduced to balance the equation.

What causes these inertial 'forces'? According to Mach, since we can observe motion only of matter relative to matter, they must arise from motion with respect to the rest of the universe. However, in Aristotle's theory, the immediate cause of the inertial 'forces' is motion relative to the privileged vector field in space–time. The two views can be reconciled only if the foliation of space–time is itself generated by matter. How then is the privileged field to be identified by observation? In principle, we are supposed to be able to determine which bodies are subject to no 'forces', thus to distinguish between

† There is a large medieval literature on the correct formulation of this law resulting essentially from confusion due to the intercession of static friction (and the fact that the law is not true anyway). This is not relevant to us.

on the one hand the case $v'=0$ because $f=0$ and $u=0$, and on the other $v'=0$ because $f=-mu\neq 0$. This presents problems because certain natural 'forces' are defined for Aristotle in terms of the motion they produce. In practice, the Earth is taken to provide an independent absolute standard of rest. This does not, however, provide a causal relation between the Earthly standard of rest and the induction of inertial forces, since the dynamical laws do not require the existence of the Earth in order to operate.

Nevertheless, the relation between the location of the Earth and the geometry of $\mathscr{A}$ is not entirely arbitrary. The question as to why the Earth should be situated at the centre of the universe leads to what one might regard as the Aristotelian substitute for a theory of gravity. According to this, there is a distinction between *natural* motions (due to natural 'forces') and unnatural, forced motions. Natural motions are of two kinds: the regular circularity of the heavens beyond the lunar sphere, and the purely radial motions beneath. We can describe this by adding vector fields to $\mathscr{A}$ to represent the natural velocity of a particle at any point. Bodies couple to these fields with appropriate strengths (so airy bodies have negative masses and rise, Earthly bodies fall). This results in the agglomeration of Earthly material at the centre of the universe and for a suitable choice of vector fields explains the location of the Earth. But the Earth merely responds to the geometry; it does not generate it. Therefore Aristotelian dynamics does not satisfy Mach's principle.

2.2. *Newtonian dynamics*

According to Newton's first law bodies subject to no forces are not necessarily at rest. This suggests that the Aristotelian space–time structure is not appropriate to Newtonian theory. In particular, it follows from the first law that a whole class of observers in uniform relative motion, the *inertial observers*, are privileged observers with respect to the dynamical laws, and that only this class as a whole should be picked out in the space–time structure. It follows that there can be no absolute location and hence no absolute space in Newtonian theory. On the other hand, there is still an absolute time and absolute acceleration. The latter is correctly demonstrated by Newton's bucket, in which the curvature of the fluid surface is a measure of the absolute acceleration, despite a long literature of criticism. The objection of Mach and Berkeley, that one should look to the effect of the stars, is not relevant, since we are dealing here with a local theory for which there need be no stars. Note that the existence of absolute space does not follow from absolute acceleration, a circumstance which is responsible for much of the confusion in the discussion of Newtonian absolutes. On the other hand, absolute acceleration does imply an absolute time, since the vanishing of the acceleration, $\mathrm{d}^2x/\mathrm{d}t^2=0$, is a condition which is invariant under only a trivial rescaling of time, $t\to at+b$, a and b constants.

What then is the geometrical structure of Newtonian space–time? To see this, note that the vanishing of acceleration connects space–time *velocities* at different events, so we can refer to 'the same velocity at different points' (in contrast to an absolute zero of velocity which connects space–time *points*). We might therefore seek to describe the geometry in terms of a vector field on a space of velocities. But this is not quite the whole story, since zero acceleration implies zero rotation of the local frame of reference, as well as zero linear acceleration. So we want to describe Newtonian geometry in terms of a connection between moving frames.

This can be thought of in terms of privileged vector fields, as in the Aristotelian theory, if we imagine these fields on a space, the points of which are each reference frames at a point of space–time (figure 2).

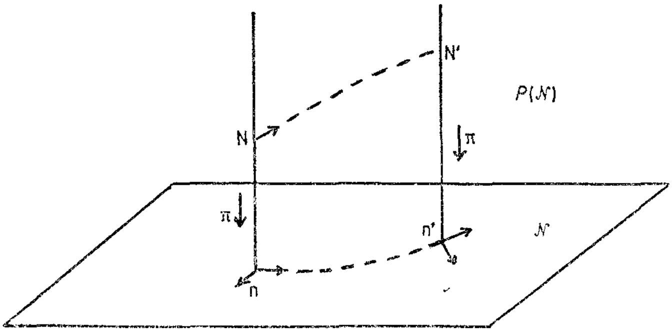

Figure 2. The affine connection on the frame bundle $P(\mathcal{N})$ of Newtonian space–time, $\mathcal{N}$. The fibre, $\pi^{-1}(n)$, over $n \in \mathcal{N}$ is a $4(4-1)=12$-dimensional space if $\mathcal{N}$ has four dimensions. A curve, NN′ in $P(\mathcal{N})$ projects into a curve, nn′, and a parallel propagated frame in $\mathcal{N}$.

The appropriate space is the frame bundle $P(\mathcal{N})$ to the Newtonian space–time manifold $\mathcal{N}$. Each point N of $P(\mathcal{N})$ consists of a point $n \in \mathcal{N}$ and a frame of reference, a set of four independent (unit) vectors, at n (figure 2). Since four unit vectors are specified by $16-4=12$ numbers, each point in $P(\mathcal{N})$ is specified by $4+12=16$ coordinates. There is an obvious projection π from $P(\mathcal{N})$ into $\mathcal{N}$ at each point, $\pi : N \to n$, and the subspace $\pi^{-1}(n)$ is called a fibre of $P(\mathcal{N})$. $P(\mathcal{N})$ can be shown to be a differential manifold, so we can consider tangent vector fields on $P(\mathcal{N})$. The zero-acceleration tangent vector fields on $P(\mathcal{N})$ connect non-rotating frames in uniform relative motion and, in particular, inertial frames at different points. There are four such independent zero-acceleration vectors at each point. One connects non-rotating frames along the space–time worldline in $\mathcal{N}$ of an inertial observer. The remaining three can be taken to connect parallel frames for different observers at the same absolute time (or, equivalently, frames on inertial trajectories in the limit of infinite velocity). In the space sections of absolute simultaneity of $\mathcal{N}$ this concept of parallel frames must agree with the standard parallelism of vectors in Euclidean geometry. By extending our concept of geometry to measurements in space–time, we obtain an extension of the idea of parallelism to space–time frames of reference.

One can think of the privileged (inertial) observers in Newtonian theory, those having zero absolute acceleration, as moving along curves in $P(\mathcal{N})$ to which a privileged vector field is tangent. This geometrical picture can also be described in terms of the components of an *affine connection* on $\mathcal{N}$. Vectors u^μ, $u^\mu + \delta u^\mu$ at adjacent points x^μ, $x^\mu + \delta x^\mu$, on the projection of such a curve in $\mathcal{N}$ will be *parallel* if they have the same components in the reference frame of a privileged observer at the two points. The most general form for δu^μ depends linearly on u^μ and δx^μ, so

$$\delta u^\mu = -\Gamma^\mu{}_{\lambda\nu} u^\lambda \, \delta x^\nu$$

where the negative sign is conventional. The $\Gamma^\mu{}_{\lambda\nu}$ are referred to as the components of an affine connection. A vector undergoes parallel transport therefore if

$$\partial u^\mu / \partial x^\nu + \Gamma^\mu{}_{\lambda\nu} u^\lambda = 0 \qquad (2.3)$$

since this is the condition that its components are constant with respect to a parallel-transported frame.

In particular, inertial frames are specified by zero acceleration, and hence must satisfy the condition that the frame velocity vector, $\mathrm{d}x^\mu/\mathrm{d}t$, remains parallel to itself along its trajectory. From equation (2.3) with $u^\mu = \mathrm{d}x^\mu/\mathrm{d}t$, this condition is

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}t^2} + \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0$$

in an arbitrary coordinate system. Inertial *coordinates* are defined by the requirement that inertial trajectories take the form

$$\mathrm{d}^2x^\mu/\mathrm{d}t^2 = 0$$

and hence by the requirement $\Gamma = 0$. That the coordinates can be adapted to the space–time structure in this way is useful for calculation, but is also a source of confusion, since it tends to hide the geometric structure behind the choice of coordinates.

In non-inertial frames the Γ represent inertial forces. As an example, consider a coordinate system (t, ξ, η, ζ) which has uniform acceleration g relative to an inertial system (t, x, y, z), so

$$\xi = x + \tfrac{1}{2}gt^2.$$

We obtain the only non-zero Γ,

$$\Gamma_{00}{}^x = -g$$

and the equation of an inertial trajectory for a particle of mass m:

$$m\,\mathrm{d}^2\xi/\mathrm{d}t^2 = mg. \tag{2.4}$$

The resemblance of this to a particle moving under gravity will be of significance presently.

To summarise then, the space–time of Newtonian dynamics, $\mathscr{N}$, has a projection π_T on to absolute time, T, with associated spaces of absolute simultaneity $\{\pi_T{}^{-1}(t) \mid t \in T\}$ having Euclidean metric. The frame bundle $P(\mathscr{N})$ has privileged vector fields which describe an affine connection and the associated concept of parallel vectors. The connection relating vectors at non-simultaneous points arises from dynamics and is unrelated to the space metric. For simultaneous events the usual parallelism provided by the metric and by the connection are required to be the same. For a useful brief review of the geometry of $\mathscr{N}$ see Kuchǎr (1981).

We have still to discuss how the specification of Γ is to be related to physical coordinate systems. The situation is similar to that in Aristotelian dynamics. The appropriate choice cannot be made locally unless one has a way of identifying bodies subject to no forces, and we shall see shortly that as a consequence of the principle of equivalence this is not possible. The theory is usually made to work by the extraneous introduction of the fixed stars, analogous to Aristotle's fixed Earth. For Mach, of course, this step was not extraneous, since he insisted that one should not ask for more than the relation of appearances in the presence of the universe as it is. We tend to be more demanding, since only the theory itself can tell us how the universe is.

Consider now what happens if we introduce gravity. Relative to the fixed stars we see apples fall with uniform acceleration. Newton would have us ascribe this to the force of gravity exerted by the Earth. But to confirm this by local experiment, we would have to turn off the action of gravity and compare the inertial motion that would then arise. Unfortunately, this is not possible; according to the Galilean principle of equivalence, *all bodies fall with equal acceleration under gravity independent of composition.* In the absence of a supererogatory reference to the fixed stars, it follows from equation (2.4) that we could equally attribute the apparent action of gravity to non-inertial behaviour in a uniformly accelerated frame of reference.

The distinction between gravitational and inertial forces in Newtonian theory is therefore ambiguous and depends on the frame of reference. To obtain a dynamical theory in the presence of gravitating matter we reverse the preceding argument. From the principle of equivalence we deduce that to a freely falling observer, i.e. one subject to no *non*-gravitational forces, bodies behave locally according to standard Newtonian dynamics. For this is simply a reversal of the transformation leading to equation (2.4). By the local behaviour of bodies we imply that in a mathematical development we should consider an infinitesimal neighbourhood (or take the appropriate limit); from a physical point of view we may consider a neighbourhood sufficiently small that departures from uniformity are not detectable to some pre-assigned accuracy. In the presence of gravity, then, it is the freely falling frames of reference that play the role of locally unaccelerated frames as far as the local dynamical laws are concerned. It is therefore these frames that must be connected by privileged vector fields on the frame bundle.

Four distinguished vector fields on the frame bundle again lead to the existence of 40 components of the affine connection, $\Gamma^{\lambda}{}_{\mu\nu}$, as before. The velocity vector tangent to a free-fall trajectory in $\mathcal{N}$ must be parallel to itself at events along the trajectory, since the path connects frames which are locally unaccelerated. This gives

$$\frac{d^2x^{\lambda}}{ds^2}+\Gamma^{\lambda}{}_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}=0 \qquad (2.5)$$

as the equation of free falls. Here s is a parameter along the path and we can take $x^0=t$ as usual. Of course, the existence of an absolute time leads to a simplification if we take $s=t$, and then we must have $\Gamma^{0}{}_{\mu\nu}=0$. However, the existence of a system of coordinates in which all the components of Γ vanish would imply the complete absence of gravitational effects. The presence of gravitating matter causes the convergence and divergence of free falls ('tidal forces'), as a result of the non-uniformity of the gravitational acceleration, and these can be detected on a sufficiently large (non-local) scale. Now therefore there can be no global system of coordinates in $\mathcal{N}$ in which the components of Γ vanish. In this case the affine connection is said to be non-integrable.

In the absence of gravity we saw that we can use the free falls, which are inertial motions in this case, to define a Cartesian system of coordinates in space–time. The tangent vector fields to the coordinate lines form a global system of parallel vectors, hence a unique association of frames in $P(\mathcal{N})$—the parallel frames at different points of $\mathcal{N}$. One can view this as providing a continuous map $\mathcal{N}\to P(\mathcal{N})$ by $n\to\{n, e_a(n)\}$, where the $\{e_a(n)\}$ are parallel to some fiducial frame $\{e_a(0)\}$ at an arbitrary origin $0\in\mathcal{N}$. A continuous map $\mathcal{N}\to P(\mathcal{N})$ is called a *section* of $P(\mathcal{N})$. In the presence of gravity the deviation of free falls leads to a dependence of the parallel transport of a vector on the path along which it is transported, hence to no unique (path-independent) way of defining the same frame at different points of $\mathcal{N}$. The presence of gravitating matter is revealed therefore through the absence of distinguished global sections of $P(\mathcal{N})$.

To connect up with the usual description of Newtonian gravity we note that there must exist a coordinate system—Galilean coordinates—in which equation (2.5) takes the form

$$\frac{d^2x^i}{dt^2}-\frac{\partial\varphi}{\partial x^i}=0$$

where φ is the gravitational potential (Misner *et al* 1973). Thus, in Galilean coordinates,

$$\Gamma^{i}{}_{00}=\partial\varphi/\partial x^i$$

and the remaining components vanish. Again, this coordinate system is usually identified with that naturally associated with the fixed stars. We see that the appropriate components of the affine connection are related to the matter distribution through Poisson's equation, $\nabla^2\varphi = 4\pi G\rho$.

In the absence of gravity the connection appears as an absolute geometrical element. Introducing gravity we obtain a relation between the connection and the distribution of matter. The position of Mach's principle in the dynamical theory can therefore now be readily clarified. According to Mach, the connection should be determined completely by the distribution of matter.

Since Poisson's equation admits source-free solutions, the inertial behaviour of bodies is not in this theory automatically determined by matter. The question arises as to whether a different theory, for example a replacement for Poisson's equation, might be constructed to admit solutions only for universes in which Mach's principle is satisfied. We shall not take up the question in this context. General relativity does indeed provide an alternative to Poisson's equation more appropriate to the discussion of cosmology and is considered next.

2.3. *General relativity*

In the space–time of special relativity, $\mathscr{M}$, we no longer have the absolute time of $\mathscr{N}$, of course, but we gain instead a proper time, s, as the measure of time-like separation, and with it a space–time metric. This brings about a remarkable simplification, since in relativistic dynamics the affine connection required to describe the inertial trajectories turns out to be related to the space–time metric! Indeed, the Minkowski coordinates (ξ^μ), $\mu = 0, 1, 2, 3$ are precisely those in which the metric coefficients take on the standard form, $(\eta_{\mu\nu}) = \mathrm{diag}\,(-1, 1, 1, 1)$, *and* the inertial paths of particles are straight lines, in the sense that along an inertial path $\xi^\lambda(s)$

$$d^2\xi^\lambda/ds^2 = 0. \tag{2.6}$$

It is then easy to show, by transforming to new coordinates $x^\lambda = x^\lambda(\xi^\mu)$, that equation (2.6) takes the form

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{2.7}$$

where the components of the connection, $\Gamma^\lambda{}_{\mu\nu}$, are related to the transformed metric coefficients,

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial\xi^\mu}{\partial x^\alpha}\frac{\partial\xi^\nu}{\partial x^\beta}$$

by

$$\Gamma^\lambda{}_{\mu\nu} = \tfrac{1}{2}g^{\lambda\alpha}\left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right). \tag{2.8}$$

The privileged vector fields on $P(\mathscr{M})$ provide a natural global section of $P(\mathscr{M})$ by the same argument as before but using Minkowski coordinates as the special global coordinate system. The affine connection is again an absolute geometrical element imposed on space–time by the first law of (relativistic) dynamics embodied in equation (2.6). The equivalence principle again provides a relation between inertial forces and gravity and leads to a geometric theory of gravitational forces.

In the presence of gravitating matter, the Galilean equivalence principle of §2.2

translates here into the *universality of free fall*, also called the *weak equivalence principle.* According to this:

> uncharged test particles projected from a given point with a given velocity follow a worldline in space–time independent of their composition

(see Thorne *et al* 1973, Will 1974). If particle dynamics were the whole of relativistic physics an argument parallel to that of §2.2 would show that gravity must be described by a non-integrable connection. But the universality of free fall does not directly rule out the possibility that by non-dynamical experiments involving, say, radio waves or nuclear forces, a local distinction between gravity and acceleration might be established.

The proposition that this is not the case is called the *strong principle of equivalence.* According to this:

> *all* the non-gravitational laws of physics take on their special relativistic form in a local freely falling frame.

There is some direct experimental evidence for this strengthened version (§4). However, there is manifestly a difficulty in that we do not yet know all the laws of physics. Nevertheless, Schiff (1960) has conjectured that the existence of a fundamental interaction violating the strong principle would entail a macroscopic violation of the universality of free fall for a test body composed of particles subject to this interaction. A restricted version has been proved by Lightman and Lee (1973).

From the strong principle of equivalence it can be shown that the motion of test bodies under gravity is described by a connection derived from the space–time metric. We summarise this by saying that we have a *metric theory of gravity.* For the existence of a non-integrable connection follows, *mutatis mutandis*, as in Newtonian theory. The relation to the metric follows because at each point equation (2.6) must hold for free falls in local Minkowski coordinates. Therefore in general coordinates we obtain equation (2.7) provided equation (2.8) holds. This must be true everywhere since both the fiducial point and the coordinate system there are arbitrary. Indeed one readily sees that the laws of physics in a gravitational field are obtained from their Minkowski form by transforming to a general frame of reference from a local freely falling one in a neighbourhood of an arbitrary point. (For a more detailed review see Will (1974, 1979).)

Strictly, the relation (2.8) holds only if we assume that the connection coefficients are symmetric, $\Gamma^{\lambda}{}_{\mu\nu}=\Gamma^{\lambda}{}_{\nu\mu}$. An antisymmetric part of Γ, the *torsion* of the connection, would not contribute in equation (2.7), but might couple to spin. Indeed in the Sciama–Kibble theory (Sciama 1962, Kibble 1961), originated by Cartan (1922, 1923) and revived and developed by Trautman and others (see Kuchowicz 1975), torsion of space–time is generated by a spin density of matter. No observations rule out this theory—essentially because there appear to be no accessible circumstances in which the effects of torsion are significant—but it provides a viable exception to the statement that the geometry of space–time must be described *entirely* by a metric. Thus, this statement is *not* what is meant by a metric theory.

Summarising, then, we see that the strong principle of equivalence (incorporating special relativity) leads to a metric space–time with test particle motions described by a connection derived from the metric. Thus, if we accept the strong principle of equivalence, Mach's principle requires that the *metric* be determined solely by the distribution of matter. How this is supposed to come about depends on one's choice of metric theory. In most of the following I shall take this to be general relativity (see §4). Some other theories will be considered in §5. All of those of which I am aware do not differ from general relativity in their failure to incorporate Mach's principle.

3. The dynamics of geometry

In metric theories of gravity the geometry of space–time itself becomes subject to dynamical laws. In general relativity the dynamics is controlled by the energy–momentum density of matter, $T_{\mu\nu}$, through the Einstein field equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa T_{\mu\nu}$$

and in other theories by other field equations with, perhaps, additional fields. Convenient as this form of the equations is for expressing the covariance under general coordinate transformations, in order to exhibit the dynamical structure a Hamiltonian formulation is more appropriate. According to this point of view, one looks at the geometry of three-dimensional space-like surfaces evolving in time. We shall assume throughout that the space–times with which we are dealing are globally hyperbolic (e.g. Hawking and Ellis 1973). This is equivalent to assuming a global topology $\Sigma\times\mathbb{R}$ where Σ is a space-like hypersurface (Geroch 1970). (But it is not equivalent to an absolute time (!) since the product structure does not arise in any unique, physically significant, or natural way.)

It was Einstein who first noted that a generally covariant field theory cannot lead to a unique determination of the field variables. For the field equations can only determine the metric coefficients up to a coordinate transformation. This means that the field equations cannot be independent; in fact, they satisfy the four *contracted Bianchi identities*, which in turn reflects the conservation of energy and momentum. It follows that the field equations must provide constraints on the field variables which represent the data on an initial hypersurface, since otherwise the ten equations would determine the time development of ten metric coefficients uniquely. The development of a Hamiltonian formulation of field theories with constraints in a Poisson bracket formalism is due to Dirac (1958). This was simplified and given an action principle form by Arnowitt *et al* (1962). We shall briefly review this now well-known analysis (see, for example, Isenberg and Nester 1980) to the point at which it becomes clear why general relativity does not satisfy Mach's principle.

3.1. Parametrised dynamics

We begin with a simple analogy (Kuchǎr 1973). A covariant form for the motion of a particle in a fixed curved space–time specified by coordinates q^{μ} is

$$\delta\int(g_{\mu\nu}q'^{\mu}q'^{\nu})^{1/2}\,\mathrm{d}\lambda=0 \tag{3.1}$$

where the primes denote differentiation with respect to the parameter λ. This action principle is invariant under changes in parametrisation. To put it into first order (Hamiltonian) form, we define

$$p_{\mu}=\partial\mathscr{L}/\partial q'^{\mu}. \tag{3.2}$$

To obtain the correct equations of motion for a particle of mass m we have to vary subject to the constraint

$$\mathscr{H}\equiv g_{\mu\nu}p^{\mu}p^{\nu}+m^2=0. \tag{3.3}$$

If we adjoin this to the variational principle by means of a Lagrange multiplier, N, we obtain

$$\delta\int\{p_\mu q'^\mu - N\mathscr{H}\}\,\mathrm{d}\lambda=0 \tag{3.4}$$

with free variations. The meaning of N is obtained by variation with respect to p_0, using the definition of proper time, s. This yields

$$N=\frac{1}{2m}\frac{\mathrm{d}s}{\mathrm{d}\lambda} \tag{3.5}$$

so N, called the *lapse function*, gives the rate of lapse of proper time with respect to an arbitrary parameter. $\mathscr{H}$ is sometimes called the *superHamiltonian* and (3.3) the superHamiltonian constraint.

The theory is *deparametrised* by choosing the coordinate time $t\equiv x^0$ in place of λ; we obtain

$$\delta\int\{p_i\dot{q}^i - H\}\,\mathrm{d}t=0$$

where $i=1, 2, 3$, a dot denotes differentiation with respect to t, and

$$H=(g_{ij}p^ip^j+m^2)^{1/2}$$

is the standard Hamiltonian expressed in terms of the dynamical variables (p^i, q_i). The superHamiltonian constraint (3.3) has the form $-p_0{}^2+H^2=0$, equivalent to $-p_0+H=0$.

3.2. *Parametrised field theory*

A variational principle for Einstein's equations is

$$\delta\int\{R+\mathscr{L}_{\mathrm{m}}\}\sqrt{-g}\,\mathrm{d}^4x=0 \tag{3.6}$$

where $\mathscr{L}_{\mathrm{m}}$ is the matter Lagrangian. This too is a parametrised theory, since the action is invariant under arbitrary transformation of the space–time coordinates. The lapse of proper time and distance with respect to the parameters, (x^μ), is expressed in four functions called the lapse, N, and shift N^i (Wheeler 1962). Of course, coordinate displacements are related to changes in proper distances through the metric, so the N, N^i must be related to the metric coefficients. In fact,

$$\mathrm{d}s^2=g_{ij}(\mathrm{d}x^i+N^i\mathrm{d}t)\,(\mathrm{d}x^j+N^j\mathrm{d}t)-(N\mathrm{d}t)^2$$

where g_{ij} is the metric of the $t=$constant hypersurfaces. We therefore expect that equation (3.6) defines the Hamiltonian evolution of the geometry of three-dimensional hypersurfaces, that the lapse and shift should appear as the Lagrange multipliers of four constraints, and that time derivatives of N and N^i should not appear.

Discarding a total divergence Arnowitt *et al* (1962) (see also Misner *et al* 1973) find that equation (3.6) reduces to

$$\delta\int\{\pi^{ij}\dot{g}_{ij}-N\mathscr{H}-N^i\mathscr{H}_i+N\sqrt{g}\,\mathscr{L}_{\mathrm{m}}\}\,\mathrm{d}^3x\,\mathrm{d}t \tag{3.7}$$

analogous to (3.4). Here π^{ij} are momentum densities conjugate to the three-space metric coefficients, g_{ij}, $g=\det g_{ij}$, and the g, π and N and N^i are to be varied freely. Of course, variation of the g_{ij} gives the definition of the π^{ij} as usual. Using explicit forms for $\mathscr{H}$ and $\mathscr{H}_i$ given below, we obtain

$$\pi^{ij}=\sqrt{g}(g^{ij}\,K-K^{ij}) \qquad K=g_{ij}K^{ij}=\tfrac{1}{2}g^{-1/2}\pi \tag{3.8}$$

where K^{ij}, the exterior curvature, is given by

$$K_{ij} \equiv -\tfrac{1}{2} \underset{\partial/\partial t}{\mathscr{L}} g_{ij} = \tfrac{1}{2} N\,[N_{i|j} + N_{i|j} - \partial g/\partial_{ji} t].$$

Note that g^{ij} here is defined as the matrix inverse of g_{ij}, not the spatial part of the space–time metric $g^{\mu\nu}$.

Thus, Einstein's equations take the Hamiltonian form

$$\frac{\partial g_{ij}}{\partial t} = \frac{\delta H}{\delta \pi^{ij}} \qquad \frac{\partial \pi^{ij}}{\partial t} = -\frac{\delta H}{\delta g_{ij}} \tag{3.9}$$

where the Hamiltonian, H, is defined by

$$H = \int \mathrm{d}^3 x (N\mathscr{H} + N^i \mathscr{H}_i + \mathscr{H}_{\mathrm{m}})$$

and $\mathscr{H}_{\mathrm{m}}$ is the Hamiltonian constructed from the matter variables. Variation of the lapse and shift leads to four constraints:

$$\mathscr{H} \equiv g^{-1/2}\{\pi^{ij}\pi_{ij} - \tfrac{1}{2}\pi^2\} - g^{1/2} R = 2\kappa g^{1/2} T^{\mu}{}_{\nu} n_{\mu} n^{\nu} \tag{3.10}$$

$$\mathscr{H}^i \equiv -2\pi^{ij}{}_{|j} = 2\kappa g^{1/2} T^{\mu i} n_{\mu}. \tag{3.11}$$

Here R is the Ricci scalar of the metric g_{ij}, and $\pi^{ij}{}_{|j}$ is the covariant divergence using the connection defined by the metric g_{ij}.

Equation (3.7) is analogous to equation (3.4). Ideally we should like to proceed to deparametrise the theory to arrive at the true dynamical variables of the gravitational field. Geometrically, such a reduction of the variables is related to the choice of coordinates in an initial hypersurface and the location of that hypersurface in time (analogous to $t = \lambda$ above). We can view this alternatively as a solution of the constraint equations and the elimination of the solved-for variables (analogous to solving $\mathscr{H} = 0$ for p^0). Choice of coordinate conditions for all time defines the Lagrange multipliers as functions of time ($t = \lambda$ implies $N = (1/2m)\,\mathrm{d}s/\mathrm{d}t$ in the particle example). Conversely, a choice of the multipliers determines how the coordinate conditions propagate.

In general relativity it is not immediately apparent that these approaches mesh together in a simple way. In general, constraints might be non-integrable, preventing the explicit elimination of a particular variable; equations might not possess solutions, or stable ones; coordinate systems might break down, becoming singular. Thus the explicit elaboration of the dynamics of geometry requires some care and is as yet incomplete. Nevertheless, it is easy to enumerate the number of degrees of freedom of the field. In the particle case, the one constraint leads to the elimination of one component of momentum and we say the particle has three degrees of freedom. In general relativity, the six momenta, π^{ij}, are restricted by four constraints, (3.10) and (3.11), leaving two components freely specifiable initially at each space point. By analogy with the relativistic particle, we say the field has two degrees of freedom at each point in space.

The future development of the geometry of an initial hypersurface, Σ_0, is governed by the evolution equations, (3.9), which leave free only the development of the coordinate system. However, it is clear that the existence of two free degrees of freedom of the field imply that the initial geometry of Σ_0 is not entirely determined by the energy and momentum of matter. The future development of Σ_0 is therefore a space–time which does not satisfy Mach's principle, and this shows, in principle at least, how general relativity does not encompass Mach's principle. Of course, the data on Σ_0 might be determined by matter at earlier times, and in practice, in realistic non-Machian space–

times one might require that an initial time be specified in some natural way. In fact, in most of the explicit solutions that exhibit non-Machian properties, the true degrees of freedom of the gravitational field vanish anyway, and it is therefore important to see that there is a further non-Machian aspect to the field equations.

The further analysis of the dynamics of general relativity divides into two problems. These are the solution of the constraint equations for suitable initial data and the evolution of the data off of the initial hypersurface. For the existence and stability of solutions of the field equations it proves expeditious to use the harmonic coordinate condition:

$$\frac{\partial}{\partial x^{\mu}} [(-g)^{1/2} g^{\mu\nu}] = 0$$

since this yields a form for the field equations which is hyperbolic and preserves the constraints and harmonic condition in the evolution (Choquet-Bruhat 1962, Choquet-Bruhat and York 1980). This choice of coordinates is not apparently useful for interpretation of the structure of the equations, since it does not appear to have a geometrical significance.

In the 'thin sandwich' formulation one attempts to understand the space–time metric as a filling-in of the four-dimensional geometry between the three-geometry on two space-like hypersurfaces (Wheeler 1964b). In the limit of an infinitesimally thin sandwich the three-metric on an initial surface g_{ij}, and its time rate of change, $\partial g_{ij}/\partial t$, are regarded as given and the constraints solved for initial values of N and N^i. Subsequently the lapse and shift are regarded as freely specifiable and the three-metric determined by the evolution equations. However, regarded as initial value equations for N and N^i, the constraints are not elliptic equations, so standard theorems cannot be used and there may be difficulties (Fischer and Marsden 1979a, b, Christodoulou and Francaviglia 1979). Thus the thin sandwich version is no longer popular.

Note that the free choice of lapse and shift in the evolution is not in any case a good approach. The simplest choice $N=1$, $N^i=0$ leads to breakdown of the coordinate system by focusing. One can either fix coordinate conditions (analogous to $q^0=t$ in §3.1) in the initial surface, and determine (N, N^i) by requiring that these conditions be preserved by the evolution to succeeding surfaces (Arnowitt *et al* 1962). Or one can impose an anti-focusing condition on the momenta which leaves the initial choice of coordinate system free but determines its development through the N and N^i (e.g. York 1979). A mixture of methods is, of course, also conceivable. The antifocusing condition $\pi=0$ (Lichnerowicz 1944, Dirac 1959, Choquet-Bruhat 1962) is possible in asymptotically flat spaces, but at most one hypersurface satisfying this condition exists in the case that Σ is compact and without boundary ('closed' universes). The condition $K=$constant can be used in general.

The field variable canonically conjugate to K is $g^{1/2}$ (where $g=\det g_{ij}$). By a canonical transformation we can interchange K and $g^{1/2}$ so $g^{1/2}$ can play the role of a momentum variable to be solved from the Hamiltonian constraints (3.10) (York 1972). With $\pi=2g^{1/2}K$ fixed, the momentum constraint (3.11) is a condition on the longitudinal (='non-divergence-free') part of $(\pi^{ij}-\frac{1}{3}g^{ij}\pi)$, the trace-free part of the momentum. This leaves the transverse (=divergence-free) components of the traceless momentum as the two freely specifiable components of momentum.

In the non-compact case the solution of the constraints depends on the imposition of non-Machian boundary conditions, and even for compact initial slices there is no guarantee that the Hamiltonian constraints do not possess non-Machian solutions. In fact, of course, Minkowski space–time arises as just such a non-Machian solution. It is difficult to see how one might impose Machian conditions directly on the constraints, since there is no natural way to separate a source-free component in the solution of a non-linear equation. Furthermore, from a practical point of view, the theory has been developed mainly for the compact and asymptotically flat cases and is therefore not appropriate to many of the cosmological models we want to discuss. We shall consider an alternative view of the structure of the dynamics of geometry in relation to Mach's principle in §5.

4. The observational status of Mach's principle

Since, according to the preceding section, Mach's principle is not incorporated in general relativity we are led to ask two questions. First, whether general relativity is the correct theory of gravity, and second, whether Mach's principle is really supported by observations. I shall briefly review the tests of general relativity under two headings: tests of the equivalence principle, and tests of the field equations. Since there are excellent recent reviews available I shall simply note some of the best results taken mainly from Will (1979). Observational support for Mach's principle is provided by limitations on cosmological models imposed by the isotropy of the microwave background.

4.1. Tests of the equivalence principle

The best result claimed for an Eötvös type test of the weak equivalence principle is by Braginsky and Panov (1972) who give a relative composition-dependent acceleration $\delta a/a < 10^{-12}$. An interesting extension of the test to the microscopic level is $\delta a/a < 3 \times 10^{-4}$ for an anomalous acceleration of neutrons (Koester 1976).

High precision tests of the strong equivalence principle were initiated by Pound and Rebka (1960) with an experiment to measure the effect of gravity on the frequency of light. This experiment measures the relative acceleration, $\delta a/a$, between a freely falling frame and a frame in which the electrodynamic laws take their special relativistic form. Pound and Snider (1965) found $\delta a/a \lesssim 10^{-2}$. Using a rocket-borne hydrogen maser Vessot and Levine (1976) obtained $\delta a/a \lesssim 2 \times 10^{-4}$.

An even stronger version of the equivalence principle, sometimes called the 'super-strong principle of equivalence' and sometimes the 'Einstein equivalence principle' because it holds in general relativity but not in many other metric theories (Dicke 1964), states that all laws of physics, including the theory of gravitation, must be the same in any freely falling frame of reference. An example of the way in which this can be violated is the *Nordtvedt effect* (Nordtvedt 1971) according to which massive bodies, for which self-gravity is important, may fall with accelerations that depend on their gravitational self-energy. Shapiro *et al* (1976) using lunar laser ranging have shown that a limit to such violations of the universality of free fall for self-gravitating bodies is $\delta a/a \lesssim 1.4 \times 10^{-13}$.

4.2. The PPN *formalism*

For weak gravitational fields the space–time metric can be expanded about the Minkowski form. The idea of treating the coefficients of the terms in such an expansion as parameters, to be fixed in any specified metric theory, in order to understand what it is that tests of gravity theories test, first occurs, in embryonic form, in Eddington (1922). For the exterior static spherically symmetric metric of a mass M we have

$$ds^2 = -\left[1 - \sum_{n=1}^{\infty} b_n \left(\frac{2GM}{rc^2}\right)^n\right] dt^2 + \left[1 + \sum_{n=1}^{\infty} a_n \left(\frac{2GM}{rc^2}\right)^n\right] dr^2 + r^2\, d\Omega^2$$

where $b_n = \delta_n{}^1$ and $a_n = 1$ in general relativity. In the parametrised-post-Newtonian (PPN) formalism developed by Nordtvedt, Will and others (Will and Nordtvedt 1972) the expression for the metric is generalised to accommodate as source either self-gravitating particles or a fluid, and for comparison with observations the motion of non-test bodies is discussed.

Possible forms for the PPN metric are restricted by coordinate conditions which require the metric coefficients to fall off as $1/r$ or faster as $r \to \infty$, and by stationarity (invariance under $t \to -t$). The order parameter $\varepsilon^2 = GM/Rc^2$ is supplemented by a velocity $v/c \sim \varepsilon$, so both even and odd powers of ε occur in the expansion. The Newtonian potential GM/r is replaced by an integral over the matter distribution

$$U = \frac{G}{c^2} \int \frac{dM}{r}.$$

At each order in the expansion quantities in addition to the potential which fall off faster than $1/r$ can be constructed from the matter density and velocity, and these are included with further unknown coefficients. If preferred velocities or locations are assumed to exist these can be used to generate additional parametrised terms. In the equations of motion the expansion to ε^4 for g_{00}, ε^3 for g_{0i} and ε^2 for g_{ij} yield terms of the same order. The PPN coefficients are the metric parameters occurring up to this order. Higher orders include radiation, and the stationarity condition then breaks down.

In the absence of preferred frame and preferred location effects, and considering only the motion of test bodies, only the parameters γ and β appear: we have

$$-g_{00} = 1 - 2U + 2\beta U^2$$

$$g_{ij} = (1 + 2\gamma U)\, \delta_{ij}.$$

These parameters are determined by the classical tests and have the value unity in general relativity. For comparison, in the Brans–Dicke theory (§5.1), $\beta = 1$, $\gamma = (1+\omega)(2+\omega)^{-1}$.

4.3. *The classical tests of general relativity*

Amongst the three classical tests of general relativity, the red-shift of spectral lines emitted from a potential well is best considered as a test of the strong equivalence principle (§4.1). To the remaining two tests, the bending of light by the Sun and the perihelion precession of Mercury, there can be added the time delay of radio waves propagating in the solar gravitational field. These have now been developed into high precision tests of gravity theories.

The deflection of radio waves by the Sun is observable with precision using very long baseline interferometry on groups of quasars. The bending due to the solar corona is frequency-dependent in contrast to the gravitational effect, so can be factored out by multi-frequency observations (Fomalont and Sramek 1977). Fomalont and Sramek (1976) obtain $\gamma = 1.014 \pm 0.018$.

The general relativistic effect of solar gravity on the propagation time of radio pulses can be separated by its dependence on orbital position (Weinberg 1972). Radar ranging of the Viking spacecraft has yielded $\gamma = 1.000 \pm 0.002$ (Reasenberg *et al* 1979).

The perihelion shift of Mercury is known to 0.5% after subtraction of the perturbations due to other planets. The residue could have a significant contribution from the quadrupole moment of the Sun, which is difficult to measure directly. If the Sun is assumed to rotate *uniformly*, with its observed angular velocity, its resulting oblateness gives a quadrupole moment which contributes $\sim 3\times 10^{-4}$ of the general relativistic precession. On this assumption, the results of Shapiro *et al* (1976) yield $\beta = 0.991 \mp 0.011$. Direct observation of the solar oblateness (Dicke and Goldenberg 1974) gave a contribution of $\sim 10\%$ from this effect. However, this was not confirmed by Hill *et al* (1974) who found the effect to be ~ 30 times smaller, within the observational errors in β. Observations of the binary pulsar (§4.5) rule in favour of the lower value.

4.4. *Preferred frames and locations*

In metric theories one cannot have fields in addition to the metric coupling directly to test bodies. Further support for this is provided by the Hughes–Drever experiment (Hughes *et al* 1960, Drever 1961) which shows that the level splitting of two spin states of the ^{7}Li nucleus is independent of orientation. This effectively rules out a second symmetric tensor field coupling directly to matter, since such a coupling must be some 10^{23} times weaker than gravity (Dicke 1964). Since a symmetric tensor field is equivalent to a preferred frame of reference in space–time, namely the frame in which the tensor is diagonal, the experiment rules out preferred frame effects acting directly. Aether drift experiments, such as that of Turner and Hill (1964), rule out the direct action of a preferred velocity.

On the other hand, alternative gravity theories have been proposed, some no doubt by Devil's advocates, in which extra geometrical structures are introduced which act indirectly through their contribution to the space–time metric (see §5.2). The PPN parameters (α_1, α_2, α_3) have been introduced to measure the effect of a velocity field relative to a preferred frame, and the parameter ξ measures the effect of a preferred location in space–time. These structures give anomalous contributions to the standard tests such as perihelion precession (but not light deflection or time delay) and one can think of the parameters as measuring the ratio of these contributions to the general relativistic values. However, limits on the parameters are best set by looking for effects which do not occur in general relativity. Thus, for example, variations in the Earth's sidereal rotation suggest that preferred frame effects are less than 2%, if our preferred velocity is taken to be our motion relative to the microwave background (§4.6). Likewise, limits on variations in the locally measured Newtonian gravitational constant yield $|\xi| \lesssim 10^{-3}$ (Warburton and Goodkind 1976).

4.5. *The binary pulsar*

At orders of approximation higher than the PPN level, metric theories predict the existence of gravitational radiation carrying off energy from time-dependent systems. Attempts to detect such radiation directly have so far not been successful, but the next best thing is to detect the damping effect of the loss of energy on a radiating system. This has now been done by observations of pulse times of a pulsar in the binary system PSR 1913+16 (Taylor *et al* 1979). Furthermore, these observations provide an accurate check on the PPN parameters.

The binary system contains a pulsar in orbit about what is probably another neutron star. The PPN order parameter $\varepsilon \sim 10^{-3}$ here, and the ellipticity is of the order $e \sim 0.617$, significantly greater than the corresponding values for Mercury ($\sim 3\times10^{-4}$ and 0.206). The system parameters are highly overdetermined by the pulsar timing observations. There is no evidence for a periastron advance different from the general relativistic value ($\dot{\omega}\sim 4.2^\circ\ \mathrm{yr}^{-1}$); the measurements yield self-consistent estimates of the stellar masses (both $\sim 1.4\ M_\odot$) and quantitative confirmation of the existence of gravitational radiation as predicted by general relativity to within a factor 1.3 ± 0.3. The importance of this last result is enhanced by the fact that competing theories of gravity predict significantly more radiation. It is perhaps not too much of an exaggeration to think of these results as playing for general relativity the role of Hertz's experiments in Maxwellian electrodynamics.

4.6. Mach's principle and cosmology

According to Mach's principle the matter distribution in the universe should determine a local inertial frame, a result that was, of course, suggested by crude observations. Thus the rotation of the plane of the Foucault pendulum and the flattening of the poles of the Earth indicate that the Earth rotates relative to the stars which therefore determine the local inertial frame; likewise, the laws of planetary dynamics and the flattening of the galaxy are determined relative to the fixed distant stars, hence the bulk of matter in the universe (Sciama 1971). Our task here is to show that this crude qualitative agreement can be refined to a detailed quantitative law.

Evidence for the large-scale distribution of matter comes from the galaxy correlation function. (For an introductory survey see Raine (1981) or Davis (1976); for detailed analysis see Peebles (1980).) This measures the tendency for pairs of galaxies to cluster together in associations which have a mean separation less than the average. This clustering tendency is quite marked on small scales. The probability of finding two galaxies 1 Mpc apart is 20 times that which one would expect for a purely random distribution of galaxies. On larger scales the correlations exhibit two features. The first is that there are no preferred correlation lengths, so that one cannot, for example, think of the universe as composed of randomly distributed clusters of galaxies. More important for us is that the degree of correlation tends to zero as the length scale increases, and is effectively zero on scales much less than the size of the visible universe. This indicates a trend to uniformity on sufficiently large scales. Furthermore, the correlations scale with the distance of the galaxy sample under investigation in the way one would expect for a homogeneous universe. These results from optical data are confirmed for analysis of radio sources (Webster 1976).

The relative rates of expansion of the universe in different directions gives a measure of its anisotropy. Peebles (1971) quotes an upper limit of anisotropy of 30% for relative variations, $\Delta H/H$, in the Hubble constant with direction. More detailed investigations have given contradictory results. Rubin *et al* (1973) find an unexplained anisotropy in the distribution of Sc galaxies out to ~ 100 Mpc, while Stenning and Hartwick (1980) have detected a different anisotropy in agreement with the results from the microwave background, which we discuss next.

The cosmic microwave background is uniform over the sky to temperature fluctuations $\Delta T/T \lesssim 10^{-3}$ on all scales. One can think of the radiation as coming to us from a 'surface of last scattering', a surface located at the red-shift at which a typical microwave photon was last scattered by matter. In the absence of an intergalactic medium this is, at the time of matter recombination, at a red-shift $z\sim 1000$. In the more likely case of a reheated intergalactic plasma the last scattering surface is at lower red-shifts ($z\sim 7$ in the

Einstein–de Sitter marginally open model). In any case, the observation that the microwave sky is at a uniform temperature restricts the possible systematic motion of matter at last scattering, since any large-scale motion would induce Doppler shifts in the scattered radiation, which would appear as temperature variations.

We assume first that the universe is homogeneous on a large scale and use the microwave data to limit possible anisotropies. In this way, Hawking (1969) and Collins and Hawking (1973a) found limits on the possible rotation (local vorticity) of the universe in the sense of a rotation of a local dynamical inertial frame at each point relative to the bulk of matter. The precise result for the allowed rotation at the present epoch depends on the evolution of the model since last scattering, and hence on the type of anisotropy assumed. An upper limit for an open model, density $\Omega\rho_c$, less than the critical density, $\rho_c = 1.2\times10^{-26}$ kg m^{-3}, is a rotation

$$\omega < \frac{3\times10^{-10}}{\Omega} \qquad \text{seconds of arc per century.}$$

The important point here is that this is much less than that corresponding to the speed of light at the edge of the visible universe, $c\,(c/H_0)^{-1} \sim 2\times10^{-3}$ seconds of arc per century. One cannot therefore argue that the rotation of the universe seems to be small because it cannot in any case be any bigger than the limit imposed by the velocity of light (McCrea 1971). Furthermore, Collins and Hawking follow the evolution of the rotation back in time to show that it must always have been significantly less than the velocity of light limit. This aspect of Mach's principle is therefore verified to a high precision.

Easier to deal with from a theoretical point of view than rotating universes are anisotropically expanding non-rotating ones. Indeed, no exact expanding and rotating non-empty solutions of Einstein's equations are known explicitly. One would expect anisotropically expanding universes not to satisfy Mach's principle because locally the shearing motions mimic a rotation of the matter relative to a dynamical inertial frame (Bondi 1960). Likewise, Einstein's equations give a relation between the shear, σ, expansion, θ, and energy density, μ, of matter in the simplest case of a Bianchi type I model (see Ellis 1971, MacCallum 1973):

$$\tfrac{1}{3}\theta^2 = \sigma^2 + 8\pi G\mu/c^2. \tag{4.1}$$

As $t\to0$ at the initial singularity, we find $\mu/\sigma^2\to0$ and $\theta\sim\sigma$ (see appendix 4). Thus, 'matter does not matter' at early times and the model behaves essentially as a vacuum, hence non-Machian, solution.

Limits on σ at the present epoch from the microwave observations again depend on the model. A natural measure of the importance of shear is the ratio of the shear, σ, to the mean Hubble expansion, θ, In the worst case Collins and Hawking (1973a) find

$$|\sigma/\theta| \lesssim 10^{-3}$$

and in the best

$$|\sigma/\theta| \lesssim 6\times10^{-8}.$$

In many cases stronger limits are imposed by the condition that the influence of the shear on the rate of expansion implied by equation (4.1) should not alter the cosmological production of helium significantly (Barrow 1976). The conclusion again is that Mach's principle is satisfied to a high precision.

Now, the universe is not exactly homogeneous. Departures from homogeneity on small scales induce local velocities, such as that of the solar system in the galaxy and the

Local Group in the Virgo Supercluster. We are therefore not stationary relative to the last scattering surface, so we expect a large-scale (360°) anisotropy in the microwave temperature of the order of the largest of these velocities (~ 300 km s^{-1}). Such a dipole anisotropy has been reported by a number of groups (Smoot and Lubin 1979, Fabbri *et al* 1980, Boughn *et al* 1981) with the general consensus that our velocity relative to the background is ~ 350 km s^{-1} in the direction $\alpha \sim 11\frac{1}{2}$ h, $-20° \lesssim \delta \lesssim 20°$. The observations also place upper limits on a quadrupole variation of temperature.

The result gives the Local Group a net velocity of 600 km s^{-1}. This is rather high for a random velocity in the Virgo Supercluster (White and Silk 1979). Wilson and Silk (1981) and Peebles (1981) have suggested that the motion induced by the clustering of matter revealed by the correlation function analysis could be responsible for both the dipole and quadrupole variation. Raine and Thomas (1981a) have investigated the possibility that it is due to the shear induced by a very large-scale density enhancement in a low-density universe with a reheated intergalactic medium (see also Warwick *et al* 1980).

In the present context one can ask what limits this observation places on the presence of *inhomogeneous* source-free shear in the universe, i.e. shear which is introduced as an arbitrary initial condition and not induced by local enhancements of the matter density. Limits to intermediate-scale inhomogeneous shear have been considered by Barrow (1976) with results similar to those in the spatially homogeneous cases. For large scales the Bondi models (Bondi 1947) provide a useful class of exact solutions. These contain a spherically symmetric expanding distribution of zero pressure fluid, the source-free shear of which is determined by a function $t_0(r)$. This is the proper time delay of the big bang at each point, which therefore occurs at time $t - t_0(r)$ in the past at each co-moving radius r (Eardely *et al* 1972). The source-free shear is related to $t_0'(r)$.

Detailed results depend on the choice of $t_0(r)$ (Raine and Thomas 1981b). In summary, for a smooth distribution of shear on scales greater than a few thousand Mpc, the constraints from helium production provide limits comparable to those in the homogeneous models. On the other hand, localised regions of large shear are possible. These would correspond to a 'contemporaneous' big bang in one part of the sky. For example, if $q_0 = 0.1$ the shear could reach its limiting value, $\sigma/\theta \to 1/\sqrt{3}$, over regions of sky $\lesssim 400$ Mpc at $z \sim 3$ and have no presently detectable influence on the microwave background. However, even this contrived situation contains significant amounts of shear over only $\lesssim 0.1\%$ of the volume of the universe sampled by the observations.

We conclude that the case for Mach's principle, like that for general relativity itself, no longer rests on a few semi-quantitative remarks, but is based on observations of high precision.

5. Pathways to Mach's principle

In the previous two sections we have seen that both Mach's principle and the general theory of relativity are well substantiated by observations, but the principle is not incorporated in the theory. It follows that either there are other reasons why Mach's principle is valid, or we need a different theory of gravity. We shall return to the first alternative in §6. The latter view meets increasing resistance as the ability of observations to rule out theories develops. At the very least, one must be clear what assumption in general relativity is being broken. Some of the suggested alternatives will be discussed here.

5.1. The role of inertial mass

An alternative approach to Mach's principle is clearly illustrated in an argument due to Narlikar. To obtain a relativity of motion a necessary condition is that the motion of a single body in an otherwise empty universe should be undetermined. This will be the case if in an empty universe the inertial mass of the test body is zero. For then an arbitrary acceleration is consistent with zero force, so inertial motion is undetermined. In general relativity inertial mass is an atomic property independent of the environment. Several alternative theories have been proposed which break this assumption.

Jennison and Drinkwater (1977) have attempted to show how inertial mass can be generated locally in a purely electromagnetic model of massive particles. The details of the model are unimportant. The claim is that the 'particle' responds to an external force by generating an inertial resistance dependent on the electromagnetic energy content, E, with, indeed, an effective inertial mass of E/c^2. In the Newtonian approximation, at least, this certainly yields the affine structure for particle dynamics (i.e. 'explains' the first law) as is claimed. But in order to do so, it is necessary to employ the equations of electrodynamics, hence the affine structure for electromagnetic theory (that the equations are valid in freely falling frames described by the appropriate connection). That the two connections turn out to be the same yields an example of Schiff's conjecture (§2.3) that violation of the strong equivalence principle entails violation of the universality of free fall, and has nothing to do with any standard view of Mach's principle (which in this case would require a demonstration that the electrodynamic connection is determined by matter in the universe).

Hoyle and Narlikar (1974) have attempted to formulate a theory of gravity in which particle masses are generated by a space–time metric which is itself determined by the motion and masses of particles. Formally, the mass m_A of a particle A is the sum of contributions from other particles B,

$$m_A=\sum_B \lambda^2 \int \mathscr{G}(a, b)\, \mathrm{d}b \tag{5.1}$$

where the integration is over particle paths with elements of proper length db, and the integrand is a Green function for the conformally invariant scalar wave equation

$$\Box\mathscr{G}+\tfrac{1}{6}R\mathscr{G}=(-g)^{-1/2}\,\delta.$$

The field equations determining the metric are then obtained rather elegantly from an action principle having an action-at-a-distance form

$$\delta\left(\sum_A\sum_B\int\int\mathscr{G}(a,b)\,\mathrm{d}a\,\mathrm{d}b\right)=0$$

by variation of the metric and particle paths. The inertial mass in equation (5.1) appears to depend on location. But a conformal rescaling of the metric, $g_{\mu\nu}\to\psi^{-2}g_{\mu\nu}$ gives $\mathscr{G}\to\psi\mathscr{G}$, $m\to\psi m$ so an appropriate choice of conformal rescaling, $\psi=m^{-1}$, can be made to obtain constant particle masses. With this choice of conformal factor one obtains field equations which agree with those of general relativity.

To justify this use of a conformal rescaling Hoyle and Narlikar argue that there is no way of comparing masses except by reference to other masses, and therefore no way of checking whether particles have the same mass at different points (as long as they all change together). Likewise with lengths and times, only ratios are measured, and so the length scale in the metric must be arbitrary. They conclude that there is no reason to prefer the general relativistic metric and propose to treat the whole conformal class as equivalent. Note that even if this argument were correct one cannot use the rescaling proposed by Hoyle (1975) to continue big bang models through the initial singularity, since in no case is a scaling of the units to zero permitted. In any case, while the premise of the argument is correct, the conclusion is false. For, if the superstrong equivalence principle holds, that the laws of physics, including gravity, are the same in all freely falling frames, then the fundamental constants are independent of location. One can therefore measure particle masses in terms of the dimensionless ratio $m/(hc/G)^{1/2}$. The connection and space–time metric are defined by the free falls. Of course, the theory can be made to look more complicated by rescaling m and G keeping these dimensionless ratios constant. But unless the scaling factor drops out of all equations, so that one has genuine conformal invariance and the conformal factor is not determined by the theory, one does not get anything new. Since the metric controls the free falls this is only possible if one can extend the class of free falls to a larger equivalence class defined by a 'conformal connection'. The standard way of doing this is to postulate a theory involving only zero-mass particles, the paths of which determine the metric only up to a conformal factor, and conformally invariant field equations. Unfortunately, the inclusion of massive particles breaks this conformal invariance.

Since the superstrong principle of equivalence is valid in the Hoyle–Narlikar theory as in general relativity, the most the theory could achieve is a reformulation of general relativity (with the possible interest in early versions of the theory (Hoyle and Narlikar 1966) that the reformulation works only for universes having large numbers of particles). Furthermore, the appearance of a long-range generation of inertial mass is spurious, for the formulation of the theory is appropriate only for a universe containing one type (mass) of particle. For additional particle types separate λ must be introduced to express the coupling. Hence λ^2 is an intrinsic inertial mass in heavy disguise. In addition, since only λ^2 and not λ itself appears there is no reason for λ to be real and the claim that gravity is attractive in this theory appears to be false, since the opposite is true if $\lambda^2 < 0$.

To express the generation of inertial mass by interactions successfully, one requires at least a long-range force generated by a new field. This can break the superstrong equivalence principle, since in view of its generation by distant matter the new field can be regarded as an aspect of gravity, the action of which in freely falling frames thereby becomes dependent on the proximity of masses in the environment. In this way one can obtain a theory which differs from general relativity.

The Brans–Dicke theory (Dicke 1964) is the best known of a class of such theories which couple a scalar field, φ, to matter through a coupling $\omega(\varphi)$ (Bergmann 1968, Wagoner 1970, Nordtvedt 1970, Bekenstein 1977). In the Brans–Dicke theory ω is a constant to be specified, and $\omega \to \infty$ is the general relativity limit. In these theories the dimensionless ratio $(G/hc)^{1/2}m$ is *not* constant. This variation can be attributed either to a varying G, or, by a conformal rescaling of the metric, to a variation of mass, $m = m(\varphi)$. The underlying idea is that if there is no matter we have $\varphi \to 0$ and $m \to 0$, and therefore a test particle would have no inertia, as required. However, the theory is not conformally invariant; the non-gravitational laws have to satisfy the strong equivalence principle and this again defines a physically preferred connection and metric. In order to avoid violating the equivalence principle the matter responds to the φ field only via the metric (i.e. we have a metric theory). The effect of the additional scalar field is therefore to act as a source of gravity through the field equations:

$$\Box\varphi=\frac{\kappa T}{2\omega+3}$$

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa\varphi^{-1}T_{\mu\nu}-\omega\varphi^{-2}(\nabla_\mu\varphi\nabla_\nu\varphi-\tfrac{1}{2}g_{\mu\nu}\nabla_\lambda\varphi\nabla^\lambda\varphi)+\varphi^{-1}(\nabla_\mu\nabla_\nu\varphi-g_{\mu\nu}\varphi).$$

Unfortunately, one also avoids incorporating Mach's principle for this reason. The most one could achieve is that the absence of matter entails $\varphi=0$ and that there should be no solution to the remaining field equations in this case. What actually transpires is that in the absence of matter one obtains, at least, the vacuum solutions of general, relativity.

5.2. *Alternative field equations*

Attempts to satisfy Mach's principle by modification of the field equations within the context of a metric theory have, of course, a noble ancestry, dating to Einstein's proposal for a cosmological term $\Lambda g_{\mu\nu}$ in the field equations. (For an early history of the Λ term see North (1965).) The idea is to obtain field equations which do not possess non-Machian solutions; in particular, do not admit vacuum solutions. As is well known, de Sitter (1917) showed that Einstein's modified field equations possess solutions even in the absence of matter. Other direct modifications of the field equations are forbidden if one requires a Hamiltonian formulation (Kuchăr 1974b) or a formal Newtonian limit, unless extra fields are introduced.

In metric theories matter cannot respond directly to any additional fields, but only indirectly through a metric which is generated in part by these fields. Several theories of this type exist with additional dynamical fields, i.e. fields determined by their own field equations, not necessarily constructed with Mach's principle in mind. The Brans–Dicke theory (§5.1) is an example of an additional scalar field and the Will–Nordtvedt (1972) theory contains an extra vector field. Alternatively, one may add additional non-dynamical fields. For a recent brief review see Will (1979). For Machian theories a favourite non-dynamical field is provided by the background cosmology, either through a background metric or a cosmic time. Gürsey (1963) appears to have initiated the study of Machian effects relative to an expanding substratum (see Reinhardt 1973), and other early discussions include Hönl and Dehnen (1963, 1964; see also Ehlers and Schücking 1967). The idea has been taken up by Goldoni (1976, 1980), unfortunately, it seems, without regard to the relation of the theory to the PPN formalism.

In the search for an alternative justification of Mach's principle, Jones (1981) has pointed out that general relativity must necessarily be modified to accommodate quantum theory, and that one might therefore see whether Mach's principle can be incorporated at this stage. His idea is that only Machian universes would appear as classical limits, i.e. as wave packets of quantum universes. This is an appealing suggestion, but its elucidation presents considerable difficulties.

5.3. *Alternative theories*

Ab initio, non-metric theories of Mach's principle are now of little interest unless they contain a discussion of how one gets round the equivalence principle. For example, Sciama's (1953) theory of inertia referred to in §1, in which a vector model of gravity was used to indicate how a long-range $1/r$ addition to the gravitational field could induce inertial forces, is now of historical interest only.

A radical alternative to general relativity has been suggested by Barbour (1974) and developed by Barbour and Bertotti (1977) and Bertotti and Easthope (1978). This overcomes the initial objection to *ab initio* theories by seeking first to rewrite the Newtonian analysis of dynamics and space–time structure. The idea is then to extend the theory to be compatible with relativity by making the local validity of special relativity a function of the actual matter distribution of our universe. This neatly avoids the equation: special relativity plus equivalence principle leads to general relativity, by incorporating a thorough-going Machian phenomenalism according to which our physical laws are not necessarily general laws at all, but relations of the appearances in the universe in which we happen to find ourselves.

To begin with, one attempts to construct a dynamical theory consistent with Leibnitz's view of a relational space–time. In this only relative distances, and hence velocities, can be measured, so absolute acceleration is impossible. Such a dynamics must do without privileged vector fields on the space–time manifold, or the frame bundle (or any higher-order bundle). It must therefore be constructed from a Lagrangian invariant under the *Leibnitz group* of transformations between frames of reference in arbitrary relative motion. In the original version of the theory Barbour (1974) achieves this by means of a product Lagrangian constructed in terms of the relative distances of particles, r_{ij},

$$L = V^{1/2}T$$

$$V = \sum_{i<j} \frac{m_i m_j}{r_{ij}}$$

$$T = \left(\sum_{i<j} \frac{m_i m_j}{r_{ij}} r_{ij}'^2 \right)^{1/2}$$

where the prime denotes a derivative with respect to a parameter λ. Note that the space–time still has an absolute time, albeit arbitrarily parametrised in the theory, and a Euclidean metric on simultaneity sections. In a later version Barbour and Bertotti (1977, 1981) claim to reproduce Newtonian dynamics in a many-particle universe, thereby satisfying Mach's principle. They also suggest generalisations in which relativity is incorporated for suitable cosmological models with a derivation of the velocity of light in terms of observable global properties of the models.

The theory certainly provides a dynamics appropriate to Leibnitzian space–time. Two possible problems might be mentioned in considering whether it does any more. First, to get back to Newtonian dynamics one chooses a frame in which certain quantities, the momentum and angular momentum of the whole universe, are put equal to zero. These quantities are supposed to be conserved as a result of symmetry under the Euclidean group (translations and rotations) on each space slice. But their conservation requires that the group should act in the same way on each simultaneity slice, for otherwise one has an infinite dimensional Lie group of symmetries and Noether's theorem yields identities, not conservation laws. Defining the same action on each slice is equivalent to introducing the Newtonian connection (or the Aristotelian one!). Put another way, one can only have a conserved total momentum and angular momentum if there is a Newtonian connection; it is arbitrary whether the vanishing of these quantities corresponds to the non-acceleration of the stars, since in a Leibnitzian theory this statement has no invariant meaning. It is not clear whether this is merely a problem of presentation (or a misinterpretation!) since the conclusion clearly contradicts the initial intentions: namely, that by using only relative configurations an appropriate choice of Lagrangian should lead automatically to Newton's laws relative to the fixed stars.

The second problem arises because the incorporation of forces into the product Lagrangian works because one can have an action-at-a-distance interaction in this formulation of the theory. In a relativistic theory one would expect to introduce a field to represent this interaction, and the free degrees of freedom of this field then, in general, yield non-Machian solutions. Indeed, there is the possibility that one returns eventually to the equivalent of a selection rule for solutions of general relativity, namely that the theory selects just those general relativistic cosmologies in which Mach's principle is valid.

Finally, we should mention other approaches to Mach's principle which are beyond the scope of this review. For example, Tipler (1978) has recently proved a version of Pirani's conjecture (Pirani 1956) to the effect that the only non-singular vacuum solution of the Einstein field equations is essentially Minkowski space. Such considerations, while both valid and interesting, have at most a semantic link with the scope of this review.

5.4. Selection rules for Machian space–times

The final approach to Mach's principle, to be treated in the remainder of this review, again goes back to Einstein. The non-Machian aspect of general relativity, Einstein noted, could be traced to the boundary conditions to be imposed on the field equations (see Grünbaum 1957). He suggested that there would be no need for these conditions in space–times with closed spatial sections, and that Mach's principle should therefore be regarded as selecting such 'physical' solutions from amongst all possible solutions. This is incorporated into Wheeler's *geometrodynamics* (Wheeler 1964b), according to which Mach's principle is expressed through the plan of general relativity as a dynamics of the evolution of three-space geometry (§3). By itself, the closure postulate is insufficient, since it allows empty spatially closed space–times obtained by identifications of points ('cutting and pasting') in Minkowski space–time. As part of the plan of geometrodynamics such solutions can be ruled out (Isenberg 1974).

The implementation of this approach requires theorems that evolve appropriate initial data into space–times, with or without the closure postulate. This formalises the idea that the distribution of matter and gravitational energy at one time determines the future evolution of the three-geometry. Isenberg labels such solutions Wheeler–Einstein–Mach space–times. Included here, in addition to the standard cosmological models, are asymptotically flat space–times (if one agrees to drop the closure postulate), certain space–times devoid of matter, and rotating cosmologies, on the grounds that in these cases the gravitational energy provides the source of inertia. A particular case is Wheeler's gravitational geon (Hartle 1960, Komar 1965). Isaacson (1968) has shown how a high-frequency gravitational wave can be separated from a slowly varying background geometry to appear as a source on the right-hand side of Einstein's equations. From this point of view the gravitational energy of the geon appears as a material particle to an outside observer (but not to an inside one!).

However, the question naturally arises as to whether one can go further and express Mach's principle in the spirit of its original presentation as a condition that the gravitational wave modes be initially unexcited. As we saw in §3, for open three-geometries the specification of appropriate initial conditions for this is an unsolved question. For, not only must the dynamical degrees of freedom vanish, but the constraints must be solved in a Machian way. Even in the case of closed three-spaces, it is not clear that one can rule out homogeneous solutions not determined directly by matter.

The basic difficulty arises from the ambiguity in the meaning of 'determined by' in expressing the idea that geometry is determined by matter in non-linear theories in which there is no superposition principle. Following earlier work of Lynden-Bell (1967) and Al'tschuler (1967), and an idea of Hoyle and Narlikar that the influence of each element of matter propagates through that space–time geometry to which the whole matter distribution gives rise, Sciama *et al* (1969) were able to exhibit a solution of this problem for the field equations of general relativity. This takes us to the theory of the next section.

6. The integral formulation of general relativity

6.1. The Sciama–Waylen–Gilman theory

This remarkable property of general relativity, that it gives rise to a self-consistent representation of the metric as a linear superposition of contributions which are propagated from their sources through the space–time having the given metric, arises from the homogeneity of the Lagrangian in the field variables. D Lynden-Bell (1969 private communication) has given a symbolic derivation as follows.

Let $L(\varphi, \nabla_\mu\varphi) - J\varphi$ be a Lagrange function for the field φ. In the application, φ will be the metric $g_{\mu\nu}$ (and J essentially the energy–momentum tensor), but we shall suppress indices here. Let L be homogeneous of degree n in φ. Then Euler's theorem gives, symbolically,

$$\frac{\delta L}{\delta\varphi}\varphi = nL.$$

Varying this equation with respect to φ, we obtain

$$\frac{\delta^2 L}{\delta\varphi^2}\varphi = (n-1)\frac{\delta L}{\delta\varphi} = (n-1)J \tag{6.1}$$

which expresses the homogeneity of degree $(n-1)$ of $\delta L/\delta\varphi$ and where the final equality follows from the field equations, $\delta L/\delta\varphi = J$. Variation of the field equations gives the propagation equation for small disturbances

$$\frac{\delta^2 L}{\delta\varphi^2}\delta\varphi = \delta J. \tag{6.2}$$

From equations (6.1) and (6.2) there follows

$$\frac{\delta^2 L}{\delta\varphi^2}(\varphi + \delta\varphi) = (n-1)J + \delta J. \tag{6.3}$$

For $n=2$ we obtain a propagation equation for small disturbances $\delta\varphi$ (equation (6.2)) of the same form, containing the same differential operator, as that for the field φ (equation (6.1)). In this sense, the equations are said to be *stable*.

In fact, to obtain the stability condition it suffices that $\delta L/\delta\varphi$ be homogeneous of degree one. This much can be achieved in general relativity by taking the basic variable to be $\varphi = g^{\mu\nu}$ with indices raised, and the field equations in the form

$$R^\mu_\nu = \kappa(T^\mu_\nu - \tfrac{1}{2}g^\mu_\nu T). \tag{6.4}$$

Variation of (6.4) with respect to $g^{\mu\nu}$ leads to the Sciama–Waylen–Gilman equation of the form (6.3) for $\hat{g}^{\mu\nu}=g^{\mu\nu}+\delta g^{\mu\nu}$:

$$L^{\mu\nu}{}_{\rho\sigma}\,\hat{g}^{\rho\sigma}\equiv\hat{g}^{\mu\nu}-2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\,\hat{g}^{\rho\sigma}+2\nabla^{(\mu}\hat{g}^{\nu)}=2\kappa K^{\mu\nu}\equiv g^{\rho(\nu}[T^{\mu)}_{\rho}+\delta T^{\mu)}_{\rho}-\tfrac{1}{2}\delta^{\mu)}_{\rho}\,(T+\delta T)] \qquad (6.5)$$

where

$$\hat{g}^{\mu}=\nabla_{\nu}(\hat{g}^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}\hat{g}).$$

For an explicit derivation see Sciama *et al* (1969). Other choices of basic variable, such as $g_{\mu\nu}$, or of forms for the field equations lead to either a loss of the stability property, that the operator $L^{\mu\nu}_{\rho\sigma}$ be also the propagator of small disturbances, or to non-self-adjoint equations (see below, equations (6.7) and (6.9)). The Sciama–Waylen–Gilman equation (6.5) uniquely satisfies these requirements (Gilman 1969). Of course, once the equation has been obtained indices are raised and lowered in the usual way.

Passing to the limit $\hat{g}_{\mu\nu}\to g_{\mu\nu}$ we obtain the Einstein field equations, since the covariant derivatives of the metric vanish. It is rather more interesting to integrate the equations first and then pass to the limit, when we obtain an integral representation of the metric. We cannot proceed directly with equations (6.5) since as they stand the equations are underdetermined and do not possess a unique solution. This indeterminacy arises from the invariance of the equations under a gauge transformation

$$\hat{g}_{\mu\nu}\to\hat{g}_{\mu\nu}+\underset{\xi}{L}g_{\mu\nu}$$
$$K_{\mu\nu}\to K_{\mu\nu}+\underset{\xi}{L}K_{\mu\nu}$$

where $\underset{\xi}{L}$ is a Lie derivative along the vector field, ξ^{λ}, which represents an infinitesimal coordinate transformation generated by ξ^{λ} and reflects the freedom to make such purely coordinate variations in $\hat{g}_{\mu\nu}$.

To fix the gauge we impose a covariant analogue of the Hilbert–de Donder harmonic condition

$$\hat{g}^{\mu}=0. \qquad (6.6)$$

Note that this restricts the variation of the coordinates in the process of varying the metric, but not the original choice of coordinates, since for $\delta g^{\mu\nu}\to 0$ it becomes an identity. The analysis is therefore manifestly covariant. The gauge condition is compatible with the evolution equations since it follows from equation (6.5) and the varied Bianchi identities that

$$\tfrac{3}{2}\Box\hat{g}^{\nu}-\tfrac{1}{2}R^{\nu}{}_{\alpha}\hat{g}^{\alpha}-\tfrac{1}{2}R\hat{g}^{\nu}=-2\kappa\delta\{\nabla_{\mu}(K^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}K)\}\approx 0.$$

Hence if $\hat{g}^{\mu}=0$ on an initial surface then $\hat{g}^{\mu}\approx 0$ to first order in $\delta g^{\mu\nu}$ for all time.

In this gauge, and assuming as always that the space–time is globally hyperbolic, we obtain a system of the hyperbolic equations for the ten field variables $\hat{g}_{\mu\nu}$ which can be inverted by means of a Green function, $G^{\alpha'\beta'}_{\mu\nu}(x, x')$ satisfying

$$\hat{L}^{\rho\sigma}_{\mu\nu}\,G^{\alpha'\beta'}_{\rho\sigma}\equiv G^{\alpha'\beta'}_{\mu\nu}-2R^{\rho}{}_{\mu}{}^{\sigma}{}_{\nu}\,G^{\alpha'\beta'}_{\rho\sigma}=-\sqrt{-g}\,\bar{g}^{\alpha'}_{(\mu}\bar{g}^{\beta'}_{\nu)}\,\delta^{(4)}(x, x') \qquad (6.7)$$

(DeWitt and Brehme 1960), where we have defined the operator $\hat{L}^{\rho\sigma}_{\mu\nu}$ to be $L^{\rho\sigma}_{\mu\nu}$ in the gauge $\hat{g}^{\mu}=0$. Here $\bar{g}^{\alpha'}_{\mu}$ is the parallel propagator introduced by Synge (1964) and is defined such that a vector $A^{\mu}(x)$ as x is propagated parallel to the vector $A^{\alpha'}(x')=\bar{g}^{\alpha'}_{\mu}\,A^{\mu}(x)$ at x' along the free fall joining x and x'. The integral representation for $\hat{g}_{\mu\nu}$ can be obtained following the standard procedure, and the Sciama–Waylen–Gilman representation is obtained in the limit $\hat{g}_{\mu\nu}\to g_{\mu\nu}$. More simply in this case we can proceed directly: multiply (6.7) by $g^{\mu\nu}(x)$ and integrate over a neighbourhood $\Omega(x')$ of x' to obtain

$$g^{\alpha'\beta'}(x') = 2\int_{\Omega} G^{\alpha'\beta'}_{\rho\sigma} R^{\rho\sigma} \sqrt{-g}\, d^4x + \int_{\partial\Omega} G^{\alpha'\beta'}_{\rho\sigma;\tau} g^{\rho\sigma} \sqrt{-g}\, n^\tau\, dS$$

where n^μ is a unit normal to the boundary $\partial\Omega$, surface element $n^\mu dS$. From now on we choose the retarded Green function. For boundary conditions on an initial hypersurface Σ, and using Einstein's equations, we obtain the Sciama–Waylen–Gilman representation:

$$g^{\alpha'\beta'}(x') = 2K\int_{\Omega} (G^{\alpha'\beta'}_{\rho\sigma} - \tfrac{1}{2}g_{\rho\sigma}g^{\mu\nu}G^{\alpha'\beta'}_{\mu\nu})\, T^{\rho\sigma} \sqrt{-g}\, d^4x + \int_{\Sigma} G^{\alpha'\beta'}_{\rho\sigma;\tau}\, g^{\rho\sigma} \sqrt{-g}\, n^\tau d\Sigma. \qquad (6.8)$$

From the self-adjoint property of the operator $\hat{L}_{\mu\nu}{}^{\rho\sigma}$

$$\int_{\Omega} u^{\mu\nu}\hat{L}_{\mu\nu}{}^{\rho\sigma}\, v_{\rho\sigma} \sqrt{-g}\, d^4x = \int v^{\mu\nu}\, \hat{L}_{\mu\nu}{}^{\rho\sigma}\, u_{\rho\sigma} \sqrt{-g}\, d^4x \qquad (6.9)$$

for differentiable functions $u^{\mu\nu}$, $v^{\mu\nu}$, with compact support, it can be shown that the surface integral over Σ in equation (6.8) is a homogeneous solution (complementary function) of (6.5) in the gauge (6.6) and so represents the contribution to $g^{\alpha'\beta'}(x')$ from matter not in Ω.

From the stability property, that $\hat{L}_{\mu\nu}{}^{\rho\sigma}$ is also the propagator of small disturbances, we have

$$\delta g^{\alpha'\beta'}(x') = 2K\int (G^{\alpha'\beta'}_{\rho\sigma} - \tfrac{1}{2}g_{\rho\sigma}g^{\mu\nu}G^{\alpha'\beta'}_{\mu\nu})\, \delta T^{\rho\sigma}\sqrt{-g}\, d^4x + \int G^{\alpha'\beta'}_{\rho\sigma;\tau}\, \delta g^{\rho\sigma} n^\tau\, d\Sigma \qquad (6.10)$$

since the terms that would involve variations of the Green function must integrate to zero. This exhibits explicitly how an element of matter, $\delta T^{\rho\sigma}$, generates an infinitesimal contribution to $g^{\alpha'\beta'}(x')$ by propagation through the space–time described by the full metric. As an application of the theory Clarke and Sciama (1971) have shown how the multipole moments at infinity of a stationary gravitational field can be related to the moments of the source distribution.

6.2. Application to Mach's principle

It would appear that Mach's principle can now be readily expressed as the condition

$$\lim \int_{\Sigma} G^{\alpha'\beta'}_{\rho\sigma;\tau}\, g^{\rho\sigma}\, n^\tau\, d\Sigma = 0 \qquad (6.11)$$

in the limit that Ω becomes the whole space–time manifold, $\mathcal{M}$, since from equation (6.8) it then follows that at any point x' in $\mathcal{M}$, the metric is determined by the matter in a well-defined way (Gilman 1970). Since not all (globally hyperbolic) space–times would be expected to satisfy this condition it defines a selection rule for Machian space–times. Indeed, one might regard the integral equations

$$g^{\alpha'\beta'}(x') = 2\kappa\int_{\mathcal{M}} (G^{\alpha'\beta'}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\, G^{\alpha'\beta'}_{\rho\sigma})T^{\mu\nu} \sqrt{-g}\, d^4x \qquad (6.12)$$

with $G^{\alpha'\beta'}_{\mu\nu}$ defined by equation (6.7) as a reformulation of the field equations of general relativity such that all solutions satisfy Mach's principle. Note that the selection criterion is 'globally local' in the sense that while it is required to hold everywhere in $\mathcal{M}$ it need only be checked in a neighbourhood of the initial hypersurface Σ.

Two important results follow. First, vacuum space–times are not Machian since from equation (6.12) $T_{\mu\nu}=0$ would imply $g^{\alpha'\beta'}=0$. In particular, this rules out Minkowski space–time. It is also reasonable to suppose that asymptotically flat space–times do not satisfy condition (6.11), since they differ arbitrarily little from Minkowski space–time as one approaches infinity. Second, Gilman (1970) has shown that Robertson–Walker space–times are Machian. A simplified proof is given in appendix 1.

Unfortunately, this theory cannot be quite correct. It provides too restrictive a criterion in the case of cosmological models having particle horizons, i.e. limits to the region of the universe causally connected to a given observer. This is important because most cosmologies proposed as realistic models for our universe are not exact Robertson–Walker solutions, do have particle horizons, and would probably be non-Machian according to Gilman's condition. Roughly expressed, the condition rules out space-times in which the metric at an observer is generated in part by the general relativistic ghost of the instantaneous $1/r^2$ contribution to the gravitational field from matter beyond the particle horizon of the observer. Imagine, for example, a perturbation in which an extra atom is added to an otherwise Robertson–Walker model. The universe is pulled towards the atom even where this is beyond the particle horizon (Ellis and Sciama 1972). This instantaneous action does not violate causality: the conservation laws prevent the spontaneous creation of an atom, which must therefore be imagined as always in existence. The influence of the atom beyond a particle horizon cannot be attributed to the volume term in equation (6.10), since $\delta T^{\rho\sigma}\sim 0$ in the domain of integration by hypothesis. It must be transmitted through the surface integral, which must therefore be admitted in a Machian solution. In appendix 2 we demonstrate this explicitly, and show that in the Robertson–Walker solutions the high symmetry implies a cancellation of contributions to the surface integral from matter beyond the horizon.

Of course, this result is a restatement of the fact that the initial data are not free, hence not freely zero, but subject to constraints depending on the matter content. In the present context, the gauge condition (6.6) can be used to eliminate time derivatives of $\hat{g}^{0\mu}$ from the evolution equations (6.5), giving four constraint equations on the initial data. We should like to analyse these constraints in a systematic way, for which the standard procedure would be to pass to a first-order formalism (coordinates and momenta instead of coordinates and velocities). Unfortunately, the Sciama–Waylen–Gilman equations (6.5) are not the Euler–Lagrange equations of any variational principle, since such equations are necessarily self-adjoint, which equations (6.5) are not. In sufficiently simple theories (flat space electrodynamics is an example) one can proceed directly to define the analogue of a Green function for an underdetermined set of equations by using a suitably modified δ function satisfying the same conservation laws as the sources. The Green function so defined is no longer hyperbolic and propagates the effects of all distant sources correctly. However, this cannot be employed here, essentially for the physical reason that the scattering of $\hat{g}$ waves by the curvature in equation (6.5) mixes the degrees of freedom in the evolution. One might try to proceed on the basis of an *ad hoc* non-covariant splitting of the dynamical degrees of freedom, but it is difficult to see how this could be guaranteed to produce physically significant results (Raine 1971).

In order to overcome this problem I proposed an indirect method (Raine (1975a); an introductory sketch may be found in Raine (1975b)). This approach divides the problem into two parts, which might be regarded as kinematical and dynamical, each of which leads to a condition to be satisfied by a Machian space–time.

Along the worldline of a freely falling observer one can introduce Fermi coordinates appropriate to his local frame of reference. In these coordinates the metric differs locally from Minkowski space–time only at second order:

$$ds^2 = -(1+R_{0l0m}x^l x^m + \ldots)\,dt^2 - (\tfrac{4}{3}R_{0ljm}x^l x^m + \ldots)\,dt\,dx^j + (\delta_{ij} - \tfrac{1}{3}R_{iljm}x^l x^m + \ldots)\,dx^i\,dx^j$$

(Misner *et al* 1973). This fixes the coordinate freedom along a free fall and shows that the dynamical part of the metric is contained in the Riemannian curvature, $R^\lambda{}_{\mu\nu\rho}$. Consequently, the metric will be determined by matter if (i) $g_{\lambda\mu}$ can be uniquely reconstructed from $R^\lambda{}_{\mu\nu\rho}$ and (ii) the curvature is determined by the energy–momentum density, $T_{\mu\nu}$.

The first condition will be satisfied if Fermi coordinates are uniquely determined by the Riemannian curvature to second order at each point. This will not be the case if and only if the space–time admits a curvature collineation

$$\underset{\xi}{L} R^\lambda{}_{\mu\nu\rho} = 0$$

(Hlavaty 1960, Collinson 1970). In fact, this leaves too vague what is meant by 'determined by'. A consequence is that Minkowski space–time appears to be admitted as Machian. Again, what is required is a *stable linear* representation of the potential $g_{\mu\nu}$ as a function of the curvature in the given metric space–time.

Rather remarkably, the Sciama–Waylen–Gilman equations (6.5) turn out to be the trace of just such a generalised relation between $\hat{g}^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}$ and $R^\lambda{}_{\mu\nu\rho} + \delta R^\lambda{}_{\mu\nu\rho}$. This generalisation is obtained by variation of the identity

$$g^{\nu\mu} R^\lambda{}_{\nu\rho\sigma} = R^{\lambda\mu}{}_{\rho\sigma}$$

and yields a wave equation for $\hat{g}^{\mu\nu}$

$$\hat{g}_{[\lambda[\rho;\mu]\sigma]} + \hat{g}_{[\rho[\lambda;\sigma]\mu]} + \tfrac{1}{2}\hat{g}_{\nu[\lambda}R^\nu{}_{\mu]\rho\sigma} + \tfrac{1}{2}\hat{g}_{\nu[\rho}R^\nu{}_{\sigma]\lambda\mu} = \tfrac{1}{2}K_{\lambda\mu\rho\sigma} \tag{6.13}$$

where

$$K_{\lambda\mu\rho\sigma} = (R^{\alpha\beta}{}_{\rho\sigma} + \delta R^{\alpha\beta}{}_{\rho\sigma})\,g_{\alpha\lambda}g_{\beta\mu} + (R^{\alpha\beta}{}_{\lambda\mu} + \delta R^{\alpha\beta}{}_{\lambda\mu})\,g_{\alpha\rho}g_{\beta\sigma}.$$

Since equations (6.13) are invariant under a further infinitesimal coordinate transformation, they are underdetermined and cannot be integrated immediately using a Green function. We do not want to introduce gauge conditions and constraints, since this would take us back to the problem from which we started. The essential point to note is that in contrast to equations (6.5) we now have 20 equations for 10 unknown functions so homogeneous solutions of (6.13) should be rather hard to find. In any case a putative homogeneous solution to (6.13) represents a part in the metric which does not contribute to the curvature so is not physically manifested in tidal forces. In general, we should be able to discard such contributions.

The discussion can be made precise through the introduction of a generalised inverse for partial differential operators and is developed in Raine (1975a). In the limit of $\hat{g}^{\mu\nu} \to g^{\mu\nu}$ we obtain the *first Mach condition*, that the metric must be a generalised inverse function of the curvature.

In general this condition is satisfied. Exceptions occur when $\underset{\xi}{L}\, R^\lambda{}_{\rho\nu\sigma} = 0$, in which case $g_{\mu\nu}$ is determined up to the addition of a finite contribution of the form $\xi_{(\mu;\nu)}$, since this is then a homogeneous solution of (6.13). These are probably the only exceptions. Included here are Minkowski space–time, for which $R^\lambda{}_{\mu\nu\rho} = 0$ and

$$\eta_{\mu\nu} = \xi_{(\mu;\nu)} \tag{6.14}$$

the generalised system (6.13) being in this case the integrability condition for (6.14), and plane-wave space–times (Ehlers and Kundt 1962). It is also reasonable to suppose that asymptotically flat space–times would be rigorously excluded.

The dynamical part of the problem involves the relation between $R^{\lambda}{}_{\mu\nu\rho}$ and $T_{\mu\nu}$. The Ricci tensor part of the curvature is given directly by the Einstein equations. The Bianchi identities can be rearranged to provide linear field equations on the given background space–time for the remaining Weyl curvature (trace-free) part of $R^{\lambda}{}_{\mu\nu\rho}$

$$\nabla_{\lambda} C^{\lambda}{}_{\mu\nu\rho} = \kappa(T_{\mu[\nu;\rho]} - \tfrac{1}{3} g_{\mu[\nu} T_{;\rho]}). \tag{6.15}$$

The essential point here is that a homogeneous solution of equations (6.15) represents a contribution to $C^{\lambda}{}_{\mu\nu\rho}$ from distant matter and not merely gauge freedom, either wholly or in part. As a consequence it turns out that constrained variables can be explicitly extracted from the initial data and a Machian condition imposed on the remaining contribution of the unconstrained data to a homogeneous solution.

This leads to the result that Robertson–Walker solutions are Machian, in agreement with the previous discussion. On the other hand, it enables us to rule out as non-Machian shearing or rotating spatially homogeneous cosmologies, and all vacuum space–times. As stated in §4.6, Bondi models which do not have vanishing shear as $t \to 0$ are also non-Machian.

The principal objection to this theory is that it is too ungainly to be true! In the next subsection I shall outline a new approach which provides Machian field equations for general relativity directly and is the subject of current research. Whether the two theories are equivalent is not established, but where the Machian classification of a space–time has been calculated in the two theories the results agree.

6.3. A new version of Mach's principle in general relativity

The inelegant apparatus of the preceding subsection is unnecessary in space–times in which there are no particle horizons, since for these the Gilman criterion is correct. The first key point to extending this to space–times with horizons is that there is no uniquely defined *retarded* solution in these cases. Consider, for example, a regular manifold $\Sigma \times \mathbb{R}$, not necessarily a solution of Einstein's equations for the moment, the part $t > 0$ of which is taken as a cosmological model. This certainly has horizons. To the unique retarded Green function of a wave equation on $\Sigma \times \mathbb{R}$ we can add any solution with sources in $\Sigma \times (0, -\infty)$ and obtain many 'retarded' Green functions on $\Sigma \times [0, \infty)$.

Note next that the reason for the difficulty in the particle horizon case is that the integral in equation (6.12) does not necessarily satisfy the gauge conditions (6.6) on the basis of which it was constructed. The constraints which arise from the gauge conditions provide the contribution to the surface integral necessary to satisfy the conditions. The key to the solution of the problem is therefore to use the freedom afforded by the choice of retarded solutions to construct an integral representation in which the gauge conditions are identically satisfied by the volume and surface terms separately.

Now we have already noted in §6.2 that one cannot construct a Green function which satisfies a divergence condition for all time. But we can find a Green function that satisfies this condition *initially*. The volume integral analogous to (6.12) but constructed using this function satisfies the gauge condition initially, and hence satisfies the constraints. But the preservation of the gauge condition then implies that even though the Green function itself does not evolve to satisfy the divergence condition, the volume integral as a whole must do so.

We call a solution of (6.7)

$$G_{\mu\nu}^{\alpha'\beta'} = M_{\mu\nu}^{\alpha'\beta'} - \tfrac{1}{2} g_{\mu\nu} g^{\rho\sigma} M_{\rho\sigma}^{\alpha'\beta'}$$

satisfying the gauge conditions

$$\nabla_{\alpha'} M_{\mu\nu}^{\alpha'\beta'} = 0 \qquad \nabla^{\mu} M_{\mu\nu}^{\alpha'\beta'} = 0$$

on the initial surface, a Mach–Green function. In appendix 3, we show formally how to construct a Mach–Green function, $M_{\mu\nu}^{\alpha'\beta'}$, such that

$$g^{\alpha'\beta'} = 2\kappa \int M_{\mu\nu}^{\alpha'\beta'} T^{\mu\nu} \sqrt{-g}\, \mathrm{d}^4x \qquad (6.16)$$

are Machian field equations for general relativity.

Flat and asymptotically-flat space–times are clearly ruled out by condition (6.16). Indeed, in these cases the condition coincides with Gilman's criterion. For the Robertson–Walker solutions the Gilman Green function is also already a Mach–Green function, so the result that these space–times satisfy Mach's principle is unchanged. In appendix 4 we show that the Bianchi type I spatially homogeneous anisotropic models are non-Machian. It is then reasonable to conjecture that all rotating or shearing spatially homogeneous models are non-Machian. For (*a*) the Mach condition is globally local (§6.2) so the global topology of the space–like hypersurfaces in these models is not important, and (*b*) no model near a type I model in the sense that the defining parameters a, n_1, n_2, n_3, Ω, $\omega^2 \sim 0$ (MacCallum 1973, Ellis and MacCallum 1969) but not near a Robertson–Walker model, $\sigma^2 \gg 0$, can be Machian. For otherwise a small variation in parameters would produce a large variation in the surface integral in contradiction to the stability of the representation. In principle, large contributions to the surface integral from large values of one or more of these parameters could cancel the large contribution from the shear implied by the result for type I models. It is difficult to see why some solutions should be picked out in this way, but the problem clearly requires further investigation.

Finally note that this formulation (6.16) of Mach's principle justifies the claim that a small matter inhomogeneity in a Robertson–Walker space–time should produce a small perturbation in the metric in a Machian solution. For if a small perturbation does produce a solution, i.e. if the space–time is linearisation-stable (D'Eath 1976, Fischer and Marsden 1979b), and if the result is Machian, then the stability of the Mach–Green function implies

$$g^{\alpha'\beta'} + \delta g^{\alpha'\beta'} = \int M_{\mu\nu}^{\alpha'\beta'} (T^{\mu\nu} + \delta T^{\mu\nu}) \sqrt{-g}\, \mathrm{d}^4x.$$

It follows that a small inhomogeneity in a Machian universe should induce small anisotropies in broad agreement with observation.

7. Holy Grail versus snare and delusion

The spherical symmetry of a sphere means that one cannot tell whether it has been rotated. It is easy to slip from this tautology to this profundity: that one should not be able to tell if it is rotating. This expresses the psychological appeal of Mach's principle, that it is at base a symmetry principle: a symmetry with respect to acceleration as deep as the symmetry with respect to velocity of the principle of relativity and to be as deeply cherished and as valiantly sought for (Raine 1979). On the other hand, to some tastes the principle of Mach offers only a misplaced nostalgia for an age when a universe might be construed in a page of algebra, an era that is now vanished in the pall of history. The title of this section juxtaposes two remarks on the subject made by distinguished relativists of those contrary persuasions.

How then is one to conclude? Certainly Mach's principle is part of *classical* physics. That would not be a problem were it not that Mach's principle purports to relate to regions of space–time where classical physics cannot be valid. For, near the big bang one expects the quantum aspects of gravity to become important. Mach's principle appears to require that free quantum oscillations of the gravitational field are not initially excited, although one cannot be precise in the absence of a quantum theory of gravity. This is obviously not the case for the quantum fields describing matter. One might more reasonably expect an initial black-body distribution of gravitational radiation by analogy with the electromagnetic background, although if this background is made as a result of non-time-reversible processes at early times, as in the grand unified theories (Turner and Shramm 1979, Weinberg 1979), this argument is less compelling. However, unless one accepts a semi-classical quantisation of gravity as the final theory, the gravitational field cannot be quantised on the basis of the integral representation (6.16) as field equations; one has to go back to the differential form of the original Einstein equations. On this view then Mach's principle may have been the holy grail, but the old order changeth.

The problem may however be even more fundamental. We have seen, rather surprisingly perhaps, that Mach's principle plays a well-defined role in the structure of space–time with regard to the relation between the affine connection and the matter content. In general relativity it receives remarkable expression in terms of the 'correct' field equations (6.16). One can then explain the absence of large shearing motions or rotation in the universe in accordance with observation. If it cannot quite explain why the universe must be approximately homogeneous, Mach's principle does at least imply that approximate homogeneity leads to approximate isotropy. Other theories have been proposed to account for these observations. They include the chaotic cosmology programme (Misner 1969), in which non-Machian anisotropies are supposed to be dissipated by viscous effects in universes without effective horizons, and an alternative mode of dissipation by quantum processes (Zeldovich 1972, Hartle and Hu 1980). These dissipation schemes produce entropy and are therefore limited by the observed radiation entropy density (Barrow and Matzner 1977). A rather different approach is through the anthropic principle (Carter 1974, Collins and Hawking 1973b). According to this scheme the universe has to have its observed symmetry in order to produce intelligent (or even unintelligent) observers.

How do we know which, if any, of these theories is true? How can we test a theory which has no effect on local physics, but which predicts that the universe is as it is because it forbids the occurrence of universes that are different? Here an intriguing possibility presents itself. An important aspect of current trends in the philosophy of science concerns the idea of an underdetermination of theories by data, an idea initiated by Duhem and argued perhaps most influentially by Quine (1969). At one level this thesis is obviously true, although it tends to raise the hackles of most physicists, who would argue that we do not even have one theory, let alone an underdetermined host. As far as I am aware, satisfactory examples of complete theories that would be undetermined by data have not been given. Mach's principle seems to present itself as a good candidate. Thus, it may explain why the compass of inertia rotates with the fixed stars and how inertia arises as a result of acceleration relative to the large-scale material universe—but we might never know it. In the context of Mach's phenomenalist thesis, it would be a savage irony indeed if this were the consequence of our 'glance ... into the depths' of the matter!

Appendices

Appendix 1. The Machian character of Robertson–Walker models

Gilman (1970) has shown that Robertson–Walker models with equation of state

$$p=(\gamma-1)\,\mu \qquad 1\leqslant\gamma<2 \tag{A1.1}$$

satisfy his Mach condition. We present here a simpler proof. This exploits the homogeneity and isotropy of the models which allows us to write the Sciama–Waylen–Gilman system of partial differential equations (6.5) in the gauge (6.6) as a pair of ordinary differential equations before passing to the integral representation. Recall that for Robertson–Walker models the three-geometry ($k=\pm 1, 0$) is uniquely related to the dynamics ($\rho\gtrless\rho_{\rm crit}$, $\rho=\rho_{\rm crit}$).

Define first an orthonormal tetrad of vectors $e_{(a)}$ ($a=0$, 1, 2, 3), with $e^{\mu}_{(0)}=\delta^{\mu}_{0}$ tangent to the fluid flow lines and such that the Robertson–Walker metric coefficients are given by

$$g_{\mu\nu}=e^{(a)}_{\mu}\, e^{(b)}_{\nu}\, \eta_{ab}$$

where η_{ab} is the Minkowski metric. Then, as in §2.2 and appendix 5, we have

$$\nabla_a u^b\equiv e^{\mu}_{(a)}\, e^{(b)}_{\nu}\, \nabla_\mu u^\nu=e^{\mu}_{(a)}\, \partial_\mu u^b+\Gamma^b_{ac}\, u^c$$

for a vector $u^a=e^{(a)}_{\mu}\, u^\mu$, where the Ricci rotation coefficients are

$$\Gamma^a_{bc}=e^{\nu(a)}\, e_{(c)\nu;\,\mu}e^{\mu}_{(b)}.$$

With these definitions the Sciama–Waylen–Gilman equations (6.5) in the gauge (6.6) become

$$\eta^{cd}\nabla_c\nabla_d\, \varphi^{ab}+2R^a{}_c{}^b{}_d\, \varphi^{cd}=2\kappa K^{ab} \tag{A1.2}$$

and yield an integral representation for η^{ab} in the limit $\delta g^{\mu\nu}\to 0$.

For Robertson–Walker metrices the coefficients in equation (A1.2) depend only on cosmic time t, so we may assume also $\varphi^{ab}=\varphi^{ab}(t)$. For simplicity we consider the $k=0$ model only. The other cases can be treated similarly, but in fact our result will hold generally since all models approach the $k=0$ case as $t\to 0$ where we check the Mach condition.

The non-zero Ricci rotation coefficients are

$$\Gamma^0{}_{ij}=\tfrac{1}{3}\theta\delta_{ij}=\Gamma_{ji0}$$

where $\theta=3\dot{R}/R$ is the expansion in a model with scale factor $R(t)$. Equations (A1.2) become

$$\begin{aligned}-\ddot{\varphi}_0-\theta\dot{\varphi}_0+\tfrac{2}{3}\theta^2\varphi_0-2\dot{\theta}\varphi_1&=2\kappa K_0\\ -\ddot{\varphi}_1-\theta\dot{\varphi}_1+\tfrac{2}{3}\theta^2\varphi_1-\tfrac{2}{3}\dot{\theta}\varphi_0&=2\kappa K_1\end{aligned} \tag{A1.3}$$

where the dot denotes differentiation with respect to t, and

$$\varphi_{ab}=\mathrm{diag}\,(\varphi_a) \qquad \varphi_1=\varphi_2=\varphi_3 \qquad K_{ab}=\mathrm{diag}\,(K_a).$$

Putting $\varphi_0=-1$, $\varphi_1=1$ and $K_0=\frac{3}{2}(\mu+3p)$, $K_1=\frac{1}{2}(\mu-p)$, we regain the field equations from (A1.3) as required. For completeness we state the gauge condition (equation (6.6)), the non-vanishing zero component of which is

$$\dot{\varphi}_0+\dot{\varphi}_1+\tfrac{2}{3}\theta(\varphi_0+\varphi_1)=0.$$

The conservation equation

$$\dot{\mu}+\theta(\mu+p)=0$$

yields

$$\mu=\mu_0R^{-3\gamma} \qquad \mu_0=\text{constant} \tag{A1.4}$$

using equation (A1.1) and the definition of θ. From equations (A1.3) we can write decoupled equations for $y_\pm=\varphi_0\pm\sqrt{3}\varphi_1$:

$$R^{3\gamma/2-4}\,(R^{4-3\gamma/2}y_\pm')'-(1/R^2)\,(6\pm3\sqrt{3})\,y_\pm=S_\pm \tag{A1.5}$$

where the prime denotes differentiation with respect to R. Note that in the general relativity limit we have

$$R^2S_\pm\sim R^2\times\text{constant}\times R^{3\gamma-2}\,\mu\sim\text{constant}$$

so $y_\pm$=constant is a particular integral of (A1.5) and there is no need to add a complementary function in order to obtain a representation of the metric. This suggests that the models are Machian.

Explicitly we construct a vector Green function (G_+, G_-) from homogeneous solutions of equations (A1.5) $u_\pm$, $v_\pm$ which are of the form R^p for suitable values of p. Set

$$u_+(R)=R^{p_1}-R_0{}^{p_1-p_2}\,R^{p_2}$$

$$v_+(R)=R^{p_2}-(p_2/p_1)\,R_0{}^{p_2-p_1}\,R^{p_1}$$

so $u_+(R_0)=0=v_+'(R_0)$, with similar expressions for u_-, v_-. Since $y_\pm'\to0$ at the initial time $R=R_0$ in the general relativity limit, the analogue of the Gilman surface integral (equation (6.11)) is

$$\mathscr{I}_\pm(R,R_0)=\left(\frac{R_0}{R}\right)^{4-3\gamma/2}\frac{u_\pm'(R_0)\,v_\pm(R)}{W_\pm(R)}$$

where $W_\pm(R)$ is the Wronskian. Choosing $p_1>0$, which is possible in the stated range of γ, we obtain

$$\lim_{R_0\to0}\mathscr{I}_\pm(R,R_0)=0$$

as required.

Appendix 2. Perturbations of Robertson–Walker space–times and the Mach condition

In this appendix we show how a perturbation to a Robertson–Walker space–time contributes to the metric beyond a particle horizon. It will follow that the Gilman surface integral is zero in these models by virtue of cancellations resulting from their symmetry.

Since the Weyl tensor is zero in Robertson–Walker models, its first-order perturbation, $\tilde{C}^{\mu\nu\rho\sigma}$, satisfies

$$\tilde{\nabla}_\mu\tilde{C}^{\mu\nu\rho\sigma}=\tilde{J}^{\nu\rho\sigma} \tag{A2.1}$$

as a consequence of the perturbed Bianchi identities. An explicit form for $\tilde{J}^{\nu\rho\sigma}$ is given in §6.2. The tildes denote quantities relating to the Robertson–Walker metric $\tilde{g}_{\mu\nu}$. This is conformally related to the Minkowski metric

$$\tilde{g}_{\mu\nu}=\Omega^2\eta_{\mu\nu}$$

where $\Omega=\Omega(r,t)$ (Infeld and Schild 1945).

We *define* the untilded quantities $C^{\lambda\mu\nu\rho}$, $J^{\mu\nu\rho}$ by

$$C^{\lambda\mu\nu\rho}=\Omega^7\tilde{C}^{\lambda\mu\nu\rho} \qquad J^{\mu\nu\rho}=\Omega^7\tilde{J}^{\mu\nu\rho}$$

which allows us to write equations (A2.1) as

$$\nabla_\mu C^{\mu\nu\rho\sigma}=J^{\nu\rho\sigma} \tag{A2.2}$$

where ∇_μ is a Minkowski space covariant derivative. Note that this transformation is *not* a conformal rescaling but merely a convenient definition. The aim now is to write these equations in a form in which they can be explicitly integrated. To do this we separate out the angular dependence using spin-weighted functions on the unit sphere, S^2 (Newman and Penrose 1966). First we need some technology.

A complex basis in the tangent space to S^2 is

$$\boldsymbol{m}=\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi} \qquad \bar{\boldsymbol{m}}=\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}.$$

Any three-space symmetric Cartesian tensor, V_{ij}, may be separated into spin-weighted components, V_s, defined by

$$V_2=V^{ij}m_im_j \qquad V_1=V^{ij}m_i\hat{r}_j \qquad V_0=V^{ij}\hat{r}_i\hat{r}_j \qquad V_{-1}=V^{ij}\bar{m}_i\hat{r}_j \qquad V_{-2}=V^{ij}\bar{m}_i\bar{m}_j$$

where $\hat{\boldsymbol{r}}$ is a unit radius vector. We choose V^{jk} to be the complex tensor

$$V^{jk}=C_0{}^j{}_0{}^k+(\mathrm{i}/2)\,\varepsilon^{lmk}\,C^j{}_{0lm}$$

where ε^{ijk} is the standard permutation tensor. We need also the spin-weighted sources $\rho_s(s=-1,0,1)$ constructed likewise from $J^{0j0}+(\mathrm{i}/2)\,\varepsilon^{jkl}\,J^0{}_{kl}$ and $J_s(-2\leqslant s\leqslant 2)$ constructed from $J^{0jk}+\mathrm{i}\,\varepsilon^{jkl}\,J^{00}{}_l$. Finally, we define the operator $\eth$ ('thop') acting on a quantity η of spin-weight s by

$$\eth\eta=-(\sin\theta)^s\,(\partial/\partial\theta+\mathrm{i}\,\mathrm{cosec}\,\theta\,\partial/\partial\varphi)\,(\sin\theta)^{-s}\,\eta$$

and the operator $\bar{\eth}$,

$$\bar{\eth}\eta=-(\sin\theta)^{-s}\,(\partial/\partial\theta-\mathrm{i}\,\mathrm{cosec}\,\theta\,\partial/\partial\theta)\,(\sin\theta)^s\,\eta.$$

Collecting these definitions, and employing the 'natural' basis in Minkowski space,

$$\boldsymbol{e}_0=\partial/\partial t \qquad \boldsymbol{e}_1=\partial/\partial r \qquad \boldsymbol{e}_2=(1/r)\,\partial/\partial\theta \qquad \boldsymbol{e}_3=(1/r\sin\theta)\,\partial/\partial\varphi$$

we can write equations (A2.2) as

$$\begin{aligned}
&\frac{1}{r}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)V_2-\eth V_1=J_2\\
&\frac{1}{r^4}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)(r^4V_1)+(1/r)\bar{\eth}V_2=J_1-\rho_1\\
&\frac{1}{r^4}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)(r^4V_0)-(1/r)\bar{\eth}V_1=J_0-\rho_0\\
&\frac{1}{r^4}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)(r^4V_0)+(1/r)\eth V_{-1}=J_0+\rho_0\\
&\frac{1}{r^4}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial r}\right)(r^4V_{-1})+(1/r)\eth V_{-2}=J_{-1}+\rho_{-1}\\
&\frac{1}{r}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial r}\right)V_{-2}-\bar{\eth}V_{-1}=J_{-2}.
\end{aligned} \tag{A2.3}$$

We now define the spin-weighted spherical harmonics $Y_{lm}^s(\theta, \varphi)$ by

$$Y_{lm}^0 = Y_{lm}$$
$$Y_{lm}^{s+1} = (l-s)^{-1/2}\,(l+s+1)^{-1/2}\,\eth Y_{lm}^s$$
$$Y_{lm}^{s-1} = -(l+s)^{-1/2}\,(l-s+1)^{-1/2}\,\bar{\eth} Y_{lm}^s.$$

The field and its sources in equations (A2.3) are expanded in series of spin-weighted harmonics, as

$$T_s = \Sigma T_{lm}^s\,(r, t)\; Y_{lm}^s(\theta, \varphi).$$

In particular, we find

$$(1/r^4)\,(\partial/\partial r)\,(r^4 V_{00}^0) = \rho_{00}^0$$

and therefore a radial field

$$V_{00}^0(r, t) = Q/4\pi r^4$$

where

$$Q = 4\pi \int \rho_{00}^0 r^4\,\mathrm{d}r.$$

We have therefore obtained an analogue of Gauss's theorem on a Robertson–Walker space–time background. It follows that a perturbation gives rise to an instantaneous component of the gravitational field and substantiates our claim that the influence of a perturbation extends beyond the particle horizon for the case of a spherically symmetric perturbation. For the Robertson–Walker model itself we regard each element of matter beyond the particle horizon as a perturbation which contributes a non-zero field, the total effect summing to zero by symmetry.

Appendix 3. General relativistic Machian field equations

We outline an integral formulation of general relativity constructed to circumvent the problem of particle horizons in a direct manner.

Suppose $\varphi_T^{\mu\nu}$ is a solution of the Sciama–Waylen–Gilman equation (6.5) with zero boundary conditions on a regular initial surface Σ_0. If $\varphi_T^{\mu\nu}$ satisfies the gauge conditions (equation (6.6)) for all choices of Σ_0, we can proceed to the limit that Σ_0 tends to the initial boundary to the whole space–time, and the Gilman Mach condition is $\varphi_T^{\mu\nu} \to g^{\mu\nu}$. If $\varphi_T^{\mu\nu}$ does not satisfy the gauge conditions as Σ_0 tends to the initial boundary, we construct a new solution which does. Thus we put

$$\bar{\psi}_T^{\mu\nu} = \psi_T^{\mu\nu} + \Lambda^{\mu\nu}$$

where $\psi^{\mu\nu} = \varphi^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\varphi$, and require

$$(\mathscr{L}\Lambda)^{\mu\nu} = 0 \tag{A3.1}$$

where $\mathscr{L}$ is the Sciama–Waylen–Gilman operator defined by equation (6.5). Equation (A3.1) is to be solved subject to initial conditions on Σ_0:

$$\nabla_\mu \Lambda^{\mu\nu} = -\nabla_\mu \psi_T^{\mu\nu} \equiv f^\nu.$$

By hypothesis, f^ν depends linearly on $T^{\mu\nu} + \delta T^{\mu\nu}$.

The initial conditions do not uniquely determine $\Lambda^{\mu\nu}$. We may add the conditions $\Lambda^{ij} = 0$, $\Lambda^{0\mu} = 0$ on Σ_0, which are consistent with $(\mathscr{L}\Lambda)^{ij} = 0$, whence (A3.1) is replaced by four equations

$$(\mathscr{L}\Lambda)^{0\mu} = 0 \tag{A3.2}$$

subject to the initial conditions

$$\Lambda^{0\mu}=0 \qquad \nabla_0\Lambda^{0\mu}=f^\mu \tag{A3.3}$$

on Σ_0. Formally we obtain a hyperbolic initial value problem, with solution a linear function of $T^{\mu\nu}+\delta T^{\mu\nu}$.

Thus formally we have

$$\psi^{\alpha'\beta'}=-2\kappa\int_\Omega M^{\alpha'\beta'}_{\mu\nu}(T^{\mu\nu}+\delta T^{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x$$

where $\Omega=\Sigma_0\times[t_0,\infty)$. Of course, on regular hypersurfaces Σ_0, $\psi^{\alpha'\beta'}$ still satisfies zero boundary conditions. But $M^{\alpha'\beta'}_{\mu\nu}$ satisfies the gauge conditions in x' as $x'\to x_0'\in\Sigma_0$; hence, $\psi^{\alpha'\beta'}$ satisfies the gauge conditions on Σ_0 (and therefore for all time) for all choices of Σ_0. In the limit $\delta g^{\mu\nu}\to 0$, $\Omega\to\mathscr{M}$ the whole space–time, we obtain an integral representation for a Machian solution

$$g^{\alpha'\beta'}=2\kappa\int_{\mathscr{M}} M^{\alpha'\beta'}_{\mu\nu}T^{\mu\nu}\sqrt{-g}\,\mathrm{d}^4x. \tag{A3.4}$$

Equivalently, equations (A3.4) can be regarded as integral field equations, all the solutions of which satisfy both the Einstein equations and Mach's principle.

Appendix 4. The non-Machian character of Bianchi type I cosmologies

I shall show that Bianchi I models with perfect fluid equations of state (equation (A1.1)) do not satisfy Mach's principle. This conclusion is independent of whether one uses the Gilman criterion or that outlined in §6.3 and appendix 3 and agrees with the result in Raine (1975a).

We again start from the Sciama–Waylen–Gilman equations in the form (A1.2) but with the basis tetrad chosen to be appropriate to the Bianchi I metric

$$\mathrm{d}s^2=-\mathrm{d}t^2+X_1^2(t)\,\mathrm{d}x_1^2+X_2^2(t)\,\mathrm{d}x_2^2+X_3^2(t)\,\mathrm{d}x_3^2.$$

The non-zero Ricci rotation coefficients are

$$\Gamma_{ij0}=\theta_{ij}=-\Gamma_{0ij}$$

where

$$\theta_{ij}=\sigma_{ij}+\tfrac{1}{3}\theta\delta_{ij}$$

is the expansion tensor given by

$$\theta_{ij}=0 \qquad i\neq j \qquad \theta_{11}=\dot{X}_1/X_1 \qquad \text{etc.}$$

It follows that equations (A1.2) can be written explicitly as

$$\begin{aligned}-\ddot{\varphi}_0+\theta\dot{\varphi}_0-2(\theta^k\theta_k)\,\varphi_0-2\dot{\theta}_k\varphi^k&=2\kappa K^0\\ -\ddot{\varphi}_i+\theta\dot{\varphi}_i-2\theta_i(\theta^k\varphi_k)-2\dot{\theta}_i\varphi_0&=2\kappa K_i\end{aligned} \tag{A4.1}$$

where

$$\varphi_{ij}=\mathrm{diag}\,(\theta_i) \qquad \theta_{ij}=\mathrm{diag}\,(\theta_i) \qquad K_{ij}=\mathrm{diag}\,(K_i).$$

Introducing an average scale factor $l(t)$ by

$$\theta=3\dot{l}/l=\dot{X}_1/X_1+\dot{X}_2/X_2+\dot{X}_3/X_3$$

we can again integrate the conservation equation to obtain

$$\mu=\mu_0 l^{-3\gamma} \qquad \mu_0=\text{constant}.$$

The Einstein field equations give

$$\dot{\sigma}_{ij} + \theta\sigma_{ij} = 0$$

from which

$$\sigma_i = {}_i/l\Sigma^3 \qquad \Sigma_i = \text{constant.}$$

The remaining field equations are

$$\tfrac{1}{3}\theta^2 = \tfrac{1}{2}(\sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2) + \mu$$

$$\dot{\theta} + \tfrac{1}{3}\theta^2 + \sigma_1{}^2 + \sigma_2{}^2 + \sigma_3{}^2 = -\tfrac{1}{2}(\mu + 3p).$$

Provided $\mu/\theta^2 \to 0$ as $l \to 0$, which is the consistent assumption for $\gamma < 2$, we find, asymptotically as $l \to 0$,

$$\theta \sim l^{-3} \qquad \dot{\theta} \sim l^{-6}.$$

Equation (A4.1) becomes, as $l \to 0$,

$$(-\varphi_0{}'l)' + 18l^{-1}\varphi_0 + 6\sqrt{6}l^{-1}\ \Sigma_k(\Sigma^2)^{-1/2}\,\varphi_k - 6l^{-1}\,(\varphi_1 + \varphi_2 + \varphi_3) = (6/\Sigma^2)\,l^5 S_0$$

$$(-\varphi_i{}'l)' + 12l^{-1}\,[\Sigma_i(\Sigma^2)^{-1/2} + 1/\sqrt{6}]\,[1/\sqrt{6}\,(\varphi_1 + \varphi_2 + \varphi_3) + \Sigma_k(\Sigma^2)^{-1/2}\,\varphi_k]$$
$$- 6l^{-1}\{(\sqrt{3}/\sqrt{2})\ \Sigma_i[(\Sigma^2)^{1/2} - 1]^{-1}\}\varphi_0 = (6/\Sigma^2)\,l^5 S_i$$

where $\Sigma^2 = \Sigma_k\Sigma^k$. Since the source terms are of the asymptotic form $l^{9-3\gamma}$, it is clear that as $l \to 0$, $\varphi^0 = -1$, $\varphi^i = 1$ is an asymptotic solution of the homogeneous systems, and we expect the models to be non-Machian.

Somewhat more explicitly, we construct a Green function from homogeneous solutions, l^{p_i}. For completeness, the non-vanishing constraint (equation (6.6)) is

$$\partial/\partial t\{l^2[\varphi^0 + \tfrac{1}{3}(\varphi^1 + \varphi^2 + \varphi^3)]\} + \sigma_i\varphi^i = 0.$$

However, since this is to be satisfied by adding solutions, $u(l)$, $v(l)$, of the homogeneous system this merely alters the coefficients in the Green function without affecting the following argument. Thus, the particular integral of (A4.2) is a sum of terms of the form

$$\int_{l_0}^{l'} \frac{l}{l'} \frac{u(l)\ v(l')l^5 S(l)}{w(l)}\,\mathrm{d}l \sim \int_{l_0}^{l'} \frac{l^{1+p+5+4-3\gamma}\,l'^{p'}}{l'\,l^{p+p'-1}}\,\mathrm{d}l$$

which is vanishingly small for l' near zero. Therefore the integral representation of $\varphi^{a'b'}(l')$ must contain a 'surface' term, and the solution is non-Machian.

Appendix 5. Notation and conventions

We use the MTW conventions (Misner *et al* 1973), so the metric has signature $(-1, +1, +1, +1)$, the Ricci tensor is

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

and the Einstein equations are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}$$

where $R = g_{\mu\nu}R^{\mu\nu}$ is the Ricci scalar, and we have arbitrarily set the cosmological constant to be zero. Greek indices, $\alpha, \beta, \ldots \mu, \nu, \ldots$ range over the values (0, 1, 2, 3) and indicate coordinate components, latin indices $a, b, \ldots$ take the same range for tetrad components, while latin indices $i, j, \ldots$ take the values (1, 2, 3).

Other notation is entirely standard and is collected here for convenience. The tetrad basis $e^{\mu}_{(a)}$ is related to the metric through the dual 1-forms, $e^{(a)}_{\mu}\,\mathrm{d}x$, by

$$\mathrm{d}s^2=\eta_{ab}e^{(a)}_{\mu}\,\mathrm{d}x^{\mu}e^{(b)}_{\nu}\,\mathrm{d}x^{\nu}.$$

The commutators are

$$\underset{e_a}{L}e_b=\gamma^{c}{}_{ab}e_c$$

from which we construct the Ricci rotation coefficients

$$\Gamma_{abc}=\tfrac{1}{2}(\gamma_{abc}+\gamma_{cab}-\gamma_{bca}).$$

Tetrad indices, $(a, b, \ldots)$, are raised and lowered using η_{ab}. The covariant derivative ∇_a is defined by its action on a vector u^b:

$$\nabla_a u^b=\partial_a u^b+\Gamma^{b}{}_{ac}u^c$$

where $\partial_a=e^{\mu}_{(a)}\partial_\mu$ and ∂_μ denotes $\partial/\partial x^\mu$. Covariant differentiation in a coordinate basis is also symbolised by a semicolon, $\nabla_\mu u^\nu\equiv u^\nu{}_{;\mu}$, and for a connection defined by a three-space metric by a bar, $u^i{}_{|j}$.

The Riemann tensor is given explicitly by

$$R^{a}{}_{bcd}=\partial_c\Gamma^{a}{}_{db}-\partial_d\Gamma^{a}{}_{cb}+\Gamma^{a}{}_{ce}\Gamma^{e}{}_{db}-\Gamma^{a}{}_{de}\Gamma^{e}{}_{cb}+\Gamma^{a}{}_{eb}\gamma^{e}{}_{dc}$$

and the Weyl tensor by

$$C_{abcd}=R_{abcd}+g_{a[c}R_{d]b}-g_{b[c}R_{d]a}+\tfrac{1}{3}Rg_{a[c}g_{d]b}$$

where the square bracket denotes anti-symmetrisation:

$$A_{[ab]}=\tfrac{1}{2}(A_{ab}-A_{ba}).$$

Symmetrisation is denoted by

$$A_{(ab)}=\tfrac{1}{2}(A_{ab}+A_{ba}).$$

For geodesic flow orthogonal to a surface of homogeneity the shear $\sigma_{\mu\nu}$ and expansion θ of the fluid velocity u^μ are defined by

$$u_{i;j}=\sigma_{ij}+\tfrac{1}{3}\theta g_{ij}.$$

In numerical results the Hubble constant is taken as $H_0=100$ km s^{-1} Mpc^{-1}.

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