# Machian Theory of Inertia and Gravitation

ROBERT C. GILMAN*
*Princeton University Observatory, Princeton, New Jersey 08540*
(Received 12 March 1970)

The generally covariant integral formulation of Einstein's field equations developed by Sciama, Waylen, and the author is used to define the source-free contribution to the inertial-gravitational field. A theory is then developed, based on Mach's ideas on the origins of inertia, that requires of acceptable cosmological models that they be entirely source-generated solutions of Einstein's field equations without the cosmological constant term. We show that many (and probably all) nonempty relativistic Robertson-Walker models with $\Lambda=0$ and $0\leq p\leq \rho c^2$ are acceptable Machian cosmologies, in terms of this theory. On the other hand, many solutions are found to be unacceptable as Machian cosmologies, including the Minkowski, Schwarzschild, Kerr, Gödel, and Kantowski-Sachs-Thorne solutions. Some of the implications of this theory are discussed.

## I. INTRODUCTION

IN a previous paper,[1] Sciama, Waylen, and the author have presented a generally covariant integral formalism based on Einstein's field equations (hereafter referred to as the SWG formalism). In this paper, the SWG formalism will be used as the basis of an integral theory of the inertial-gravitational (IG) field. The qualitative ideas behind this theory have been discussed previously by a number of authors.[2] Briefly, the idea is that, in accord with Mach's ideas on the origins of inertia,[3] we want to admit as physically acceptable only those space-times that are entirely source-generated. The importance of the SWG formalism is that is provides a means for rigorously distinguishing between the source-generated and the source-free contributions to the the IG field. In the rest of this paper we will develop the theory, apply it to a number of relativistic cosmologies, and discuss the results.

## II. THEORY

### Source-free Contribution to IG Field

In the SWG formalism, the IG potential at some space-time point $x'$ is given by a volume integral over sources in the past light cone plus a surface integral, i.e.,[4]

$$g^{\alpha'\beta'}(x')=2\kappa\int_\Omega\left\{G_\mu{}^{-\alpha'\beta'\nu}(x,'x)\left[T^\mu{}_\nu(x)-\tfrac{1}{2}T(x)\delta^\mu{}_\nu-\frac{\Lambda}{\kappa}\delta_\mu{}^\nu\right]\times[-g(x)]^{1/2}d^4x\right\}+\int_{\partial\Omega}G^{-\alpha'\beta'\nu}{}_{\nu;}{}^\sigma(x',x)\times[-g(x)]^{1/2}dS_\sigma. \quad (1)$$

For notation convenience, we will rewrite Eq. (1) as

$$g_{\alpha'\beta'}(x')={}^1g_{\alpha'\beta'}(x',\partial\Omega)+{}^2g_{\alpha'\beta'}(x',\partial\Omega), \quad (2)$$

where ${}^1g_{\alpha'\beta'}$ and ${}^2g_{\alpha'\beta'}$ represent, respectively, the volume and surface integrals in Eq. (1).

In the standard integral formulations for scalar and vector fields,[5] the volume integral is interpreted in physical terms as the contribution to the local field from the sources within the volume of integration, while the surface integral is interpreted as the contribution from sources outside the volume of integration plus any "waves coming in from infinity," i.e., the source-free contribution. An important feature of the SWG formalism is that the conditions under which it is the unique integral representation of Einstein's field equations are completely analogous to the conditions that are required to ensure that the standard integral formulations for scalar and vector fields provide unique representations for their respective field equations.[6]

This suggests, and we shall assume, that the usual physical interpretations are applicable to ${}^1g_{\alpha'\beta'}$ and ${}^2g_{\alpha'\beta'}$. Thus the source-free contribution is part of ${}^2g_{\alpha'\beta'}$, and our problem now is to separate out this part.

The integrals in Eq. (1) are well defined as long as the volume of integration $\Omega$ is globally hyperbolic.[7] The requirement of global hyperbolicity is essentially a causality requirement, since it excludes closed timelike paths. Globally hyperbolic cosmological models are generally of two types: Either they have a particle horizon in the finite proper past or they extend back into an infinite proper past. Let us define $\partial\Omega_S$ as that bounding surface which, on and within the past light cone, is either the surface formed by the union of particle horizons for points on a timelike world line containing $x'$, or the infinite past "surface," whichever is appro-

---

* Present address: Rosburg, Wash. 98643.
[1] D. W. Sciama, P. C. Waylen, and R. C. Gilman, Phys. Rev. **187**, 1762 (1969).
[2] B. L. Al'tshuler, Zh. Eksperim. i. Teor. Fiz. **51**, 1143 (1966) [Soviet Physics JETP **24**, 766 (1967)]; D. Lynden-Bell, Monthly Notices Roy. Astron. Soc. **135**, 413 (1967); D. W. Sciama, P. C. Waylen, and R. C. Gilman (Ref. 1); R. C. Gilman, Ph. D. thesis, Princeton University, 1969 (unpublished).
[3] E. Mach, *The Science of Mechanics* (Open Court, Chicago, 1919).
[4] The SWG formalism and the notation employed in it are explained in Ref. 1.
[5] See B. S. DeWitt and R. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[6] For a more detailed discussion of this point, see R. C. Gilman, Ph.D. thesis, Princeton University (unpublished), as well as Ref. 1.
[7] Y. Choquet-Bruhat, in *Battele Rencontres*, edited by C. A. DeWitt and J. A. Wheeler (Benjamin, New York, 1968), p. 84.

priate for a given cosmology. Let us also define

$$I_{\alpha'\beta'}(x') \equiv \lim_{\partial\Omega \to \partial\Omega_S} {}^2g_{\alpha'\beta'}(x',\partial\Omega). \tag{3}$$

When we take the limit $\partial\Omega \to \partial\Omega_S$, $\Omega$ comes to contain all of the sources in the accessible past light cone, so $I_{\alpha'\beta'}$ must be interpreted (in analogy with the usual interpretation in the case of scalar and vector fields) as "waves coming from $\partial\Omega_S$," i.e., contributions to the local field that cannot be attributed to any of the observable sources. Since it is not source-generated, $I_{\alpha'\beta'}$ must be *the source-free contribution to the* IG *field at* $x'$.

### Classification Scheme for Relativistic Cosmologies

With the source-free contribution to the field now defined, we can briefly state our Machian theory. *Every acceptable cosmological model must*: (1) be a solution of Einstein's field equations with $\Lambda=0$, and (2) contain no source-free contributions to the IG field, i.e., $I_{\alpha'\beta'}=0$ everywhere. Thus the acceptable cosmologies in this theory are a subset of the relativistic cosmologies. We require $\Lambda=0$ because within the SWG formalism $\Lambda$ must be treated as a source term; yet $\Lambda$ is clearly not a Machian source term.

What is the program for developing this theory? The condition $I_{\alpha'\beta'}=0$ is a boundary condition, and if this theory were linear, we could use this boundary condition and the SWG integral equations to compute the solutions that are acceptable cosmologies. Even with the nonlinearity of the theory, it may be possible to do this, but it is not yet clear how. We will instead start with a *known* cosmological solution of Einstein's field equations and thus also of the integral equations. We can then (1) compute the Green's functions, (2) compute ${}^2g_{\alpha'\beta'}$ for $\partial\Omega \neq \partial\Omega_S$, and (3) compute $I_{\alpha'\beta'}$. We will then use the scheme in Table I to classify the solution. This classification scheme is very elementary, but it is probably wise to wait until more is known about the general theory of Green's function integral formulations and how they apply to general relativity before constructing a more elaborate scheme. For many cosmologies it will not be necessary to carry out all three of the above steps. There will also be some cases in which it is impossible to do so because, for example, not all cosmologies are globally hyperbolic. We can, nevertheless, classify *all* relativistic cosmologies with the above scheme. The division between Classes I and II and Class III is immediately obtainable on the basis of $\Lambda$, and if $I_{\alpha'\beta'}$ is not well defined for some solution, then the solution can not be of Class I. Thus our provisional program for the development of the theory requires the classification of known relativistic cosmologies according to the above scheme.

We can in fact immediately classify many solutions by inspection. Thus, in Class II are all solutions with $\Lambda=0$ and $T_{\mu\nu}=0$ everywhere; in Class III are all solutions with $\Lambda\neq 0$, notably the Gödel solution. It should be noted that the Gödel solution is not of Class I not only because of its nonzero $\Lambda$, but also because, with its closed timelike world lines, it is not globally hyperbolic.

TABLE I. Classification scheme for cosmological models.

| Class | $I_{\mu\nu}$ | $\Lambda$ | Proposed interpretation |
|---|---|---|---|
| I | zero everywhere | zero | fully Machian cosmologies |
| II | not everywhere zero | zero | non-Machian as cosmologies, but possibly Machian as pieces of Class-I cosmologies |
| III | anything | nonzero | non-Machian |

### Some General Equations

Before going on to consider specific cosmologies it will be helpful to record here some general equations needed for the evaluation of $I_{\alpha'\beta'}$ and not reported in our previous paper.[8] The surface integral ${}^2g_{\alpha'\beta'}$ involves only the partially contracted retarded Green's function which, within a normal neighborhood of $x'$, can be written as[9]

$$G^{-\alpha'\beta'\nu}{}_{\nu} = (1/2\pi)[\delta^{-}(\Gamma)g^{\alpha'\beta'}\Sigma + H^{-}(\Gamma)V^{\alpha'\beta'\nu}{}_{\nu}]. \tag{4}$$

This does not involve the bi-vectors of geodetic parallel displacement and thus it is simpler to compute than the full Green's function. The boundary values for $V^{\alpha'\beta'\nu}{}_{\nu}$ on the light cone have the simple form

$$V^{\alpha'\beta'\nu}{}_{\nu}[1-(1/2\Sigma)\Sigma_{;\rho'}\Gamma^{;\rho'}] + V^{\alpha'\beta'\nu}{}_{\nu;\rho'}\Gamma^{;\rho'} = -\tfrac{1}{2}g^{\alpha'\beta'}\Box'\Sigma + R^{\alpha'\beta'}\Sigma. \tag{5}$$

## III. APPLICATIONS

### Robertson-Walker Cosmologies

The simplest cosmological models consistent with our understanding of the universe have metrics of the Robertson-Walker type,[10] i.e., with time-orthogonal spacelike hypersurfaces which are homogeneous and isotropic. We will begin our analysis of specific cosmologies with this important group. Of the many forms of this metric, the most useful for our purposes can be expressed in terms of the line element

$$ds^2 = A^2(\tau)[d\tau^2 - d\chi^2 - S^2(\chi)d\Omega^2], \tag{6}$$

where $\tau$ is a timelike coordinate ($x^0$), $\chi$ is a radial spacelike coordinate ($x^1$), $\theta$ and $\varphi$ are angular coordinates ($x^2$ and $x^3$), and

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2. \tag{7}$$

There are three forms for $S(\chi)$, depending on the

[8] D. W. Sciama, P. C. Waylen, and R. C. Gilman (Ref. 1.).
[9] For definitions and notation, see Ref. 1.
[10] H. P. Robertson, Astrophys. J., 82, 284 (1935); A. G. Walker, Proc. London Math. Soc. 42, 90 (1936)

curvature, i.e.,

$$S(\chi)=\sin\chi \quad \text{for} \quad k=+1$$
$$=\chi \quad \text{for} \quad k=0 \qquad (8)$$
$$=\sinh\chi \quad \text{for} \quad k=-1,$$

where $k$ is the curvature index.[11]

The scale function $A(\tau)$ can be obtained by solving Einstein's field equations; it depends on $k$, the energy density $\rho c^2$, and the pressure $p$. It also depends on $\Lambda$, but since any solution with $\Lambda\neq 0$ belongs to Class III, we will consider only $\Lambda=0$ solutions here. The behavior of $A(\tau)$ can be illustrated by taking an equation of state of the form

$$p(\tau)=\gamma\rho(\tau)c^2, \qquad (9)$$

where physical considerations require $0\leq\gamma\leq 1$.[12] Models with this type of equation of state have to be discussed thoroughly by Vajk[13] and others, and we give here, without proof, some well-known results. This equation of state has the conservation law

$$\rho_0=\rho(\tau)A^{(3\gamma+3)}(\tau)=\text{const}. \qquad (10)$$

$A(\tau)$ is given by[14]

$$k=+1: A(\tau)=A_0{}^{m/2}\sin^m(\tau/m), \qquad (11)$$

$$k=0: \quad A(\tau)=A_0{}^{m/2}(\tau/m)^m, \qquad (12)$$

$$k=1: \quad A(\tau)=A_0{}^{m/2}\sinh^m(\tau/m), \qquad (13)$$

where

$$m=2/(3\gamma+1), \quad A_0=\tfrac{1}{3}\kappa\rho_0c^2.$$

The zero for $\tau$ has been chosen so that $A(\tau)\to 0$ as $\tau\to 0$. The most general nonempty models of the Robertson-Walker type have equations of state of the form

$$p(\tau)=\gamma(\tau)\rho(\tau)c^2, \qquad (14)$$

where again physical considerations require $0\leq\gamma(\tau)\leq 1$. Allowing $\gamma$ to vary with time is physically reasonable, but complicates the mathematics without changing the qualitative features of importance here. It should be noted that all nonempty Robertson-Walker models have singular constant-$\tau$ hypersurfaces at $\tau=0$.

The empty Robertson-Walker model is Minkowski space-time. This can be written in terms of our line element in either of two forms:

$$k=0, \quad A(\tau)=C \qquad (15)$$

or

$$k=-1, \quad A(\tau)=Ce^{\tau}, \qquad (16)$$

where $C$ is some arbitrary nonzero constant.

We will find it convenient, for explicit calculations, to work in terms of orthonormal tetrad (ONT) components, i.e., we relate tensor components to ONT components by[15]

$$W_{\sigma\tau}=e_\sigma{}^ie_\tau{}^jW_{ij}, \qquad (17)$$

where $W_{\sigma\tau}$ is some arbitrary second-rank tensor. We will, for the rest of this paper, use Greek indices for tensor components and Latin indices for ONT components. The $e$'s for our form of the Robertson-Walker metric are

$$\begin{aligned} &e_0{}^0=e_1{}^1=A(\tau),\\ &e_2{}^2=A(\tau)S(\chi),\\ &e_3{}^3=\sin\theta\, e_2{}^2,\\ &e_\mu{}^j=0 \quad \text{for} \quad \mu\neq j. \end{aligned} \qquad (18)$$

We will distinguish between tensor and ONT for explicit components with a tilde, i.e., $\tilde{W}_{00}$ for tensor and $W_{00}$ for ONT.

Finally, we will use the convention[16] that, for $Q$ some arbitrary function of $\tau$ and/or $\chi$,

$$\dot{Q}\equiv\partial Q/\partial\tau, \quad Q'\equiv\partial Q/\partial\chi.$$

### *Bi-Scalars* $\Gamma$ *and* $\Sigma$

We now proceed with the evaluation of the quantities necessary to determine $I_{\alpha'\beta'}$ for Robertson-Walker models. Let $x_0$ be a local variable point and let $x_f$ be some other (fixed) space-time point. Let the radial coordinate system be fixed at $x_f$ (i.e., $\chi_f=0$) and

$$\begin{aligned} \Delta\tau&\equiv\int_{\tau_f}^{\tau_0}d\tau=\tau_0-\tau_f,\\ \Delta\chi&\equiv\int_{\chi_f}^{\chi_0}d\chi=\chi_0, \end{aligned} \qquad (19)$$

Along radial geodesics these are related by[17]

$$\Delta\chi=\pm\int_{\tau_f}^{\tau_0}[1+\beta^2A^2(\tau)]^{-1/2}d\tau, \qquad (20)$$

where we adopt the convention that the top sign refers to the $\Delta\tau>0$ case. The bi-scalar $\beta(x_0,x_f)$ is constant along the geodesic and equal to zero on the light cone. The geodesic interval $s(x_0,x_f)$ is given by

$$s=\pm\beta\int_{\tau_f}^{\tau_0}[1+\beta^2A^2(\tau)]^{-1/2}A^2(\tau)d\tau. \qquad (21)$$

[11] H. P. Robertson (Ref. 10).

[12] See B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Chicago U. P., Chicago, 1965).

[13] J. P. Vajk, Ph.D. thesis, Princeton University, 1968 (unpublished).

[14] For $k\neq 0$, Vajk gives $A(\tau)$ for only certain values of $m$. The general forms (11) and (13) were obtained by Dr. J. E. Gunn and the author and are apparently new.

[15] Cf. H. Flanders, *Differential Forms* (Academic, New York, 1963).

[16] We are also using the prime (′) to denote a particular space-time point, $x'$ and its associated tensor indices (see above), but this double usage does not overlap and should not cause any confusion.

[17] H. P. Robertson, Rev. Mod. Phys. 5, 62 (1933).

Let us define

$$I_n(\tau_0,\tau_f)\equiv\int_{\tau_f}^{\tau_0} A^n(\tau)d\tau\,. \tag{22}$$

Then we find from Eq. (20) that

$$\begin{aligned}\beta^2(x_0,x_f)=(2/I_2)(\Delta\tau\mp\Delta\chi)+(3I_4/I_2^3)(\Delta\tau\mp\Delta\chi)^2\\+(9I_4^2/I_2^5-5I_6/I_2^4)(\Delta\tau\mp\Delta\chi)^3\\+O[(1\mp\Delta\chi/\Delta\tau)^4].\end{aligned} \tag{23}$$

From the definition $\Gamma(x_0,x_f)=\pm s^2(x_0,x_f)$, and Eqs. (21) and (23), we find

$$\begin{aligned}\Gamma(x_0,x_f)=2I_2(\Delta\tau\mp\Delta\chi)-(I_4/I_2)(\Delta\tau\mp\Delta\chi)^2\\+(I_6/I_2^2-I_4^2/I_2^3)(\Delta\tau\mp\Delta\chi)^3\\+O[(1\mp\Delta\chi/\Delta\tau)^4].\end{aligned} \tag{24}$$

The bi-scalar $\Sigma$ is related to $\Gamma$ by[18]

$$\Sigma(x_0,x_f)=\tfrac{1}{4}[g(x_0)g(x_f)]^{-1/4}[-\det(\Gamma_{;\mu_0;\nu_f})]^{1/2}, \tag{25}$$

and we find

$$\Sigma=\frac{I_2}{A(\tau_0)A(\tau_f)S(\Delta\chi)}\left[1-\frac{A^2(\tau_0)+A^2(\tau_f)}{2I_2}(\Delta\tau\mp\Delta\chi)+O((1\mp\Delta\chi/\Delta\tau)^2)\right]. \tag{26}$$

On the light cone,

$$\Box_0\Sigma=-\frac{\Sigma}{A^2(\tau_0)}\left(\frac{\ddot{A}(\tau_0)}{A(\tau_0)}+k\right). \tag{27}$$

From the general rules for transforming Dirac $\delta$ functions[19] and the above results, we find the following simple expression for the light-cone part of the Green's function, i.e.,

$$\delta^-(\Gamma)\Sigma(x_0,x_f)=\frac{\delta(\Delta\tau-\Delta\chi)}{A(\tau_0)A(\tau_f)S(\Delta\chi)}. \tag{28}$$

The terms in the denominator can be interpreted as follows. One of the $A$'s is necessary for bookkeeping reasons since $\delta^-(\Gamma)$ is a bi-scalar but $\delta(\Delta\tau-\Delta\chi)$ is a bi-density. The other $A$ combines with the $S$ to give a proper radius; i.e., $4\pi A^2S^2$ is the proper surface area of a sphere centered on the source, going through the observer, and evaluated at the time of the reception of the signal at the observer.[20]

### *Bi-Tensor* $V^{\alpha'\beta'\nu}{}_{\nu}$

This part of the Green's function is a solution to the homogeneous field equations.[21] To solve for it in the case of the Robertson-Walker metrics, we need the explicit form for $D_{\mu\nu}{}^{\sigma\tau}w_{\sigma\tau}$, where $D_{\mu\nu\sigma\tau}$ is the basic operator of the SWG formalism,

$$D_{\mu\nu\sigma\tau}=\tfrac{1}{4}(g_{\mu\sigma}g_{\nu\tau}+g_{\mu\tau}g_{\nu\sigma})\nabla^\rho\nabla_\rho-\tfrac{1}{2}(R_{\mu\sigma\nu\tau}+R_{\mu\tau\nu\sigma}),$$

and $w_{\sigma\tau}$ is some arbitrary symmetric second-rank tensor. We shall assume $\partial w_{\sigma\tau}/\partial\varphi=0$, since it is easily shown that this is the case for the tensors and bi-tensors of interest here. We find

$$\begin{aligned}\tilde{D}_{00}{}^{\sigma\tau}w_{\sigma\tau}=\frac{1}{2}\bigg\{&\ddot{w}_{00}-w_{00}''-\frac{1}{S^2}\frac{\partial^2w_{00}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{00}-2\frac{S'}{S}w_{00}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{00}+4\frac{\dot{A}}{A}w_{01}'+\frac{4\dot{A}}{AS}\frac{\partial}{\partial\theta}w_{02}-6\left(\frac{\dot{A}}{A}\right)^2w_{00}\\&+8\frac{\dot{A}}{A}\frac{S'}{S}w_{01}+4\frac{\dot{A}}{A}\frac{\cot\theta}{S}w_{02}-2\left(\frac{\dot{A}}{A}\right)^2(w_{11}+w_{22}+w_{33})+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right](w_{11}+w_{22}+w_{33})\bigg\},\end{aligned} \tag{29}$$

$$\begin{aligned}\tilde{D}_{01}{}^{\sigma\tau}w_{\sigma\tau}=\frac{1}{2}\bigg\{&\ddot{w}_{01}-w_{01}''-\frac{1}{S^2}\frac{\partial^2w_{01}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{01}-2\frac{S'}{S}w_{01}-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{01}+2\frac{\dot{A}}{A}(w_{00}'+w_{11}')+\frac{2}{S}\frac{\partial}{\partial\theta}\left(\frac{\dot{A}}{A}w_{02}+\frac{S'}{S}w_{12}\right)\\&-6\left(\frac{\dot{A}}{A}\right)^2w_{01}+2\left(\frac{S'}{S}\right)^2w_{01}+2\frac{\dot{A}S'}{AS}(2w_{11}-w_{22}-w_{33})+\frac{2\cot\theta}{S}\left(\frac{\dot{A}}{A}w_{12}+\frac{S'}{S}w_{02}\right)+\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]w_{01}\bigg\},\end{aligned} \tag{30}$$

$$\begin{aligned}\tilde{D}_{02}{}^{\sigma\tau}w_{\sigma\tau}=\frac{1}{2}S\bigg\{&\ddot{w}_{02}-w_{02}''-\frac{1}{S^2}\frac{\partial^2w_{02}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{02}-2\frac{S'}{S}w_{02}'-\frac{\cot\theta}{S^2}\frac{\partial w_{02}}{\partial\theta}+2\frac{\dot{A}}{A}w_{12}'+\frac{2}{S}\frac{\partial}{\partial\theta}\left[\frac{\dot{A}}{A}(w_{22}+w_{00})-w_{01}\frac{S'}{S}\right]\\&-\left[6\left(\frac{\dot{A}}{A}\right)^2-\left(\frac{S'}{S}\right)^2-\frac{\cot^2\theta}{S^2}\right]w_{02}+6\frac{\dot{A}}{A}\frac{S'}{S}w_{12}+2\frac{\dot{A}}{A}\frac{\cot\theta}{S}(w_{22}-w_{33})+\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]w_{02}\bigg\},\end{aligned} \tag{31}$$

[18] Cf. Refs. 1 and 5.
[19] Cf. B. Friedman, *Principles and Techniques of Applied Mathematics* (Wiley, New York, 1956), p. 136.
[20] Cf. H. Bondi's discussion of luminosity distances, *Cosmology* (Cambridge U. P., Cambridge, England, 1961), p. 107.
[21] See Ref. 1.

$$\tilde{D}_{03}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S\sin\theta\left\{\ddot{w}_{03}-w_{03}''-\frac{1}{S^2}\frac{\partial^2 w_{03}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{03}-2\frac{S'}{S}w_{03}'-\frac{\cot\theta}{s^2}\frac{\partial w_{03}}{\partial\theta}+2\frac{\dot{A}}{A}w_{13}'+\frac{2\dot{A}}{SA}\frac{\partial}{\partial\theta}w_{23}\right.$$
$$\left.-\left[6\left(\frac{\dot{A}}{A}\right)^2-\left(\frac{S'}{S}\right)^2-\frac{\cot^2\theta}{S^2}\right]w_{03}+6\frac{\dot{A}}{A}\frac{S'}{S}w_{13}+4\frac{\dot{A}}{A}\frac{\cot\theta}{S}w_{23}+\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)\right]^2w_{03}\right\},\quad(32)$$

$$D_{11}{}^{\sigma\tau}w_{\sigma\tau}=\frac{1}{2}\left\{\ddot{w}_{11}-w_{11}''-\frac{1}{S^2}\frac{\partial^2 w_{11}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{11}-2\frac{S'}{S}w_{11}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{11}+4\frac{\dot{A}}{A}w_{01}'+4\frac{S'}{S^2}\frac{\partial}{\partial\theta}w_{12}+2\left(\frac{S'}{S}\right)^2(2w_{11}-w_{22}-w_{33})\right.$$
$$\left.-2\left(\frac{\dot{A}}{A}\right)^2(w_{11}+w_{00})+4\frac{S'}{S^2}\cot\theta\, w_{12}-2\left[\left(\frac{\dot{A}}{A}\right)^2+k\right](w_{22}+w_{33})+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)\right]^2w_{00}\right\},\quad(33)$$

$$\tilde{D}_{12}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S\left\{\ddot{w}_{12}-w_{12}''-\frac{1}{S^2}\frac{\partial^2 w_{12}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{12}-2\frac{S'}{S}w_{12}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{12}+2\frac{\dot{A}}{A}w_{02}'+\frac{2}{S}\frac{\partial}{\partial\theta}\left[\frac{S'}{S}(w_{22}-w_{11})+\frac{\dot{A}}{A}w_{01}\right]\right.$$
$$\left.+\left[5\left(\frac{S'}{S}\right)^2-2\left(\frac{\dot{A}}{A}\right)^2+\frac{\cot^2\theta}{s^2}\right]w_{12}-2\frac{S'}{S}\frac{\dot{A}}{A}w_{02}+2\frac{S'}{S}\frac{\cot\theta}{S}(w_{22}-w_{33})+\left[\left(\frac{\dot{A}}{A}\right)^2+k\right]w_{12}\right\},\quad(34)$$

$$\tilde{D}_{13}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S\sin\theta\left\{\ddot{w}_{13}-w_{13}''-\frac{1}{S^2}\frac{\partial^2 w_{13}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{12}-2\frac{S'}{S}w_{13}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{13}+2\frac{\dot{A}}{A}w_{03}'+\frac{2S'}{S^2}\frac{\partial}{\partial\theta}w_{23}\right.$$
$$\left.+\left[5\left(\frac{S'}{S}\right)^2-2\left(\frac{\dot{A}}{A}\right)^2+\frac{\cot^2\theta}{S^2}\right]w_{13}-2\frac{S'}{S}\frac{\dot{A}}{A}w_{03}+4\frac{S'}{S}\frac{\cot\theta}{S}w_{23}+\left[\left(\frac{\dot{A}}{A}\right)^2+k\right]w_{13}\right\},\quad(35)$$

$$\tilde{D}_{22}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S^2\left\{\ddot{w}_{22}-w_{22}''-\frac{1}{S^2}\frac{\partial^2 w_{22}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{22}-2\frac{S'}{S}w_{22}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{22}+\frac{4}{S}\frac{\partial}{\partial\theta}\left[\frac{\dot{A}}{A}w_{02}-\frac{S'}{S}w_{12}\right]+2\left(\frac{S'}{S}\right)^2(w_{22}-w_{11})\right.$$
$$\left.-2\left(\frac{\dot{A}}{A}\right)^2(w_{22}+w_{00})+4\frac{\dot{A}}{A}\frac{S'}{S}w_{01}+2\frac{\cot^2\theta}{S^2}(w_{22}-w_{33})-2\left[\left(\frac{\dot{A}}{A}\right)^2+k\right](w_{11}+w_{33})+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]w_{00}\right\},\quad(36)$$

$$\tilde{D}_{23}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S^2\sin\theta\left\{\ddot{w}_{23}-w_{23}''-\frac{1}{S^2}\frac{\partial^2 w_{23}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{23}-2\frac{S'}{S}w_{23}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{23}+\frac{2}{S}\frac{\partial}{\partial\theta}\left[\frac{\dot{A}}{A}w_{03}-\frac{S'}{S}w_{13}\right]\right.$$
$$\left.+2\left[\left(\frac{S'}{S}\right)^2-\left(\frac{\dot{A}}{A}\right)^2+\frac{2\cot^2\theta}{S^2}\right]w_{23}-2\frac{\cot\theta}{S}\left[\frac{\dot{A}}{A}w_{03}-\frac{S'}{S}w_{13}\right]+\left[\left(\frac{\dot{A}}{A}\right)^2+k\right]w_{23}\right\},\quad(37)$$

$$\tilde{D}_{33}{}^{\sigma\tau}w_{\sigma\tau}=\tfrac{1}{2}S^2\sin^2\theta\left\{\ddot{w}_{33}-w_{33}''-\frac{1}{S^2}\frac{\partial^2 w_{33}}{\partial\theta^2}+2\frac{\dot{A}}{A}\dot{w}_{33}-2\frac{S'}{S}w_{33}'-\frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}w_{33}+4\frac{\cot\theta}{S}\left[\frac{\dot{A}}{A}w_{02}-\frac{S'}{S}w_{12}\right]\right.$$
$$+2\left(\frac{S'}{S}\right)^2(w_{33}-w_{11})-2\left(\frac{\dot{A}}{A}\right)^2(w_{33}+w_{00})+4\frac{\dot{A}S'}{AS}w_{01}+\frac{2\cot^2\theta}{S^2}(w_{33}-w_{22})$$
$$\left.-2\left[\left(\frac{\dot{A}}{A}\right)^2+k\right](w_{11}-w_{22})+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]w_{00}\right\}.\quad(38)$$

All the $w$'s on the right-hand sides of these equations are tetrad components. Note that the set of 10 equations breaks up into two independent sets: three equations for $w_{03}$, $w_{13}$, and $w_{23}$, and seven equations for the other components. As a test of the correctness of the algebra, the author has twice computed the above from their definitions. Also the contraction of the $\square w_{\sigma\tau}$ part of these equations gives the correct form for $\square w^{\tau}{}_{\tau}$, and when $w_{\sigma\tau}=g_{\sigma\tau}$, the above expressions yield $-R_{\mu\nu}$ as they should.

The functions $V^{\alpha'\beta'}{}_{\nu}$ satisfy the boundary-value equation (5) and

$$D_{\sigma'\tau'\alpha'\beta'}V^{\alpha'\beta'}{}_{\nu}=0.\qquad(39)$$

For the evaluation of these functions in the case of the Robertson-Walker metrics, it will be useful to make a change of variables defined by

$$V_{\alpha'\beta'}{}^{\nu}{}_{\nu}=\frac{f_{\alpha'\beta'}}{A(\tau')A(\tau)S(\Delta\chi)}. \tag{40}$$

In terms of the $f$'s the characteristic boundary-value equation is

$$f_{\alpha'\beta';0'}\pm f_{\alpha'\beta';1'}=\tfrac{1}{2}g_{\alpha'\beta'}(\ddot{A}/A+k)+A^2R_{\alpha'\beta'}. \tag{41}$$

Let $f_{ij}$ be the ONT representation of $f_{\alpha'\beta'}$, where we have suppressed the primes for the sake of clarity. It follows from Eqs. (29)–(41) that

$$\begin{gathered}\partial f_{ij}/\partial\theta=0,\\ f_{02}=f_{03}=f_{12}=f_{13}=f_{23}=0,\\ f_{22}=f_{33}.\end{gathered} \tag{42}$$

The nontrivial equations for the $f_{ij}$ are thus

$$0=\ddot{f}_{00}-f_{11}''-f_{00}\left(\frac{\ddot{A}}{A}+k\right)+4\frac{\dot{A}}{A}\left[f_{01}'+\frac{S'}{S}f_{01}\right]-6\left(\frac{\dot{A}}{A}\right)^2 f_{00}-2\left(\frac{\dot{A}}{A}\right)^2[f_{11}+2f_{22}]+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right][f_{11}+2f_{22}], \tag{43}$$

$$\begin{aligned}0=\ddot{f}_{11}-f_{11}''-f_{11}\left(\frac{\ddot{A}}{A}+k\right)+4\frac{\dot{A}}{A}\left[f_{01}'-\frac{S'}{S}f_{01}\right]+4\left(\frac{S'}{S}\right)^2[f_{11}-f_{22}]-2\left(\frac{\dot{A}}{A}\right)^2(f_{11}+f_{00})\\ -4\left[\left(\frac{\dot{A}}{A}\right)^2+k\right]f_{22}+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]f_{00},\end{aligned} \tag{44}$$

$$\begin{aligned}0=\ddot{f}_{22}-f_{22}''-f_{22}\left(\frac{\ddot{A}}{A}+k\right)+2\left(\frac{S'}{S}\right)^2(f_{22}-f_{11})-2\left(\frac{\dot{A}}{A}\right)^2(f_{22}+f_{00})+4\frac{\dot{A}S'}{AS}f_{01}\\ -2\left[\left(\frac{\dot{A}}{A}\right)^2+k\right](f_{11}+f_{22})+2\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]f_{00},\end{aligned} \tag{45}$$

$$\begin{aligned}0=\ddot{f}_{01}-f_{01}''-f_{01}\left(\frac{\ddot{A}}{A}+k\right)+2\frac{\dot{A}}{A}\left[f_{00}'+f_{11}'-\frac{S'}{S}(f_{00}+f_{11})\right]\\ +\left[2\left(\frac{S'}{S}\right)^2-6\left(\frac{\dot{A}}{A}\right)^2\right]f_{01}+4\frac{\dot{A}S'}{AS}(f_{11}-f_{22})+\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]f_{01}.\end{aligned} \tag{46}$$

The characteristic boundary-value equations are

$$f_{00}\pm f_{00}'\mp 2\frac{\dot{A}}{A}f_{01}=\frac{1}{2}\left\{\frac{\ddot{A}}{A}+k+6\left[\frac{\ddot{A}}{A}-\left(\frac{\dot{A}}{A}\right)^2\right]\right\}, \tag{47}$$

$$f_{01}\pm f_{01}'\mp\frac{\dot{A}}{A}(f_{00}+f_{11})=0, \tag{48}$$

$$f_{11}\pm f_{11}'\mp 2\frac{\dot{A}}{A}f_{01}=-\frac{1}{2}\left[3\frac{\ddot{A}}{A}+5k+2\left(\frac{\dot{A}}{A}\right)^2\right], \tag{49}$$

$$f_{22}\pm f_{22}'=-\frac{1}{2}\left[3\frac{\ddot{A}}{A}+5k+2\left(\frac{\dot{A}}{A}\right)^2\right]. \tag{50}$$

For Minkowski space-time, the right-hand sides of Eqs. (47)–(50) are all zero, and thus all of the $f_{ij}$ are zero. (In fact, it is easily shown that all of the $V_{\alpha'\beta'\mu\nu}$ are zero for Minkowski space-time.)

For the nonempty models, the metric is well behaved when $\tau$ is in the range $0<\tau<\infty$ ($k=0, -1$) or $0<\tau<m\pi/2$ ($k=+1$). If both $\tau_0$ and $\tau_f$ are in this range then $f_{ij}(x_0,x_f)$ is well defined, and when $A(\tau)$ is analytical in the interval $\tau_0$ to $\tau_f$, $f_{ij}(x_0,x_f)$ is also.[22] However, for the evaluation of $I_{\alpha'\beta'}$, we need to know the behavior of $f_{ij}(x_0,x_f)$ in the limit as $\tau_f\rightarrow 0$, and the metric is not well behaved in that limit. We shall now prove that, despite the singular behavior of the metric at $\tau=0$, $f_{ij}$ remains finite in the limit $\tau_f\rightarrow 0$.

Suppose to the contrary that the $f_{ij}$ are not all finite in the limit $\tau_f\rightarrow 0$. Then let $f_*(\tau_*,\chi_*,\tau_f,0)$ be the most singular $f_{ij}$ in that limit where $x_*$ is some point (with $\tau_*>\tau_f\geqslant 0$) that maximizes the order of the singularity of $f_{ij}$ in that limit. Then let us define, for $\tau_*>\tau_f\geqslant 0$,

$$C(\tau_f)\equiv[|f_*(x_*,x_f)|+1]^{-1}$$

and

$$a_{ij}(x_0,x_f)\equiv C(\tau_f)f_{ij}. \tag{51}$$

The range of $C(\tau_f)$ is clearly 0 to 1. The $a_{ij}$ are nonsingular in the limit $\tau_f\rightarrow 0$ and are not all zero in that limit. The assumption that $f_*$ is singular in the limit $\tau_f\rightarrow 0$ implies that

$$\lim_{\tau_f\rightarrow 0} C(\tau_f)=0. \tag{52}$$

We turn now to the differential and boundary-value equations (43)–(50). Note that these are local equations

[22] Cf. J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Yale U. P., New Haven, 1923); and Ref. 5.

at $x_0$, not $x_f$, and the boundary is the future light cone of $x_f$. The $f_{ij}$ must satisfy these equations for $\tau_f>0$, and so must satisfy them in the limit $\tau_f\to 0$. Because these are equations at $x_0$, we are free to multiply them by $C(\tau_f)$, thus changing variables from $f_{ij}$ to $a_{ij}$. The $a_{ij}$ must then satisfy the transformed equations for $\tau_f>0$ and so also in the limit $\tau_f\to 0$. The boundary values for $f_{ij}$ are nonsingular in the limit $\tau_f\to 0$. The boundary values for the $a_{ij}$ are equal to the boundary values for the $f_{ij}$ multiplied by $C(\tau)$, and so are all zero in the limit $\tau_f\to 0$ whenever $C(\tau)$ is also zero in that limit. But the unique solution with all zero boundary values is zero, i.e., all of the $a_{ij}$ must be zero in the limit $\tau_f\to 0$. Yet the $a_{ij}$ cannot all be zero in that limit if $f_*$ is singular in that limit. Thus all of the $f_{ij}$ must be nonsingular in the limit $\tau_f\to 0$. Then $C(\tau_f)$ is nonzero in that the limit and the contradiction is avoided.

If we write

$$f_{ij}(\tau_0,\chi_0,\tau_f,0)=f_{ij}{}^0(\tau_0,\chi_0)+\tau_f f_{ij}{}^1(\tau_0,\chi_0,\tau_f,0)\,, \quad (53)$$

where

$$f_{ij}(\tau_0,\chi_0,0,0)=f_{ij}{}^0(\tau_0,\chi_0)\,,$$

then it is easily shown that $f_{ij}{}^1$, and thus $\partial f_{ij}/\partial\tau_f$, are nonsingular in the limit $\tau_f\to 0$. The proof goes as follows. For any given point $x_0$, both $f_{ij}$ and $f_{ij}{}^0$ must satisfy the differential equations (43)–(46) regardless of the value of $\tau_f$ (we continue to require $\tau_f<\tau_0$). Thus $f_{ij}{}^1$ must also satisfy these differential equations. As for the boundary conditions, we have to first order in $\tau_f$,

$$\text{B.C.}[f_{ij}(\tau_0,\chi_0,\tau_f,0)]=\text{B.C.}[f_{ij}{}^0(\tau_0,\,\chi_0+\tau_f)]\,, \quad (54)$$

which implies, again to first order in $\tau_f$,

$$\text{B.C.}[f_{ij}{}^1(\tau_0,\chi_0,\tau_f,0)]=\text{B.C.}\left[\frac{\partial}{\partial\chi_0}f_{ij}{}^0(\tau_0,\chi_0)\right]. \quad (55)$$

Here B.C. is the operator used in the left-hand sides of Eqs. (47)–(50). Because of the nonsingular behavior of $f_{ij}{}^0$, and because of the differential equations that it satisfies, the right-hand side of Eq. (55) must be nonsingular. Thus the boundary values for $f_{ij}{}^1$ remain nonsingular as $\tau_f\to 0$, and as we saw above in the case of $f_{ij}$, this requires that $f_{ij}{}^1$ remain nonsingular in that limit.

### *Surface Integral*

We shall choose our volume of integration to be such that in and on the past light cone, $\partial\Omega$ is a surface of constant $\tau$. This is done both for simplicity and because $\partial\Omega_S$ is a surface of constant $\tau$, i.e., $\tau=0$. Because of the spatial homogeneity of the Roberston-Walker metrics, we can write

$${}^2g_{\alpha'\beta'}(x',\partial\Omega)={}^2g_{\alpha'\beta'}(\tau',\tau)\,. \quad (56)$$

With this choice of $\partial\Omega$, the spatial isotropy of the Roberston-Walker metrics requires, for the ONT components, that

$${}^2g_{11}={}^2g_{22}={}^2g_{33}\,. \quad (57)$$

In the SWG formalism, the surface integral is a solution to the homogeneous wave equation, i.e., $D_{\mu\nu\sigma\tau}\,{}^2g^{\sigma\tau}=0$. This property of the surface integral and Eqs. (33), (36), (42), and (57) require that ${}^2g_{01}=0$. Thus for the Robertson-Walker metrics and with $\partial\Omega$ a surface of constant $\tau$, ${}^2g_{ij}$ is diagonal with two independent functions ${}^2g_{00}$ and ${}^2g_{11}$.

This surface integral can be written as [see Eqs. (1), (4), and (28)]

$${}^2g_{ij}(\tau',\tau)=-A^2(\tau)\frac{\partial}{\partial\tau}\left[\int_0^\infty\left(\frac{\eta_{ij}\delta(\Delta\tau-\Delta\chi)}{A(\tau')A(\tau)S(\Delta\chi)}+V_{ij}{}^\nu{}_\nu H^-(\Gamma)\right)S^2(\chi)d\chi\right]. \quad (58)$$

We are here using a spherical coordinate system centered at $x'$. The points $(x_0,x_f)$ are here replaced by $(x',x)$. The shift of coordinate center from $x_f$ to $x_0$ is easily handled by the tensor calculus and turns out to be trivial leaving the functions of interest unchanged. We are continuing to suppress the primes on the ONT component indices, but it should be understood that these indices refer to $x'$. We have integrated over $\theta$ and $\varphi$ since $G^-{}_{ij}{}^\nu{}_\nu$ is independent of them. $\eta_{ij}$ is the Minkowski metric diag$(1,\,-1,\,-1,\,-1)$. For the first part of the integral we find

$$\int_0^\infty\frac{\eta_{ij}\delta(\Delta\tau-\Delta\chi)S^2(\chi)}{A(\tau')A(\tau)S(\Delta\chi)}d\chi=\frac{\eta_{ij}S(\Delta\tau)}{A(\tau')A(\tau)}. \quad (59)$$

For the second part of the integral, it is convenient to introduce a change of variables such that

$$f_{ij}(\tau',\chi',\tau,0)=(1/\Delta\tau)F_{ij}(\tau',y,\tau)\,, \quad (60)$$

where $y=\chi'/\Delta\tau$. Let us define

$$J_{ij}(\tau',\tau)\equiv\int_0^\infty\frac{S(\Delta\chi)}{S(\Delta\tau)}f_{ij}H^-(\Gamma)d\chi=\int_0^1\frac{S(\Delta\chi)}{S(\Delta\tau)}F_{ij}dy. \quad (61)$$

Then Eq. (58) becomes

$${}^2g_{ij}=\left(\frac{A(\tau)}{A(\tau')}\frac{\partial S(\Delta\tau)}{\partial\tau}+\frac{S(\Delta\tau)A(\tau)}{A(\tau')}\right)(\eta_{ij}+J_{ij})-\frac{S(\Delta\tau)A(\tau)}{A(\tau')}\frac{\partial}{\partial\tau}J_{ij}. \quad (62)$$

We can apply Eq. (62) immediately to Minkowski space-time. Because it is an empty space-time, we know that we must have, in terms of ONT components,

$${}^2g_{ij}=g_{ij}=\eta_{ij}\,. \quad (63)$$

This is easily confirmed from Eq. (62). Since all of the $V_{\alpha'\beta'\sigma\tau}$ are zero for Minkowski space-time, $J_{ij}$ is also zero. Thus Eq. (63) will hold if the first bracket in Eq. (62) is equal to 1. Both forms of the metric [Eqs. (15) and (16)] provide the desired result. This provides a nice check on our algebra and the covariance of our equations. Clearly, Minkowski space-time is a Class-II solution.

We turn now to the nonempty Robertson-Walker models. Applying the definition of the source-free contribution to our particular case, we have

$$I_{ij}(\tau')=\lim_{\tau\to 0}{}^2g_{ij}(\tau',\tau). \tag{64}$$

From the behavior of $f_{ij}$, we know that we can write

$$J_{ij}(\tau',\tau)=J_{ij}{}^0(\tau')+\tau J_{ij}{}^1(\tau',\tau), \tag{65}$$

where both $J_{ij}{}^0$ and $J_{ij}{}^1$ are nonsingular. Since $A(\tau)\to 0$ as $\tau\to 0$, the limit reduces to

$$I_{ij}(\tau')=(\eta_{ij}+J_{ij}{}^0)\frac{S(\tau')}{A(\tau')}\lim_{\tau\to 0}A(\tau). \tag{66}$$

This is zero when $\dot{A}(0)$ is zero (e.g., $m>1$), or when $J_{ij}{}^0=-\eta_{ij}$. We shall now show that even when the first condition does not hold, the second does.

### *k*=0 *Case*

We begin by considering the simple but illustrative group of metrics that are flat ($k=0$) and arise from a $\gamma$-law equation of state [Eq. (9)]. For these metrics the field equations for $f_{ij}$ are homogeneous[23] in $\chi'$ and $\tau'$ and the boundary values are homogeneous of degree $-1$. This homogeneity and the uniqueness of the Green's functions imply that $f_{ij}$ can be expressed as

$$f_{ij}(\tau',\chi')=(1/\tau)F_{ij}(y). \tag{67}$$

With

$$dF_{ij}/dy\equiv F_{ij}',\quad d^2F_{ij}/dy^2\equiv F_{ij}'', \tag{68}$$

the nontrivial equations for $F_{ij}(y)$ are [from Eqs. (43)–(46)]

$$\begin{aligned}0=(y^2-1)F_{00}''+4yF_{00}'-(7m^2-m-2)F_{00}\\+4m(F_{01}'+F_{01}/y)-2(m^2+m)(F_{11}+2F_{22}),\end{aligned} \tag{69}$$

$$\begin{aligned}0=(y^2-1)F_{11}''+4yF_{11}'-(3m^2-m-2-4/y^2)F_{11}\\+4m(F_{01}'-F_{01}/y)-4(m^2+1/y^2)F_{22}\\-2(m^2+m)F_{00},\end{aligned} \tag{70}$$

$$\begin{aligned}0=(y^2-1)F_{22}''+4yF_{22}'-(5m^2-m-2-2/y^2)F_{22}\\+4mF_{01}/y-2(m^2+1/y^2)F_{11}-2(m^2+m)F_{00},\end{aligned} \tag{71}$$

$$\begin{aligned}0=(y^2-1)F_{01}''+4yF_{01}'+(2-m^2+2/y^2)F_{01}\\+2m(F_{00}'+F_{11}'-(1/y)(F_{00}+F_{11}))\\+(4m/y)(F_{11}-F_{22}).\end{aligned} \tag{72}$$

[23] I am indebted to Prof. Martin Kruskal for this insight.

The regularity of $V_{\alpha'\beta'\mu\nu}$ implies that all of the $F_{ij}$ are zero at $y=0$. The boundary values at $y=1$ are [from Eqs. (47)–(50)]

$$F_{00}+2mF_{01}=\tfrac{1}{2}[7m-m^2], \tag{73}$$

$$F_{01}+m(F_{00}+F_{11})=0, \tag{74}$$

$$F_{11}+2mF_{01}=\tfrac{1}{2}[5m^2-3m], \tag{75}$$

$$F_{22}=\tfrac{1}{2}[5m^2-3m]. \tag{76}$$

We can obtain $J_{ij}{}^0$ from these equations by multiplying by $y$ and integrating from zero to one [see Eq. (61)]. Combining Eqs. (74) and (75), and making use of the boundary conditions, we find

$$\begin{aligned}7m-m^2=(7m^2-m)J_{00}{}^0\\+2(m^2+m)(J_{11}{}^0+2J_{22}{}^0),\end{aligned} \tag{77}$$

$$\begin{aligned}3(5m^2-3m)=(7m^2-m)(J_{11}{}^0+2J_{22}{}^0)\\+6(m^2+m)J_{00}{}^0.\end{aligned} \tag{78}$$

From Eqs. (57) and (66) we have $J_{11}{}^0=J_{22}{}^0$, so the above equations require, for all $m$,

$$J_{00}{}^0=-1,\quad J_{11}{}^0=+1. \tag{79}$$

Thus for $k=0$ and a $\gamma$-law equation of state $J_{ij}{}^0(\tau')=-\eta_{ij}$ and $I_{ij}(\tau')=0$.

### *General Case*

Let us define the functions $Q_{ij}(\tau')$ by

$$Q_{ij}(\tau')=[\eta_{ij}+J_{ij}{}^0(\tau')]S(\tau'). \tag{80}$$

From Eq. (66) we see that $I_{ij}(\tau')$ equals zero whenever $Q_{ij}(\tau')$ equals zero and/or $\dot{A}(\tau)$ equals zero in the limit $\tau\to 0$. Because of the properties of ${}^2g_{ij}$, $Q_{ij}(\tau')$ must be diagonal with two independent functions $Q_{00}$ and $Q_{11}$, and must satisfy the following differential equations [see Eqs. (29) and (33)]:

$$0=\ddot{Q}_{00}-Q_{00}\left[\frac{\ddot{A}}{A}+k+6\left(\frac{\dot{A}}{A}\right)^2\right]+6\left[\frac{\ddot{A}}{A}-2\left(\frac{\dot{A}}{A}\right)^2\right]Q_{11}, \tag{81}$$

$$0=\ddot{Q}_{11}-Q_{11}\left[\frac{\ddot{A}}{A}+5k+6\left(\frac{\dot{A}}{A}\right)^2\right]+2\left[\frac{\ddot{A}}{A}-2\left(\frac{\dot{A}}{A}\right)^2\right]Q_{00}. \tag{82}$$

Consider those Roberston-Walker cosmologies for which $A(\tau)$ can be expressed as a generalized power series in $\tau$, i.e.,

$$A(\tau)=A_0{}^{m/2}\left(\frac{\tau}{m}\right)^m\sum_{i=0}^{\infty}a_i\tau^i. \tag{83}$$

This class of cosmologies includes those given by Eqs. (11)–(13), as well as many models with smoothly varying equations of state. For these models, Eqs. (81) and (82) can be written as

$$0=\ddot{Q}_{00}-\frac{(7m^2-m)}{(\tau')^2}B_1(\tau')Q_{00}-6\frac{(m^2+m)}{(\tau')^2}B_2(\tau')Q_{11}\,,\quad (84)$$

$$0=\ddot{Q}_{11}-\frac{(7m^2-m)}{(\tau')^2}B_3(\tau')Q_{11}-2\frac{(m^2+m)}{(\tau')^2}B_2(\tau')Q_{00}\,,\quad (85)$$

where the functions $B_1(\tau')$, $B_2(\tau')$, and $B_3(\tau')$ can all be given in terms of power series in $\tau'$, and are normalized so that

$$B_1(0)=B_2(0)=B_3(0)=1\,. \quad (86)$$

Under these conditions, Eqs. (84) and (85) can be solved by means of a generalized power series of the form[24]

$$Q_{ij}(\tau')=\sum_{n=0}^{\infty}Q_{ij}{}^n(\tau')^{n+\rho}\,, \quad (87)$$

where $Q_{ij}{}^n$'s are constant, and $\rho$ is given by

$$\rho=\tfrac{1}{2}\pm\tfrac{1}{2}\{1+4[7m^2-m\mp2\sqrt{3}(m^2+m)]\}^{1/2}. \quad (88)$$

The two $\pm$ signs are independent, leading to four values of $\rho$ for each value of $m$. The $Q_{ij}{}^n$ are obtained from recursion relations that leave only the $Q_{ij}{}^0$ terms unspecified.

We now can compare this series representation with the corresponding series obtainable from Eq. (80). $S(\tau')$ can be given by an ordinary power series in $\tau'$ [see Eq. (8)]. Likewise, it is easily shown from Eqs. (83), (79), (61), and (43)–(50) that $J_{ij}{}^0(\tau')$ has the form

$$J_{ij}{}^0(\tau')=-\eta_{ij}+\sum_{n=1}^{\infty}K_{ij}{}^n(\tau')^n. \quad (89)$$

Thus $Q_{ij}(\tau')$ must be expressible as an ordinary power series with the lowest power being greater than or equal to 2.

The generalized exponent $\rho$ takes on an integer value greater than or equal to two for two values of $m$ between $\frac{1}{2}$ and 1. The first value is

$$m_1=\frac{[(2\sqrt{3}-1)^2+24(7+2\sqrt{3})]^{1/2}-(2\sqrt{3}-1)}{2(7+2\sqrt{3})}$$

$$\approx0.648\,. \quad (90)$$

With $m=m_1$, the largest value of $\rho$ is 3. The other three values of $\rho$ are noninteger or smaller than two. The other value is

$$m_2=\frac{[(2\sqrt{3}-1)^2+48(7+2\sqrt{3})]^{1/2}-(2\sqrt{3}-1)}{2(7+2\sqrt{3})}$$

$$\approx0.959\,. \quad (91)$$

With $m=m_2$, the largest value of $\rho$ is 4, and the other three values of $\rho$ are noninteger or smaller than two.

We have now shown that $Q_{ij}(\tau')$ must be expressible by both an ordinary power series and the generalized power series, Eq. (87). This requires that, unless $m$ equals $m_1$ or $m_2$, $Q_{ij}(\tau')$ must be zero for $\frac{1}{2}\leq m\leq 1$. For $m$ equal to $m_1$ or $m_2$, $Q_{ij}(\tau')$ could be anything. If we assume, however, that $\partial Q_{ij}(\tau')/\partial m$ is nonsingular in the range $\frac{1}{2}<m<1$, then $Q_{ij}(\tau')$ must be zero throughout the range $\frac{1}{2}\leq m\leq 1$. There are no apparent physical singularities associated with changes in $m$ in the range $\frac{1}{2}<m<1$, so this assumption should hold.

To summarize, we have now proven that $I_{ij}=0$ for the following types of Robertson-Walker metrics that are solutions of Einstein's field equations with $\Lambda=0$ and which are nonempty: (1) all metrics with $\lim_{\tau\to0}\partial A(\tau)/\partial\tau=0$; (2) all flat models with a $\gamma$-law equation of state [$A(\tau)$ given by Eq. (12)]; (3) All metrics with $A(\tau)$ given by a generalized power series in $\tau$ [Eq. (83)] and with $\frac{1}{2}\leq m<m_1$, $m_1<m<m_2$, or $m_2<m\leq 1$.

These groups of metrics cover a broad range and suggest that probably all nonempty relativistic Robertson-Walker models with $\Lambda=0$ and $0\leq p\leq\rho c^2$ have $I_{ij}=0$, and are thus Class-I solutions.

### Schwarzschild-Type Cosmologies

We now investigate what we shall call Schwarzschild-type solutions. That is, we consider metrics of the form

$$g^{\mu\nu}=\eta^{\mu\nu}+h^{\mu\nu}\,, \quad (92)$$

where, in terms of our above spherical coordinate system with $\chi'=0$,

$$h^{\mu\nu}=a^{\mu\nu}/\chi\approx O(1/\chi)\quad\text{for large }\chi. \quad (93)$$

Let us define

$$\bar{a}^{\mu\nu}=\bar{a}^{\mu\nu}(\tau,\chi)\equiv\frac{1}{4\pi}\oint a^{\mu\nu}\sin\theta\,d\theta d\varphi \quad (94)$$

and let us assume that

$$\lim_{\chi\to\infty}\frac{\partial}{\partial\chi}\bar{a}^{\mu\nu}=\lim_{\chi=\infty}\frac{\partial}{\partial\tau}\bar{a}^{\mu\nu}=0\,. \quad (95)$$

That is, metrics of this type approach flatness asymptotically at least as fast as the Schwarzschild solution.

We can determine $I_{\mu\nu}$ for these metrics by using the first-order integrals given by Sciama, Waylen, and the author[25] together with the Green's function for Minkow-

[24] Cf. M. H. Protter and C. B. Morrey, *Modern Mathematical Analysis* (Addison-Wesley, Palo Alto, 1964), p. 720.

[25] See Ref. 1.

ski space, which is

$$G_{\mu\nu\sigma'\epsilon'}=\frac{1}{2\pi}\frac{\eta_{\mu\nu}\eta_{\sigma'\epsilon'}}{\Delta\chi}\delta(\Delta\tau-\Delta\chi)\,. \tag{96}$$

Thus, to first order in $h^{\mu\nu}$, we have

$$^2g^{\sigma'\epsilon'}=\eta^{\sigma'\epsilon'}\left[1+\eta_{\mu\nu}\frac{\partial}{\partial\Delta\tau}\bar{a}^{\mu\nu}(\tau,\Delta\tau)+\eta_{\mu\nu}\frac{\partial}{\partial\tau}\bar{a}^{\mu\nu}(\tau,\Delta\tau)\right]. \tag{97}$$

By Eqs. (3) and (95),

$$I^{\sigma'\epsilon'}=\lim_{\tau\to-\infty}{}^2g^{\sigma'\epsilon'}=\eta^{\sigma'\epsilon'}+O((h^{\mu\nu})^2)\,. \tag{98}$$

Cosmological models with this Schwarzschild-type metric are spatially infinite with Minkowskian boundary conditions. Many of them have $T_{\mu\nu}\neq0$ in only a finite region of space (e.g., the Schwarzschild and Kerr solutions with appropriate interior solutions). They are all of either Class II or Class III since Class I solutions must have $I_{\mu\nu}=0$ everywhere while Schwarzschild-type solutions have $I_{\mu\nu}\sim\eta_{\mu\nu}$, if not everywhere, then at least throughout most of their volume.

### Kantowski-Sachs-Thorne Cosmologies

Finally, let us consider a class of homogeneous but anisotropic cosmological models due to Kantowski and Sachs[26] and Thorne,[27] i.e.,

$$ds^2=d\tau^2-f(\tau)d\chi^2-S^2(\tau)d\Omega^2\,, \tag{99}$$

where

$$\tau=\alpha(\beta+\sin\beta)\,, \tag{100}$$

$$S(\tau)=\alpha(1+\cos\beta)\,, \tag{101}$$

$$f(\tau)=A_0\left(2+\frac{\beta\sin\beta}{1+\cos\beta}\right)+C\frac{\sin\beta}{1+\cos\beta}, \tag{102}$$

and $\alpha$, $C$, and $A_0$ are constants.[28] This metric is for a dust-filled (i.e., zero-pressure) model. $A_0$ is related to the matter density in such a way that when $A_0=0$, the model is empty. In this case the model reduces to the interior Kruskal solution. For that part of the interior Kruskal solution with $dS(\tau)/d\tau>0$ (the lower quadrant of the Kruskal diagram), the solution acts as a full cosmological model for our purposes. That is, the past light cone of any point in that part of the solution is contained entirely within the part of the solution. Since the Kruskal solution is empty, it is entirely source-free, and so it is of Class II. It is clear from the above equations that adding matter to the Kruskal solution, i.e., $A_0\neq0$, does not alter the qualitative character of the metric enough to eliminate the source-free contribution to the field. Thus we conclude that these Kantowski-Sachs-Thorne models are also of Class II. (We expect in general that any class of cosmological models with the property that, going from no matter-energy to some matter-energy leaves the qualitative features of the solution largely unchanged, will be of Class II or Class III.)

[26] R. Kantowski and R. K. Sachs, J. Math. Phys. **7**, 443 (1966).
[27] K. S. Thorne, Astrophys. J. **148**, 51 (1967).
[28] We are using Vajk's (Ref. 13) form of this metric.

## IV. DISCUSSION

The results obtained above can be summarized as follows:

*Class-I Solutions:* Many (probably all) nonempty general relativistic Robertson-Walker models with $\Lambda=0$ and $0\leq p\leq\rho c^2$.

*Class-II Solutions:* Minkowski space, the Schwarzschild and Kerr solutions with appropriate interior solutions, and the Kantowski-Sachs-Thorne solutions.

*Class-II or Class-III Solutions:* All Schwarzschild-type solutions, and all vacuum cosmological solutions ($T_{\mu\nu}=0$ everywhere).

*Class-III Solutions:* All solutions with $\Lambda\neq0$, including the Gödel rotating model.

These preliminary results are encouraging since the supposedly Machian class (Class I) contains the observationally plausible Robertson-Walker models, while it excludes the traditionally anti-Mach Minkowski, Schwarzschild, and Gödel models. Yet many questions remain. For example, are there any homogeneous but anisotropic Class-I solutions? Are there any inhomogeneous Class-I solutions?

If in fact the only Class-I solutions are of the Robertson-Walker type, this would pose a problem for the theory since the real world is not strictly homogeneous. It is, however, important to remember that the theory presented here is classical, i.e., nonquantum. In a quantum theory of the IG field, we would expect the condition $I_{\mu\nu}=0$ to be replaced by $\langle I_{\mu\nu}\rangle=0$, where the brackets indicate the expectation value. This change would presumably leave the global character of Class-I solutions unchanged, but permit small-scale inhomogeneity. In this connection, it is interesting to note that Harrison[29] has suggested that quantum fluctuations in the IG field during the early moments of the universe may be required to explain the existence of galaxies. This is still all speculation since there is as yet no quantum theory of the IG field, but it suggests that inhomogeneity even on the scale of galaxies may be of too small a scale to be considered in classical cosmological theories.

## ACKNOWLEDGMENTS

This work was based on a thesis submitted to Princeton University in partial fulfillment of the requirements for the degree of Doctor of Philosophy, and was

[29] E. R. Harrison, Phys. Rev. D 1, 2726 (1970).

done during the tenure of a Princeton National Fellowship and a National Science Foundation Fellowship, which are gratefully acknowledged. Especially in view of our different approaches to Mach's principle, Professor John A. Wheeler, my thesis advisor, has been a marvelous mentor, and I wish to thank him for his encouragement and patience. I also wish to thank Dr. James F. Gunn for many helpful discussions.

PHYSICAL REVIEW D VOLUME 2, NUMBER 8 15 OCTOBER 1970

# Uniqueness of the Response to the Variation of an External Gravitational Field

J. C. Katzin and William B. Rolnick
*Department of Physics, Wayne State University, Detroit, Michigan 48202*
(Received 21 May 1970)

The response of operators of a quantum field theory to variations of the metric tensor is investigated. A class of operators, defined in a previous publication, is shown to include all physically relevant fields and currents. The response of such objects to variations of $g_{\mu\nu}$ is found and shown to be unique. The commutators $[T^{\mu\nu}(x),A(x')]$, where $A$ is any such operator, are discussed. For $\mu=0$ (or $\nu=0$), these commutators are displayed and depend only upon the tensor structure of $A$ and its equation of motion. However, for $\mu,\nu=i,j$ we find that the time derivative of the varied quantity appears in the field equations, for which the usual derivation of equal-time commutators is incomplete.

## I. INTRODUCTION

The variational action principle has been employed to obtain equal-time commutators (ETC's) of various operators.[1] How an external gravitational field would couple to matter is presumed to be contained in the generally covariant form of the theory. However, the behavior of the fundamental fields (appearing in the Lagrangian) upon variation of $g_{\mu\nu}$ is not evident, but is crucial to the application of the action principle. Although it is believed that this behavior may be chosen at will, we shall present a physical argument which uniquely specifies it.

In a previous work,[2] we have investigated the behavior of tensors and spinor-tensors upon variation of $g_{00}$. The criterion required for our results to be valid was that the objects be tensors (or spinor-tensors) for local coordinatelike transformations which may change the curvature but leave the description of the point under consideration fixed. We shall hereafter use the words "point-local" tensor (or spinor-tensor) to designate such objects. In Sec. II we elaborate upon the validity of our criterion. We show that our physical argument justifies our assumption that the fields are point local.

In this work, we also investigate the uniqueness of the choice of the local coordinate transformation to be employed to mimic an external gravitational field. For variations of $g_{\mu\nu}$, we shall see that the choice is limited, and this determines a unique behavior for point-local objects. The uniqueness arises from the nature of the action principle (viz., the derivation of the ETC's[1]). We proceed to treat variations of $g_{0i}$, and obtain results analogous to those of Ref. 2. We find that for variations of $g_{ij}$, $\partial_0\delta g_{ij}$ appears in the field equations and the derivation of ETC's in Ref. 1 is not complete. In another work, these cases will be investigated.

We proceed in this paper, as was done in Ref. 2 as well, to ignore the question of limits of nonlocal expressions which may be necessary to represent physically meaningful objects. As indicated there, a complete reformulation of quantum field theory may be required. Since the problem is due to the meaninglessness of products of distributions, it is a disease of any theory employing such objects (e.g., the Sugawara model, as discussed by Coleman, Gross, and Jackiw[3]). If the quantum action principle can be given a consistent foundation, at a certain stage of the procedure, point-local objects will be present. Furthermore, if the variation of $g_{\mu\nu}$ may be taken after the limit of the nonlocal expression under consideration, then the results of this work will be applicable.[4]

## II. POINT LOCALITY

The response of point-local operators to a variation of $g_{00}$ was found in Ref. 2 by varying the coordinate system at the space-time point in a particular way.

Our approach there[2] is summarized as follows: We consider a flat space (with a coordinate system denoted by barred quantities). We introduce at the point under consideration ($P$), a local observer (with a coordinate system denoted by unbarred quantities). The description of the point $P$ is the same to both observers, viz.,

$$x_\mu(P)=\bar{x}_\mu(P),$$

[1] J. Schwinger, Phys. Rev. **130**, 406 (1963).
[2] J. C. Katzin and W. B. Rolnick, Phys. Rev. **182**, 1403 (1969).
[3] S. Coleman, D. Gross, and R. Jackiw, Phys. Rev. **180**, 1359 (1969).
[4] M. Sheinblatt and R. Arnowitt, Phys. Rev. D **1**, 1603 (1970).