# THE DEFINITION OF MACH'S PRINCIPLE<sup>1</sup>

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Abstract. Two definitions of Mach's principle are proposed. Both are related to gauge theory, are universal in scope and amount to formulations of causality that take into account the relational nature of position, time, and size. One of them leads directly to general relativity and may have relevance to the problem of creating a quantum theory of gravity.

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# 1 Introduction

Ernst Mach's suggestion that inertial motion is not governed by Newton's absolute space and time but by the totality of masses in the universe [1, 2, 3, 4] was the primary stimulus to Einstein's creation of general relativity and

<sup>&</sup>lt;sup>1</sup>It is a pleasure to celebrate with this paper the 80th birthday of my PhD supervisor Peter Mittelstaedt, who has taken a life-long interest in Mach's principle.

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suggested its name. In 1918 Einstein [5], [6] (p. 185/6) coined the expression *Mach's principle* for Mach's idea and attempted a precise definition of it in the context of general relativity. Somewhat ironically, Einstein later disowned Mach's principle [7], but it has continued to fascinate researchers.

So many definitions of Mach's principle have been proposed that it has often been dismissed as incapable of precise formulation. I shall argue that this is because the issue at stake has not been addressed at a sufficiently basic level. For this reason, I shall not discuss the numerous attempted definitions but instead propose two candidate definitions, both of which arise from very basic considerations related to observability and the definition of causality (more precisely, determinism) in the classical dynamics of either particles or fields. I believe that the correct definition of Mach's principle is important precisely because it relates to basic issues that are likely to play an important role in the creation of quantum gravity.

The essential content of this paper could be expressed in a quarter of its length, but I have opted for a discursive presentation. This is so that I can put the definition of Mach's principle in an adequate historical perspective that distinguishes transient concepts of 'what the world is made of' from the normative principles of empirical adequacy and causality, which, if not eternal, have been a key part of the scientific outlook since it came into being. The fact is that Mach formulated his principle in terms of Newtonian masses and interaction at a distance, and Einstein eventually came to believe that the principle was made obsolete by the rise of field theory and local interaction. But Einstein thereby confused ontology-independent principles with their application to transient ontology. A key aim of this paper is to identify first principles that do not need to be abandoned when more superficial concepts change.

To that end, I shall begin by emphasizing that Newton introduced absolute space and time to define velocity, which is displacement in unit time, as the first step in creating dynamics. Having understood this, we can ask if there is an alternative to Newton's absolute definition. At this point it is important to establish what the alternative should achieve. Mach was primarily concerned with empirical adequacy: displacement should be relative to something observable and time must be derived from actual change. These are ontology-independent requirements. But they are in fact met *indirectly* in Newtonian theory if properly interpreted, as Mach came to recognize and I shall show. Mach's instinct still told him Newtonian theory needed modification to close the gap between its basic notions and direct empirical input, but he failed to do this or to provide a criterion that would confirm success in the enterprise. This is where the formulation of causality enters the story. Drawing on a penetrating but largely overlooked analysis by Poincaré [8, 9], I shall suggest that Mach should have required Newtonian dynamics to be replaced by a theory that, in a well-defined sense, has maximal predictive power. In other words, the theory should be as strongly causal, in a sense that I shall make precise, as one can make it. The important thing about this principle is that it turns out to be applicable not only to the Newtonian ontology of masses and interactions at a distance, but to all conceivable ontologies of the world provided only that they are subject to continuous symmetries. The rise of the theory of Lie groups and its widespread application in the gauge theories of modern physics has shown that such ontologies are ubiquitous. It is in this sense that I aim to persuade the reader of the universality of Mach's principle as presented here. It is intimately related to the gauge principle.

In fact, as already indicated, I shall propose two possible definitions of Mach's principle. The first does have maximal predictive power, while in the second that is weakened marginally. I leave the explanation for this to the end of the paper, and merely comment here that the alternative definition has mathematical virtues that could outweigh its slightly weaker predictive strength. Moreover, the alternative definition is realized in general relativity in an intriguing manner.

# 2 Newton's Argument for Absolute Space

In 1644 Descartes published his *Principles of Philosophy* [10, 11], in which he argued that motion is relative. He did this in order to advance an essentially Copernican scheme without offending the Inquisition [12]. I will not detail here his resulting inconsistencies but merely note that he begins by asserting among other things that

• The position and motion of any considered body is defined only relative to other bodies.

• Since there are infinitely many bodies in the universe that are all moving in different ways and any of these can be taken as a reference body, any considered body has infinitely many positions and motions.

He then proposed certain laws of motion, the most important of which exactly anticipated Newton's first law: a free body will either remain at rest or move rectilinearly at a uniform speed. He did not attempt to reconcile this with his 'official' relationalism.

At some unknown date before the mid 1680s, Newton wrote a Latin

text *De gravitatione*, first published in 1962 [13]. Much more clearly than the Scholium to the *Principia* (1687) [14], it spells out Newton's reason for introducing absolute space (see [12]).

Like Descartes, Newton accepted that bodies move relative to each other in space. The bodies are visible, but space is not. Newton was clearly inspired by Descartes's idea of doing for motion what Euclid had done for geometry: formulate axioms. More clearly than many modern authors, to say nothing of his contemporaries, Newton saw that the first task in such an undertaking is *the definition of velocity*. He noted especially a fatal flaw in Descartes's relationalism: it would be impossible to say that any given body moves rectilinearly – the bodies used to define its motion could be moving arbitrarily and, moreover, chosen arbitrarily.

He saw the need for the definition of what may be called *equilocality*: the identification of points that can be said to have the same place at different times. In a solid, each of its points can be assumed distinguishable and fixed. Rectilinear motion relative to them is well defined. But in the Cartesian universe the bodies, even if assumed distinguishable, merely have observable mutual separations  $r_{ij}$  in otherwise invisible space. One can say how the  $r_{ij}$  change but not how any body moves.

Newton understood this very clearly. It is possible that he considered the possibility of defining the motion of any particular body relative to the totality of bodies in the universe, but he must have dismissed that for two reasons. First, according to the prevailing mechanical philosophy the universe should be infinite and contain infinitely many bodies. One could never include all the reference bodies needed to define displacement. Second, in any one instant a given body would have a definite position relative to all the other bodies in the universe. If they were all to remain fixed relative to each other, the displacement of the considered body relative to them would be uniquely defined. But, as Newton noted, velocity is defined as the ratio of a displacement in a given time. During this time, the reference bodies would be moving in all sorts of different ways, making the definition of a unique displacement virtually impossible. Faced with these seemingly insuperable difficulties, Newton concluded his discussion of motion in *De gravitatione* by stating

it is necessary that the definition of places, and hence of ... motion, be referred to some motionless thing such as extension alone or space in so far as it is seen to be truly distinct from bodies.

Newton's 'motionless thing' acquired the grand title *absolute space* in the famous Scholium that immediately follows the definitions at the start of the *Principia*, to which we now turn.

## 3 The Scholium Problem and Inertial Systems

Newton acknowledged that only relative positions and times are observable. He argued that his *invisible* absolute motions could be deduced from the *visible* relative motions and concluded the Scholium in the *Principia* by claiming that how this is to be done "shall be explained more at large in the following treatise. For to this end it was that I composed it." In fact, he never returned to the issue in the body of the *Principia* – and it has been remarkably neglected ever since. Let me formulate this problem, of which Newton was acutely aware, in modern terms:

The Scholium Problem. Given only the successive separations  $r_{ij}$  of a system of particles that form a closed dynamical system in Euclidean space and told that there does exist an inertial frame of reference in which the particles obey Newton's laws and are interacting in accordance with his law of universal gravitation, how can one confirm this fact and find the motions in, for definiteness, the system's centre-of-mass inertial system? For simplicity and without loss of insight, it may be assumed that the particle masses are given.

This problem, whose solution will be discussed below, could only be neglected because nature provided *material* substitutes of absolute space and time with respect to which Newton's laws were found to hold with remarkable accuracy: the fixed stars as a spatial frame of reference and the rotation of the earth with respect to the stars as a measure of time [15]. It was only in the second half of the 19th century that scientists began to take a more critical attitude to the foundations of dynamics and consider how the Scholium Problem could be solved.

In fact, it was Mach's qualitative critique of Newton's concepts – I shall come to it shortly – that stimulated Lange [16] to attack the problem. He supposed three particles ejected from some common point that then move freely (force-free particles) and took their successive positions to define a material spatiotemporal frame of reference that he called an *inertial system*. By Galilean relativity, such a system can only be determined up to Galilean transformations. However, with respect to any such system one can verify whether other bodies, including ones with nontrivial interactions, do move according to Newton's laws.

Even though the notion of a force-free particle is not unproblematic, Lange deserves great credit for this partial solution to the Scholium Problem, and his term *inertial system*, or *inertial frame of reference*, has become standard. However, his actual method is rather cumbersome and mechanical, and a conceptually much cleaner and more illuminating procedure had already been proposed by Tait in 1883 [17]. I shall discuss this in a generalization that takes into account not only the relativity of position and time but also scale. The important thing is to establish *the amount of information* needed to determine an inertial system. Most textbooks nowadays define one simply as a frame of reference in which Newton's laws hold; they seldom describe its actual determination or the observational input that it needs. We shall see that this last is the key to the definition of Mach's principle in either of the two forms that I propose.

In the spirit of Tait's note, suppose a system of N point particles that are said to be moving inertially. We are handed 'snapshots' of them taken at certain unspecified instants by a 'God-observer'. Since only dimensionless quantities have physical meaning, we take them to give us dimensionless separations  $\hat{r}_{ij}$ :

$$\hat{r}_{ij} = \frac{r_{ij}}{r}, \quad r = \sqrt{\sum_{i < j} r_{ij}^2},$$
 (1)

where  $r_{ij}$  are the separations measured with some arbitrary scale.

In a Cartesian representation, the particles have 3N coordinates, but the  $\hat{r}_{ij}$  contain only 3N - 7 objective (observable) data: three are lost because the position of the centre of mass is unknown, three because the orientation is unknown, and one because the scale is unknown. Equivalently we can say that the positions are known only up to Euclidean translations (3), rotations (3) and dilatations (1). We have gauge redundancy corresponding to the similarity group of Euclidean space.

Tait's problem is: how can one use the snapshots to confirm that the particles are moving inertially, and how many snapshots are needed? The solution to the problem is simple but instructive.

By Galilean relativity, we can certainly take particle 1 to be perpetually at rest at the origin of the frame that is to be found. Next, at some instant, particle 2 will, in that frame, pass through its least separation from particle 1, which can be taken to be the distance 1 (this fixes the unit of distance). Further, we can choose the coordinate axes and the unit of time such that particle 2 moves with unit velocity in the xy plane along the line x = 1, y = t. The coordinates of these two particles are therefore fixed to be (0, 0, 0) and (1, t, 0); we have eliminated irrelevant ambiguity. However, at the instant t = 0 all the other particles can have arbitrary positions and velocities, so that a generic inertial solution needs  $6 \times (N-2) = 6N - 12$  data to be fully specified.

We have noted that each snapshot contains 3N-7 independent objective data, i.e., dimensionless separations. However, the times at which they are taken are unknown, so in fact we have only 3N-8 real data. Thus, two snapshots contain 6N-16 usable data, which is 4 short of the number 6N-12 needed to construct the frame. However, provided N is large enough, three snapshots contain enough information to construct the data and provide independent checks that the particles are all moving inertially in a common spatiotemporal frame.<sup>3</sup> Note that the determination of time is inseparably tied to the determination of the spatial frame and that both can only be found because a definite law of motion is operative.

Tait's result is characteristic, and it shows that for the simplest (inertial) Newtonian problem 'two-snapshot' relational initial data do not suffice to predict the evolution. There is a four-parameter *shortfall* in the relational data. In the realistic case in which interactions are present (gravity is never absent), there is a five-parameter shortfall. From the Newtonian perspective, the information that is lacking is: the value of the angular momentum  $\mathbf{L}$ , the amount of kinetic energy in overall expansion or contraction of the system (because there is no absolute scale), and the value of the dimensionless instantaneous ratio T/V, where T is the total kinetic energy of the system and V is its potential energy.

It is a reflection on the way in which dynamics is taught that, in my experience, even distinguished scientists struggle to get to grips with Tait's problem and identify the reasons for the shortfall. That it is an issue is even new to them. Note also that 4/5 of the shortfall already arises for pure inertial motion; only 1/5 arises from interactions and accelerations. This shows that, contrary to what is frequently said, the problem of absolute vs relative motion is not the difference between inertial and accelerated motion. Moreover, Newton introduced absolute space to define inertial motion, as is clear from his discussion in *De gravitatione* (see my discussion in [12]).

To conclude this section, it needs to be emphasized that there is no epistemological defect in Newtonian mechanics; it is possible to construct inertial systems from observed relative motions [18], p. 44. If there is a defect, it resides in the curious shortfall just established, as we shall see when we come to Poincaré's critique.

<sup>&</sup>lt;sup>3</sup>Fully relational dynamics – with no absolute space (i.e., no a priori equilocality relation), time or scale – has no content unless  $N \ge 3$ ; for N = 3 and N = 4 more than three snapshots are needed to solve Tait's problem.

#### 4 Mach's Intuitive Critique

Mach's contribution [1, 2, 3, 4] to the debate about the nature of motion was threefold but mostly took the form of suggestive comments rather than precise prescriptions. Mach's fundamental objection was to Newton's reliance on structure not directly derived from observation: in my terminology, equilocality at different times and the metric of time. Although aware of Riemann's revolutionary ideas about geometry, he did not object to the use of Euclidean space at a given time; he frequently talks about the observable separations between bodies in a manner which clearly implies that these separations are compatible with Euclidean geometry.

In fact, Mach's contribution was threefold:

• The dynamics of the universe should be described directly and solely in terms of the changes in the observable separations. Speaking about the Copernican revolution, he said [4], p. 284: "The universe is not *twice* given, with an earth in rest and an earth in motion; but only *once*, with its *relative* motions, alone determinable."

• The measure of time must be derived from change: "It is utterly beyond our power to measure the changes of things by time. Quite the contrary, time is an abstraction at which we arrive by means of the changes of things" [4], p. 273.

• The specific behaviour identified by Newton as inertial motion could arise from some causal action of all the masses of the universe; this could lead to some observable effects different from Newtonian theory: "Newton's experiment with the rotating vessel of water simply informs us that the relative rotation of the bucket with respect to the sides of the vessel produces no noticeable centrifugal forces ... No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick" [4], p. 284.

The first two of these had been anticipated in Newton's time by Leibniz [19] and others. The third was revolutionary. Combined with the first two, it suggests a new concept of motion. The Newtonian view is captured in the subtitle "How solitary bodies are moved" of another unpublished paper 'The laws of motion' [20], p. 208, that seems to be earlier than *De gravitatione*. By seeking laws governing individual (solitary) bodies, Newton had no option but to assume that they move in space as time passes. As we shall see, implementation of Mach's principle requires us to consider, not the motion of bodies in space and time, but *positions of the universe in its configuration* 

space. This claim needs some amplification.

The development of variational mechanics (described beautifully by Lanczos [21]) led to the notion of the configuration space Q of a closed dynamical system. In this formalism, the points in Q that represent the realized instantaneous configurations of the system lie on a curve in Q called the *dynamical orbit* of the system. It should be noted that when Newtonian mechanics is represented in this manner, the definition of Q is based on positions in an inertial frame of reference. As we have seen, this is problematic since such frames are obtained through a nontrivial process. The first step in the definition of Mach's principle will be the identification of an appropriate configuration space in which it is to be implemented.

It will also be necessary to think about time. In the standard accounts of dynamics, the point in Q that represents the instantaneous configuration of the system moves along the orbit as the time t passes. But what is this t? If textbooks address this question at all, it is generally said that t is provided by a clock external to the system. But if we have no access to anything outside the considered system, as for example a solar system surrounded by opaque dust clouds or, more relevantly, the universe, a measure of time must somehow be extracted from within the system itself. However, all the changes to which Mach referred are encoded in the curve in Q. We seem to be confronted with a vicious circle – we cannot determine the evolution parameter, which must now be extracted from differences along the curve, before the curve itself has been determined. Two different ways to resolve this problem will be presented later, both involving information encoded in a curve in Q. For the moment, I simply want to point out that Mach criticized Newton's concept of absolute time as forcefully as the concept of absolute space. Few people have noted this, though Mittelstaedt coined the expression second Mach's principle [22, 23] in order to draw attention to the issue.

Finally, a problem with Mach's writings – and the cause of much confusion and dispute – is that he did not provide a criterion that would establish when a theory is Machian. In fact, it is possible to rewrite the content of Newton's laws directly in terms of the observable separations – I shall discuss this directly – and thus seemingly meet Mach's requirement without changing the physical content of Newtonian theory. This is in fact what Lange, stimulated by Mach's critique, showed: Newtonian dynamics can be put on a sound epistemological basis, i.e., related to directly observable quantities. (This, of course, presupposes that Newtonian theory is physically correct. I shall discuss relativity later.)

Much more could be said about Mach and the way in which he has been

misunderstood, but it will be more helpful to move on directly to Poincaré.

# 5 Poincaré's Strengthened Relativity Principle

In his Science and Hypothesis [8, 9], Poincaré has some pertinent things to say about the problems of absolute and relative motion, which he comments have been much discussed in recent times. He says that it is repugnant to the philosopher to imagine that the universe can rotate in an invisible space but then poses the decisive question: what precise defect, if any, arises within Newtonian dynamics from its use of absolute space? It is the answer to this question that will provide us with both the definitions of Mach's principle to be proposed in this paper.

For the purposes of his discussion, Poincaré assumes that a definition of time is given and that the distances between bodies in (Euclidean) space can be directly measured. Thus, he presupposes the existence of a standard clock and rod. In an extension of his way of thinking, I shall, when formulating Mach's principle, dispense with these as in my discussion of Tait's problem. However, I shall stick with Poincaré's assumptions for the moment. What he in effect said was this. Let us suppose particles of known masses  $m_i$ , i =1, 2, ..., N, in space between which observers can, at any instant, observe the inter-particle separations  $r_{ij}$  and, a clock being granted, the rates of change  $\dot{r}_{ij}$  of the  $r_{ij}$ . Thus, at any instant the observers have access to  $r_{ij}$ ,  $\dot{r}_{ij}$ . Poincaré also assumes that they know that the bodies satisfy Newton's laws of motion and are governed by the law of universal gravitation. He then asks: given such initial data, is it possible to predict uniquely the evolution of the  $r_{ij}$ , i.e., the observable evolution?

The answer is no, and the reason is evident from the discussion of Tait's problem: the data  $r_{ij}$ ,  $\dot{r}_{ij}$  contain no information about the angular momentum **L** in the system. That this is so is readily seen after a moment's reflection on the two-body Kepler problem. At perihelion or aphelion any planet is moving at right angles to the line joining it to the sun, so that the planet–sun separation r is not changing:  $\dot{r} = 0$ . But the initial data  $r, \dot{r} = 0$  can lead to all possible Keplerian motions, including both circular motion and direct fall into the sun. It is the angular momentum, invisible in the data  $r, \dot{r} = 0$ , that makes the difference. Since three pieces of information are encoded in the vector **L**, two in its direction and one in its magnitude, we arrive at this important conclusion: in the generic – and archetypal – problem of N bodies interacting with any central forces, specification of initial data in the form  $r_{ij}, \dot{r}_{ij}$  will leave a three-parameter unpredictability

in the evolution. When the invisibility of scale and time are taken into account, this becomes a five-parameter unpredictability, as is clear from the discussion of Tait's problem.

This unpredictability arises exclusively from the difference between specifying purely relative quantities and specification in an inertial frame of reference, for which perfect Laplacian determinism and causality holds. The shortfall in predictive power is the price that has to be paid for Newton's introduction of absolute kinematic structure that is independent of the contents and relative motions of the objects in the universe. Thus, the case for modifying Newtonian theory is not to improve its epistemological status but its predictive power. It is an argument from causality, not epistemology. Before making this idea explicit, I want to point out that Poincaré devised a way to characterize the nature of theories that is different from the one that has become universal through the manner in which Einstein created his theories of relativity.

Einstein's approach focussed all attention on the symmetries of the laws of nature: under what transformation laws do they retain the same form? In his attempt to implement Mach's ideas, Einstein sought to make the transformation laws as general as possible. He was initially [24] convinced that general covariance was a powerful physical principle but later [5] accepted Kretschmann's argument [25] that it had only formal mathematical significance. Since then there has been much inconclusive debate about the meaning of relativity principles and general covariance. I do not wish to get into it here because I believe that in his Science and Hypothesis Poincaré proposed a more fruitful and unambiguous approach. This was to define relativity in terms of the amount of information needed to be specified in coordinate-independent (gauge-invariant) form if the evolution is to be predicted uniquely. I shall strengthen Poincaré's formulation in order to be able later to define Mach's principle in the strongest and cleanest form.

Suppose we live in a Newtonian universe and consider a dynamically isolated N-body subsystem within it. There are two ways in which we can study the subsystem. The first is in the standard Newtonian manner in an inertial frame of reference, while the second concentrates exclusively on what is actually observable within the subsystem.

Now, as Poincaré noted, certain details of the subsystem's initial state when specified relative to its inertial system have no effect on its subsequent directly observable evolution. Thus, one can imagine the initial configuration of the system rotated and translated without this having any effect. By Galilean relativity, a uniform velocity imparted to the subsystem also has no effect. Let us do some counting. Three numbers specify the position of the origin of the inertial frame relative to the subsystem, three more specify the orientation of its axes, and three more the translational velocity of the origin. All of these have no observable effect within the subsystem. But whereas one can boost the origin of the inertial frame, in a passive transformation, or equivalently the centre of mass of the subsystem, in an active transformation, one cannot 'boost the orientation'. This is because of the dynamical effect of angular momentum, which is not encoded in the  $r_{ij}$ ,  $\dot{r}_{ij}$  but shows up in the second and higher derivatives  $\ddot{r}_{ij}$ ,... as the system evolves.

This is the nub. Poincaré argued that this breakdown of relational predictability is the only 'defect' in Newtonian dynamics that arises from its use of absolute space. He said that, "for the mind to be fully satisfied", a strengthened form of the relativity principle must hold: the relational data  $r_{ij}$ ,  $\dot{r}_{ij}$  should determine the evolution uniquely. However, he noted with regret that the solar system is an effectively isolated system and manifestly does not satisfy such a strengthened relativity principle; nature does not work the way philosophers would like.

This is the place to mention the famous study of the Newtonian threebody problem made by Lagrange in 1772 [26]; it was surely at the back of Poincaré's mind when formulating his critique of Newtonian dynamics. In his study, Lagrange assumed Newtonian gravitational dynamics to be correct but then reformulated its equations in terms of the separations  $r_{ij}$ between the three particles. He obtained three equations that contain *third* derivatives with respect to the time and thus for their solution need specification of  $r_{ij}$ ,  $\dot{r}_{ij}$ ,  $\ddot{r}_{ij}$  as initial data. This showed that Newtonian theory with nontrivial interactions (and not only in the case of inertial motion as in Tait's problem) could be perfectly well expressed in terms of directly observable quantities.<sup>4</sup> It has no epistemological defect, as has already been noted. What remains very curious is that when N is large only very few of the second derivatives  $\ddot{r}_{ij}$  need to be specified in addition to  $r_{ij}$ ,  $\dot{r}_{ij}$ . It is also manifestly clear that a theory in which only  $r_{ij}$ ,  $\dot{r}_{ij}$  need to be specified will have greater predictive power.

In *Science and Hypothesis*, Poincaré does not mention Mach's ideas about the origin of inertia. As they were well known, this surprises me. If Poincaré was unaware of them, this may explain why he admitted regretfully that nature seemed to have no respect for his strengthened relativity principle and that we would simply have to accept what nature tells us. But

<sup>&</sup>lt;sup>4</sup>Lagrange's work was extended to arbitrary N by Betti [27] and, using powerful modern techniques, by Albouy and Chenciner [28].

Mach made it clear that we could only hope to gain a satisfactory understanding of dynamics by including the *whole universe* in our considerations. This opens up the possibility that the universe evolves in accordance with a stronger relativity principle than subsystems within it. Its overall behaviour need not be so philosophically repugnant as Poincaré believed.

To conclude this part of the discussion, Poincaré's analysis has two virtues: 1) it employs coordinate-free language and is expressed solely in terms of gauge-invariant (observable) quantities; 2) it provides the *precise criterion* that Mach failed to formulate.

#### 6 Configurations and Gauge Redundancy

As preparation for the definition of Mach's principle and to demonstrate that it has universal applicability, I want to draw attention to a widespread phenomenon in nature, introducing it by asking this question: what empirical content underlies Euclidean geometry?

Rods that remain mutually congruent to a good accuracy are crucial. Using them, we can measure angles and ratios of lengths. These dimensionless quantities have objective physical meaning. In fact, I suspect that all quantitative measures in science reduce ultimately to measurement of angles and length ratios. (Of course, one can *count* apples, but I am referring to measurement of their size.) It is certainly true that ancient astronomy right up to Kepler's discoveries relied entirely on angle measurements, including those used to obtain time from the observed revolution of the stars.

We can think of Euclidean geometry in the following terms. By means of a rod, we can measure the distances between  $N, N \ge 5$ , particles at some instant. We obtain N(N-1)/2 positive numbers, the inter-particle distances. They could have been arbitrary, but empirically (for  $N \ge 5$  in three dimensions) they satisfy algebraic equations (and inequalities). These empirical relations may be called *the Euclidean rules*.

They permit remarkable data compression in the representation of facts. In a globular cluster containing a million stars there are at any instant  $\approx 10^{12}$  distances between them. All this in principle independent information can be encoded by  $3 \cdot 10^6$  Cartesian coordinates. This is a colossal reduction, but it comes with inescapable redundancy and *frame arbitrariness*. One can pass freely between Cartesian frames by Euclidean translations and rotations; there is a 3 + 3 = 6-fold degeneracy. In fact, since the choice of the rod (which defines the unit of length) is arbitrary, dilatations must be taken into account, and there is a 3 + 3 = 1-fold degeneracy. Thus, for N

particles, there are 3N Cartesian coordinates but only 3N - 7 true (frameindependent) degrees of freedom. This 'economic representation with group redundancy' will be decisive when we come to consider dynamics.

In fact, the very possibility of having dynamics is intimately related to another facet of geometry, its 'procreative' capacity. One single set of distances between N points, defining a relative configuration, is sufficient to establish the possibility of placing them in a Cartesian frame with coordinates  $\mathbf{x}_i$ . The original empirically determined distances are then given by  $r_{ii} = |\mathbf{x}_i - \mathbf{x}_i|$ . Then infinitely many other relative configurations can be generated by simply specifying freely any N Cartesian vectors, which form a *Cartesian configuration*, and calculating the distances between them by the rule just given. A whole space of relative configurations can be generated in this way. It is, however, an inescapable fact that many distinct Cartesian configurations give rise to one and the same relative configuration. This is so whenever two Cartesian configurations are congruent, so that one can be carried into exact coincidence with the other by Euclidean translations and rotations (and scaling if we include that). A group of motions acts on the configurations, leaving invariant the the interparticle separations, which we measure more or less directly and regard as physical. In contrast, the Cartesian coordinates are *gauge dependent*, being changed by the action of the group of motions.

In the light of the above remarks, it is helpful to distinguish three spaces. The first is the familiar Newtonian configuration space Q, which for N particles has 3N Cartesian coordinates. If we identify all configurations in Q that can be carried into exact congruence by the transformations of the Euclidean group, we obtain the *relative configuration space* R. This is not a subspace of Q but a distinct quotient space. If, in addition, we identify all configurations that have the same shape (adding dilatations to the group of motions), the identified configurations form *shape space* S, which again is a distinct space, not a subspace of Q.

If we are considering a *single* configuration, the economic bonus of using Cartesian coordinates vastly outweighs any inconvenience in the arbitrariness. However, if distances between N objects are established when they are in two *different* relative configurations there is in principle no connection between the Cartesian frames chosen to represent each configuration. This gives rise to serious frame arbitrariness and the first fundamental problem of motion [12]. The second such problem relates to time: two such configurations contain no information about the lapse of time between them.

The 'economic representation with group redundancy' discussed above is a characteristic feature of nature that has stimulated much mathematics, above all the theory of Lie groups and algebras. Further examples with different groups of motion are:

• It can be established by measurement that the magnetic field  $\mathbf{B}(x_i)$ , i = 1, 2, 3, is always such that div  $\mathbf{B} = 0$ . One can therefore represent this empirical fact in a mathematically convenient form by introducing a vector potential  $\mathbf{A}(x_i)$  such that

$$\operatorname{curl} \mathbf{A} = \mathbf{B},\tag{2}$$

but A carries redundant gauge information, and

$$\operatorname{curl}\left(\mathbf{A} + \frac{\partial \phi}{\partial \mathbf{x}}\right) \equiv \operatorname{curl} \mathbf{A} = \mathbf{B},$$

so that the gauge transformation

$$\mathbf{A} \to \mathbf{A} + \frac{\partial \phi}{\partial \mathbf{x}},\tag{3}$$

where  $\phi(x_i)$  is an arbitrary scalar function, leaves the represented empirical data unchanged. The space of 3-vector fields **A** with gauge redundancy can be quotiented by the symmetry to obtain the space of gauge-invariant magnetic fields **B**. In contrast to the spatial gauge redundancy of the first example, this is the simplest [SU(1), Abelian] example of an internal symmetry. The non-Abelian gauge symmetries of Yang–Mills fields are further examples. The group of motions that here acts on the configurations is infinite dimensional and hence a generalization of a Lie group of motions.

• Infinitely many distance measurements between all pairs of points in a three-dimensional manifold  $\mathcal{M}$  can in principle reveal empirically that it carries a Riemannian 3-geometry.<sup>5</sup> This can be represented by means of a 3-metric  $g_{ij}$ . However, the six components of the symmetric  $g_{ij}$  include not only information about the geometry but also about the coordinates used to represent it. Any two metrics related by a coordinate transformation will represent the same 3-geometry, and the six components of  $g_{ij}$  will represent three geometrical degrees of freedom and three coordinate, or gauge, degrees of freedom. Again we have a remarkably convenient and economic representation but with redundancy. The group of motions in this case is the group of three-dimensional diffeomorphisms.

The configuration space with redundancy is called Riem; it is the space of all (suitably continuous) Riemannian 3-metrics defined on  $\mathcal{M}$ . The space of

<sup>&</sup>lt;sup>5</sup>The manifold  $\mathcal{M}$  is assumed to be compact without boundary to provide the basis for a dynamically closed universe.

gauge-invariant 3-geometries is obtained by quotienting by 3-diffeomorphisms and is called *superspace*:

Superspace := 
$$\frac{\text{Riem}}{3\text{-diffeomorphisms}}$$

• One could also argue that all distance measurements are local and only establish ratios. Then one would conclude that any two 3-metrics

$$g_{ij}(\mathbf{x}), \ \varphi(\mathbf{x})^4 g_{ij}(\mathbf{x})$$

related by a *conformal* transformation

$$g_{ij}(\mathbf{x}) \to \varphi(\mathbf{x})^4 g_{ij}(\mathbf{x}),$$
 (4)

generated by the positive scalar function  $\varphi$  represent the same physical configuration.<sup>6</sup> This further quotienting by three-dimensional conformal transformations leads to the infinite-dimensional space known as *conformal superspace*. It has precisely two (local) degrees of freedom per space point.

These examples show that the original property of nature that gave rise to Mach's critique of Newtonian mechanics is ubiquitous. It is economic representation of instantaneous configurations with redundancy and suggests that the correctly formulated Mach's principle should be universal in scope and apply in all cases in which such redundancy occurs. In fact, I shall argue that whenever this characteristic redundancy is recognized in individual configurations, it predetermines a unique dynamical theory of such configurations that is of gauge type.<sup>7</sup> In this connection, Ó Raifeartaigh notes [29], p. 13, that "the curl property of the magnetic field was known quite early and led, for example, to Stokes' theorem". In fact, coupled with the definition of Mach's principle given later, it could have led directly with no further input to Maxwell's equations in vacuum [30].

It is worth noting here that physical conceptions have changed greatly since Mach's time. It is therefore necessary to reconsider his critique in the light of developments and above all identify the central issue. If we

<sup>&</sup>lt;sup>6</sup>The purely conventional fourth power of  $\varphi$  is chosen to simplify the transformation behaviour of the (three-dimensional) scalar curvature R under (4). In four dimensions, the second power is chosen for the same reason.

 $<sup>^{7}</sup>$ It is unfortunate that the word 'gauge' is used in several different senses and often loosely, which leads to much confusion. In my view the decisive thing is the fact that one starts with instantaneous spatial configurations on which a group of motions acts. It generates gauge transformations of the configurations. A dynamical theory of gauge type arises from this fact.

accept, as is implicit in Mach's writings, that dynamics is to be built up on the notion of configurations of the universe, the central problem is always essentially the one that Newton recognized so clearly: how do you define velocity in a relational context? Once Faraday and Maxwell had introduced the notion of fields, this problem simply became that of defining rates of change of fields. In all cases, the problem is the same and arises from the fact that *physically* (as opposed to gauge related) different configurations do not come with equilocality markings or preassigned time differences between them.

In the light of this comment, it is interesting to consider a comment that Einstein made in 1949 [7], p. 29:

Mach conjectures that in a truly rational theory inertia would have to depend upon the interaction of the masses, precisely as was true for Newton's other forces, a conception which for a long time I considered as in principle the correct one. It presupposes implicitly, however, that the basic theory should be of the general type of Newton's mechanics: masses and their interactions as the original concepts. The attempt at such a solution does not fit into a consistent field theory, as will be immediately recognized.

Einstein here appears insensitive to the deep problem that Newton identified and Mach equally clearly recognized. The problem of defining change exists in essentially the same form whatever the ontology of the universe. The obsolescence of Mach's ontology did not change the underlying problem. Mach's principle needs to be defined accordingly.

For this, we must also consider time. The above examples are all of instantaneous *spatial* configurations on which a group of motions acts. The great work of Lie, Klein and others in the last decades of the 19th century led to the clear recognition that the mathematical representation of one and the same spatial structure can be changed by the action of a group of motions. The physical object is unchanged by this action. Minkowski and Einstein extended this principle dramatically when they fused time with space and, in spacetime, made it a further dimension barely distinguishable from a spatial one. In this new picture, it was entirely natural on Einstein's part to seek to characterize the structure of spacetime by laws that did not depend on arbitrary coordinate representation.

However, the concept of spacetime marked a radical departure from the 'royal highroad of dynamics'<sup>8</sup> laid out by Newton, Euler, Lagrange, Hamilton and Jacobi. This had led to the key concepts of the configuration and

<sup>&</sup>lt;sup>8</sup>Wheeler's coining I believe.

phase spaces, in which dynamical histories are parametrized by time. All of this work had been done in the framework of the absolute space and time inherited, admittedly often with unease, from Newton. A few dynamicists were just beginning to consider the Machian issues when, in creating special relativity, Einstein and Minkowski took physics down a new road. The ways parted. The spacetime route led to general relativity, while Hamilton and Jacobi's synthesis of classical dynamics led to quantum mechanics in both of its incarnations: matrix and wave mechanics. Quantum mechanics and relativity have since coexisted uncomfortably, as one sees in the immense difficulty in the creation of a quantum theory of gravity.

I believe that one source of this difficulty could be Minkowski's virtual elimination of the difference between time and space. In his picture one cannot begin to formulate a Machian theory in which time is derived from change. The time dimension is there from the beginning and exists independently of the world's happenings. Of course, Einstein modified this picture very significantly in general relativity but not in a way in which one can readily see whether time is derived from change.

Therefore, in order to gain a clear notion of time, there is a good case for formulating Mach's principle in terms of a configuration space Q, in which, as I noted earlier, a measure of time can be extracted from the realized curve in Q. However, Q cannot be the standard one based on absolute space. We need one that is defined relationally by quotienting with respect to the group of motions corresponding to the assumed ontology. As final preparation for the definition of Mach's principle, the following analysis of velocities defined by frames of reference will highlight the difficulties that arise whenever the configurations of physical systems have a gauge redundancy associated with a group of motions.

#### 7 The Decomposition of Velocities

To this end, it will be sufficient to consider the velocity decomposition that can be made in the N-body problem, for which I draw on Saari's discussion [31]. Let the bodies have masses  $m^a$ , a = 1, 2, ..., N, position vectors  $\mathbf{x}_i^a$ , i = 1, 2, 3, in an inertial frame of reference with origin permanently at the system centre of mass, and velocities  $\mathbf{v}_i^a$  at some instant. Then  $\mathbf{x}_i^a$  and  $\mathbf{v}_i^a$  can be 'packaged' as  $3 \times N$ -dimensional vectors  $\mathbf{X}$ ,  $\mathbf{V}$ , where  $X_i^a = x_i^a$  and similarly for  $\mathbf{v}_i^a$ . For such vectors the symmetries of Euclidean space define a natural inner product in configuration space:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j} m_j \mathbf{a}_j \cdot \mathbf{b}_j, j = 1, 2, ..., 3N.$$
 (5)

The moment of inertia I and kinetic energy T of the system are then

$$2I = \langle \mathbf{R}, \mathbf{R} \rangle, \ 2T = \langle \mathbf{V}, \mathbf{V} \rangle.$$

With respect to the inner product (5) there exists a unique decomposition of  $\mathbf{V}$  relative to  $\mathbf{X}$  into three orthogonal components:  $\mathbf{V}_r$ ,  $\mathbf{V}_d$ ,  $\mathbf{V}_s$ , the parts of  $\mathbf{V}$  that correspond to a rotation of the system as a rigid body ( $\mathbf{V}_r$ ), to a dilatation ( $\mathbf{V}_d$ ) of the system, and to a change of its intrinsic shape ( $\mathbf{V}_s$ ). As I will shortly present an intuitive representation of the velocity decomposition that will simultaneously suggest how Mach's principle is to be implemented, I refer the reader to [31] for the formal proof, which uses explicit group transformations to determine  $\mathbf{V}_d$  and  $\mathbf{V}_r$ . The intrinsic change of shape  $\mathbf{V}_s$  is what remains:

$$\mathbf{V}_s = \mathbf{V} - \mathbf{V}_d - \mathbf{V}_r. \tag{6}$$

It is important that the decomposition (6) relies on orthogonality with respect to (5), which can only be established when all components of the considered vectors are known and included. The decomposition is holistic and respects Mach's dictum that "nature does not begin with elements" and that ultimately we must take everything into account ("the All") [4], pp. 287/8.

The velocity decomposition is important in the N-body problem because the three velocity components interact as the system evolves. In the context of Mach's principle the decomposition is valuable because it provides a way to combine the advantages of representation in a definite frame of reference while identifying changes that are independent of the frame. Thus, given any **X** in an arbitrary frame and any **V** in the same frame, one can establish uniquely the part  $\mathbf{V}_s$  of the latter that represents the *intrinsic change* of **X**. Expressed in terms of the position and momentum vectors of the individual particles,  $\mathbf{x}^a$  and  $\mathbf{p}^a$ , the parts  $\mathbf{p}_s^a$  of the momenta that express pure change of shape must satisfy

$$\sum_{a} \mathbf{x}^{a} \times \mathbf{p}_{s}^{a} = 0.$$
<sup>(7)</sup>

$$\sum_{a} \mathbf{x}^{a} \cdot \mathbf{p}_{s}^{a} = 0, \qquad (8)$$

That (8) and (7) must hold is readily seen. For example, under an infinitesimal dilatation  $\mathbf{x}^a \to (1+\epsilon)\mathbf{x}^a$ , so that the corresponding momenta will be proportional to  $\mathbf{x}^a$ . Then all the scalar products in (8) will all be nonvanishing and have the same sign, so that (8) cannot vanish. Similarly, under a rigid-body rotation, (7) cannot vanish.

The whole mystery of absolute and relative motion is reflected in this velocity decomposition. Suppose we have N particles of known masses and can determine the dimensionless separations (1) between them at any instant. From the separations at any one instant 1, we can, having chosen some scale, construct a Cartesian representation of the system. This gives us the position vector  $\mathbf{X}_1$  at that instant. It is natural to place the origin of the frame at the centre of mass. For one single configuration, we feel no unease about choosing the orientation of the axes relative to the configuration in any arbitrary way.<sup>9</sup> However, three real difficulties arise if we now consider a second relative configuration 2 that differs slightly from the first and wish to regard it as having arisen from the first through motion.<sup>10</sup> How is the second configuration to be placed relative to the first? What scale is to be chosen for it? How much time do we believe has elapsed between the two configurations? There is nothing intrinsic in the two configurations that provides an answer to these questions. Different answers lead to different velocity vectors and, in the N-body problem, very different evolutions.

Let us look at this difficulty a little closer and agree to adopt the  $\mathbf{X}_1$  frame and an external measure of time. We can then let the particles move with arbitrary velocities in that frame. But if we were to take the dimensionless relative configuration when it is slightly different from  $\mathbf{X}_1$  and give it to different mathematicians, they would be incapable of reconstructing velocity vectors  $\mathbf{V}$  guaranteed to be the same. As we allow knowledge of the masses, all they could be sure of agreeing on is the position of the centre of mass. The orientation and scale of the second configuration relative to the first and the separation in time between them are not fixed.

However, there is something on which they could agree, which is the amount, assumed infinitesimal, by which the *shape* of the second configuration has changed compared with the first. They could do this simply by noting the changes to the dimensionless separations (1), but this would not suggest anything that one could call a natural quantitative measure of the change. Fortunately, the velocity decomposition theorem does suggest

 $<sup>^{9}</sup>$ We could consider choosing the axes along the principal axes of inertia but, except in the case of rigid bodies, such a choice has no dynamical significance and, moreover, leads to problems if the configuration happens to be symmetric.

<sup>&</sup>lt;sup>10</sup>One could equally well suppose that the first had arisen from the second.

a family of natural measures.

Represent once and for all the first configuration by a vector  $\mathbf{X}_1$  in some Cartesian frame of reference. Represent the second configuration in the same frame with its centre of mass in any arbitrary position, with any orientation and with any scale. This will give the vector  $\mathbf{X}_2$ . Choose an arbitrary 'time difference'  $\delta t$  and call  $\mathbf{V} = (\mathbf{X}_2 - \mathbf{X}_1)/\delta t$  the velocity of the system. Then for all possible positions of the centre of mass, orientation and scale of the second configuration, calculate the scalar product  $\langle \mathbf{V}, \mathbf{V} \rangle$  (note that this brings in a weighting with the masses) and find the necessarily unique position, orientation and scale for which this positive-definite quantity is minimized. It is readily seen that this 'best-matching' procedure, which reduces the 'incongruence' of the two figures to a minimum, brings the centres of mass to coincidence and ensures that for the extremalized quantities  $\mathbf{p}^a = m^a \delta \mathbf{x}^a/\delta t$ the relations (8) and (7) hold with  $\mathbf{x}^a$  the position vectors of the particles in the first configuration.

Note that the choice made for  $\delta t$  merely changes the magnitude of the *intrinsic velocity*  $\mathbf{V}_s$  that results from the extremalization, not the fact that for it the relations (8) and (7) hold. Similarly, the position in which (8) and (7) hold would still be the same if we were to multiply  $\langle \mathbf{V}, \mathbf{V} \rangle$  by any function of the configuration such as its Newtonian potential energy. Because of this freedom, we do not obtain a unique measure of change but all such measures are natural in that they derive from the metric (5), which in turn derives from the fundamental Euclidean metric suitably modified to take into account the presence of masses. The most important thing is that the intrinsic velocity  $\mathbf{V}_s$  that is obtained is orthogonal to the velocity components generated by the gauge (symmetry) transformations whatever 'best-matching' measure is chosen. Different choices of the measure will be significant in dynamics, but at any instant they merely change the magnitude of the intrinsic velocities, not their directions.

We can summarize this state of affairs by saying that the extremal  $\mathbf{V}_s$  is a tangent vector on shape space. It has a magnitude and a direction. In a space of a high number of dimensions, the direction contains much more information than the magnitude, which is always represented by just one number.

#### 8 The Definition of Mach's Principle

It is implicit, and often explicit, in Mach's writings that motion of any individual body is to be defined with respect to the entire universe ([4], pp. 286/7). This makes it possible to avoid the difficulty of Cartesian relationalism, according to which a body has as many different motions as potential reference bodies; a multiplicity of motions is replaced by a single averaged motion. However, the Machian view point is only possible if the unverse is a closed dynamical system. I shall say something about the possibility of a truly infinite universe at the end of this paper.

If we do suppose that the universe is a closed system, we can attempt to describe it by means of a relational configuration space obtained by some quotienting with respect to a group of motions. One of the open issues in the implementation of Mach's principle is the extent to which this quotienting should be taken. In all the examples listed earlier, angles (length ratios) are taken to be fundamental. I have three arguments, none decisive it must be admitted, for not considering quotienting that attempts to reduce physics to something more basic than them. The first is instinctive: angle observations do seem to be primal. The second is that if we take angles to be fundamental, the resulting theories will be built up on a *dimensionless* basis.<sup>11</sup> This is clearly an attractive, indeed necessary principle. My third reason for taking angles to be fundamental is empirical: the existence of fermions, which are described mathematically by spinors with finitely many components. This is only possible if they are defined relative to an equivalence class of orthonormal frames, which presuppose an inner product. The scale (orthonormal) is for convenience in any particular case; it is the *orthogonality* and associated invariance of angles defined by the inner product that is essential.<sup>12</sup> Thus, I believe that there is a case for taking angles to belong to the irreducible geometrical bedrock of physics, but we shall see that, as of now, angles alone do not seem to be sufficient.

We are now in a position to define Mach's principle in both the stronger and weaker forms that I propose. Its definition consists of two parts: a) the identification of cases in which it is to be invoked and b) the stipulation of what it is to achieve.

#### The Definition of Mach's Principle: a) The application of Mach's

<sup>&</sup>lt;sup>11</sup>Time will not occur in the foundations of the Machian theory, and in particle mechanics the masses can be made dimensionless by dividing the action by the total mass of the universe without changing the observable behaviour. Angles and length ratios are obviously dimensionless.

<sup>&</sup>lt;sup>12</sup>It is possible that the currently observed fermions described by finitely many components are merely low energy excitations of spinorial entities with infinitely many components. These do not need an orthogonal frame for their definition. I am indebted to Friedrich Hehl for this observation.

principle is to be considered whenever direct observations or theoretical considerations suggest that the physical configuration space of a closed dynamical system is to be obtained by group quotienting of a larger configuration space that contains redundant information unobtainable by direct observation; b) once the quotiented configuration space Q has been selected, the dynamical theory defined on it is to be such that either 1) specification of an initial point  $q \in Q$  together with a direction d in Q at q defines a unique curve in Q as the evolution of the system given the directly observable initial data q, d or 2) instead of a point q and direction, a point and tangent vector at q are specified.

The decisive difference between both definitions and the corresponding Newtonian law is reduction in the amount of information needed to specify unique evolution. If one projects Newtonian histories down to shape space, the preferred Q in this case, a five-parameter family of Newtonian solutions can all pass through any given point q of Q in a given direction d at that point. They are initially tangent to each other, but then 'splay out' in a five-parameter family. In contrast, in the case of 1) there is a unique curve that passes through q in the direction d. In case 2), a one-parameter family of curves passes through q, all being tangent to each other at q. Although case 2) leads to a less predictive theory than case 1), which until recently I regarded as *the* formulation of Mach's principle, it still corresponds to more predictive power than Newtonian mechanics and has certain virtues:

• Vectors are amenable to mathematical manipulation in a way that directions are not.

• If momentum vectors as opposed to normalized momenta (directions) are allowed fundamental status, the momentum space of the system has the same number of dimensions as the configuration space. This is important in the transformation theory of quantum mechanics and might be significant in quantum gravity.

• As my collaborator Ó Murchadha and I have recently shown [32], general relativity in the case when space is closed without boundary satisfies condition 2).

#### 9 The Implementation of Mach's Principle

Even with the strong Poincaré sharpening of causality and the relativity principle that suggested the formulation of the previous section, there are many different possible theories that implement the above Mach's principles for particle models. However, nearly all of them predict an anisotropic effective mass. That this could lead to a conflict with observation was already foreseen by the first creators of such theories: Hofmann, Reissner and Schrödinger (their papers are translated in [6]). Since then, the extraordinarily accurate Hughes–Drever null experiments [33, 34] have completely ruled out such theories. In order to overcome this difficulty, Bertotti and I proposed in 1982 [30] a universal method for creating theories that implement Mach's principle based on the notion of best matching as described above; this avoids the mass-anisotropy problem, can be used whenever there is 'economic representation with redundancy', exploits the fundamental mathematics of Lie groups, and is intimately related to gauge theory.

Best matching resolves the problem of defining change in the presence of gauge redundancy in the definition of configurations but not the representation of time (the second Mach's principle). For the stronger form of Mach's principle, an external time is eliminated by defining a geodesic principle on the considered relative configuration space. In this, the line element (action) is taken equal to the square root of an expression quadratic in the velocities that is subjected to best matching as described above. Then only intrinsic differences defined without any external time contribute to the action. Since the initial condition for a geodesic consists of a point and a direction, the strong form of Mach's principle is implemented. Examples of this can be found in [30, 35], in which it is shown how Newtonian theory can be recovered either exactly or to a very good approximation for island universes subject to the Machian conditions that they have exactly vanishing values for their total angular momentum (7) and dilatational momentum (8). Best-matching principles are applied to the dynamics of geometry in superspace and conformal superspace for the stronger form of Mach's principle in [30, 36, 37, 38] and in [32] for the weaker form. The cited geometrodynamic papers present four main results. Having summarized them, I will end the paper by considering what value they might have for future research.

The first thing that the papers show is that general relativity and a completely scale-invariant theory rather similar to it can be derived from Machian first principles. Whereas application of Machian principles to particles in Euclidean space leads to global conditions [the vanishing of (7) and (8) for the complete universe], in the theories with dynamical geometry local conditions are obtained. Indeed, best matching on superspace leads to general relativity, in which the  $G^{00} = 0$  and  $G^{0i} = 0, i = 1, 2, 3$ , members of Einstein's field equations  $G^{\mu\nu} = 0, \mu, \nu = 1, 2, 3, 4$ , arise as Machian conditions. The equation  $G^{00} = 0$  is a *local* expression of the fact that no external time appears in the theory, while the equations  $G^{0i} = 0$  arise from

best matching and are local counterparts of the condition (7).

Second, the Machian approach reverses the historical discovery of special and general relativity and puts the derivation of gauge theory in an interesting perspective. One first seeks to create a Machian theory of the evolution of Riemannian 3-geometry on a closed manifold and finds that the simplest nontrivial such theory with *local* elimination of an external time is general relativity. If one then attempts to couple matter fields to the dynamical geometry, it turns out that in all the simple ways in which this can be done the matter fields must propagate with the same limiting speed as the geometrical perturbations: the matter fields are forced to share the same 'light cone' as the geometry. In this way we find a Machian explanation of the local validity of special relativity. Equally striking is the fact that the simplest theories of a single 3-vector field interacting with Machian geometry and of a set of 3-vector fields interacting with geometry and among themselves turn out to be Maxwell and Yang–Mills fields, respectively. These results, which show that all the well-established modern dynamical theories can be derived from Machian principles, are obtained in [36, 39]. It should be said that some of the uniqueness claims made in [36] relied on unjustified tacit assumptions that ruled out more complicated possibilities, as is noted in the careful study of [40, 41]. However, with the appropriate restrictions to the simplest possibilities, the claim just made is, I believe, warranted.

Third, whereas the results so far described for Machian dynamics on superspace could be seen as mainly of retrospective interest, those for conformal superspace [37, 38, 32] could be relevant to current research. This is because of the light they cast on the failure of general relativity to be conformally invariant (a topic of interest ever since Weyl identified this as a problem in 1918 [42, 43]) and because the simplest Machian theory on conformal superspace introduces a dynamically distinguished notion of simultaneity that is nevertheless compatible with classical general relativity and, hence, the local validity of special relativity. This is because the symmetry breaking in the conformal approach arises of necessity through the best-matching variation, which imposes not only a conformal constraint but also a dynamically imposed gauge fixing. Since the aim in quantum theory is to quantize only true degrees of freedom, this suggests that the gravitational degrees of freedom are precisely the two conformal degrees of freedom within a Riemannian three-geometry. If one insists on regarding relativity of simultaneity as fundamental, it is not possible to identify definite degrees of freedom in this manner. It may also be noted that theories in which spacetime symmetry is broken by the introduction of a distinguished notion of simultaneity have recently attracted much attention following the publication of Hořava's paper [44]. Whereas the symmetry breaking is simply imposed in Hořava's approach, it arises in the conformal approach of necessity through a dynamically imposed gauge fixing.

Detailed discussion of the implications of the conformal results goes beyond the scope of this paper, but I would like to draw attention to [32], which shows that although general relativity in its spacetime formulation is not conformally covariant, as Weyl noted, it can be represented as a theory on conformal superspace, where it appears to meet Weyl's objections and is an example of a theory in which a point and tangent vector on the relevant quotient space determine the evolution. Only the local shapes of space and their rate of change, expressed by a tangent vector, play a role. They appear as the true dynamical degrees of freedom of the theory; the local scale factor is emergent, appearing merely in a distinguished gauge representation of the theory. The status of the tangent vector in the initial data is also very interesting. It is defined naturally and purely geometrically on conformal superspace by analogy with the manner in which intrinsic change of shape is defined in the velocity decomposition theorem discussed in Sec. 7. It is therefore in essence a velocity vector, which implicitly presupposes a time variable (since a velocity vector is by definition a displacement in unit time). As implemented in general relativity, the implicit time is also emergent and defined by the change of the emergent scale factor.

To conclude this paper, let me say a word about the overall structure of the universe. As we have seen, Mach's principle is difficult to formulate if the universe has infinite extent. If we assume a finite (spatially closed) universe, we then find that the geometrical evolution of such a universe in accordance with Mach's principle leads to general relativity and that Einstein's field equations are a direct expression of the Machian nature of the theory. These are, however, local conditions and are perfectly compatible with a universe of infinite extent. Thus, what seems to be a great success of the theory threatens to undermine one of the key assumptions used to derive the theory in the first place. I will not pretend to have an easy response to this difficulty. As of now, the only suggestion that I can make is that quantum gravity might dictate its resolution. Moreover, there is something intuitively appealing about self-contained systems since, as Einstein said of Mach's idea, it closes "the series of causes of mechanical phenomena" [45], p. 62. As we grope for a quantum theory of the universe, a quantum Mach's principle could be a crucial normative principle.

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