Diagrams for relativistic length contraction and time dilation

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When a rod of length L_0 (at rest) moves uniformly with speed v in the x direction parallel to its axis, that rod contracts relativistically to a length

$$L = L_0\sqrt{1-(v/c)^2}, \qquad (1)$$

where c is the speed of light.[1] In this note we propose a simple diagram[2] that shows the relativistic length contraction in geometric proportion to the speeds v and c. The diagram may be considered complementary to the familiar diagrams of moving reference frames and space-time (Minkowski) diagrams used in introductory texts.[3–6]

We construct the diagram by drawing the first quadrant of a coordinate system with a distance abscissa and a speed ordinate (see Fig. 1). We scale the axes so that c units on the speed axis have the same length as L_0 units on the distance axis. Thus a quarter circle about the origin O with radius L_0 intercepts the speed axis at c. A horizontal line from speed v on the ordinate over to point P on the quarter circle then has the length L.

The proof uses Pythagoras' theorem and the fact that the quantities c and L_0 have the same length in the diagram yielding $L = vP = [(\mathrm{OP})^2 - v^2]^{1/2} = [c^2 - v^2]^{1/2} = c[1-(v/c)^2]^{1/2} = L_0[1-(v/c)^2]^{1/2}$.

Figure 1 illustrates clearly that the contracted length L is close to the rest length L_0 for small speeds v. When, on the other hand, v approaches the speed of light c, the length L shrinks dramatically.

Another relativistic phenomenon is time dilation.[1] For example, an observer measures the period T_0 of a pendulum swinging about a fulcrum that is at rest in his reference frame. However, when the fulcrum moves uniformly with speed v, the observer measures a longer period,

$$T = T_0/\sqrt{1-(v/c)^2}. \qquad (2)$$

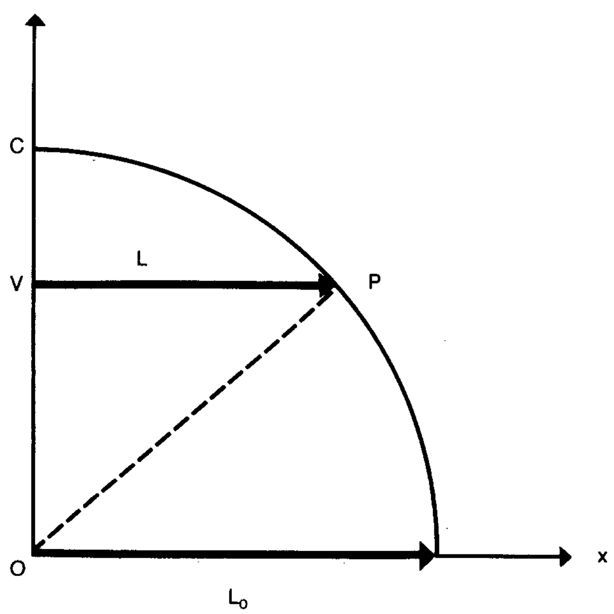

Fig. 1. Diagram for relativistic length contraction showing rest length L_0 and contracted length L of a rod moving with speed v in the x direction. The speed of light is denoted by c.

In order to illustrate relativistic time dilation with a diagram,[2] we again draw the first quadrant of a coordinate system and a quarter circle with radius c around the origin O (see Fig. 2). The ordinate represents speeds v up to the speed of light c. The abscissa measures the time elapsed between events in the reference frame of the observer. A (dotted) horizontal line from v on the speed axis over to the right intersects the circle at point P. We next draw a (dashed) vertical line to the right of the circle at a distance T_0, intersecting the abscissa at point R. A slanted straight line through O and P intersects the dashed line at M. The length OR represents then the "rest period" T_0, and OM represents the dilated period T of the traveling pendulum.

The proof employs an auxiliary point a that is the orthogonal projection of point P on the abscissa. Using similarity of the triangles OaP and ORM together with Pythagoras' theorem, we get $T/T_0 = c/a = c/[c^2-v^2]^{1/2} = [1-(v/c)^2]^{-1/2}$.

When the pendulum travels with a slow speed v, point P is slightly above the abscissa and the slowly rising slanted line OM $= T$ is only a little longer than the horizontal line OR $= T_0$. This situation corresponds to a small time dilation. When, on the other hand, the pendulum travels with v close to the speed of light c, point P is near the top of the circle. Now the slanted line is almost parallel to the dashed vertical line resulting in a very distant intersection M. The distance OM is then much longer than OR. This case illustrates a very large time dilation.

Having introduced time dilation with periods T_0 and T of pendulums swinging about resting and moving ful-

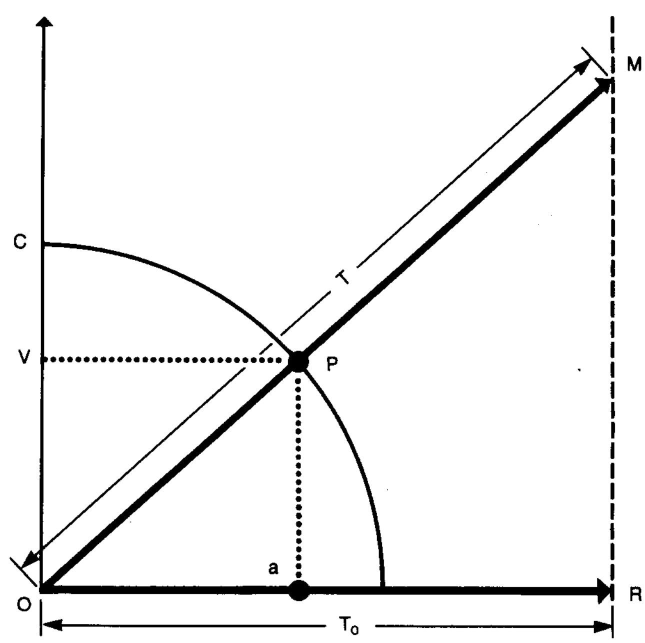

Fig. 2. Diagram for relativistic time dilation showing periods T_0 and T of a pendulum by the lengths of arrows OR and OM, respectively. The pendulum has a period T_0 when it swings about a fulcrum at rest in the observer's reference frame. When the fulcrum moves uniformly with relative speed v, a dilated period of the pendulum T is observed. The speed of light is denoted by c.

crums, we consider the flow of elapsed time, Δt_0 and Δt, between events in the observer's frame of reference and in a frame moving uniformly with relative speed v. This may be visualized via the distances OR and OM, respectively, by imagining that the dashed vertical line moves uniformly to the right starting from O at the initial event.

Since the relativistic mass,

$$m = m_0/\sqrt{1-(v/c)^2}, \quad (3)$$

has the same speed dependence as time dilation, Eq. (2), Fig. 2 may also be used for a geometric illustration of relativistic mass increase. In that case, the length OM represents m and OR the rest mass m_0.

Moreover, we can use Fig. 2 to illustrate the relativistic energy[1]

$$E = mc^2 = [(pc)^2 + (m_0c^2)^2]^{1/2} \quad (4)$$

by associating E with the length OM, the rest energy m_0c^2 with OR, and the quantity pc with RM. Here, p is the relativistic momentum.

On a slightly more abstract level than the present diagrams, the speed dependence of relativistic length contraction, time dilation, and mass, Eqs. (1)–(3), is occasionally illustrated in the literature[7] by graphs of $(1-\beta^2)^{1/2}$ and $(1-\beta^2)^{-1/2}$ vs β, respectively, where $\beta = v/c$. One of these graphs, $(1-\beta^2)^{1/2}$, is similar to Fig. 1 when coordinate axes are interchanged. The application of the diagram in Fig. 2 to Eq. (4) was inspired by a mnemonic device in Resnick's text.[8]

[1] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950), 1st ed., pp. 192–204.

[2] The purpose of this diagram is not to replace the derivations of Eqs. (1)–(3) from relativity theory, but to give the equations a geometric interpretation.

[3] R. Resnick, *Introduction to Special Relativity* (Wiley, New York, 1968).

[4] K. R. Atkins, *Physics* (Wiley, New York, 1965), pp. 425–490.

[5] H. Arzeliès, *Relativistic Kinematics* (Pergamon, Oxford, 1966).

[6] J. T. Schwartz, *Relativity in Illustrations* (New York U.P., New York, 1962).

[7] Reference 3, p. 65; Ref. 4, p. 481; Ref. 5, p. 108.

[8] Reference 3, p. 123.

On the helicity of an elliptically polarized electromagnetic wave

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This note presents a simple analytic formula for the helicity[1,2] of an elliptically polarized electromagnetic wave.

Let us consider an electromagnetic wave whose electric field is given by

$$\mathbf{E} = \hat{x}A\cos(kz - \omega t + \alpha) + \hat{y}B\cos(kz - \omega t + \beta), \quad (1)$$

where A, B, α, β, k, and ω are real constants. The angular frequency of the wave is ω and the wave vector is $\mathbf{k} = \hat{z}k$.

In general, expression (1) describes an elliptically polarized wave[1–5] such that, at a fixed point in space, the electric field rotates and its magnitude varies in time. The rotation direction of the electric field depends on the values of α and β.

A mechanical analogy is useful to show the connection between the rotation direction of $\mathbf{E}$ and the values of α and β.

Imagine a particle whose position is given by

$$\mathbf{r} = \hat{x}a\cos(\psi - \omega t) + \hat{y}b\cos(\psi - \omega t + \gamma), \quad (2)$$

where a, b, ψ, ω, and γ are real constants.

In general, such a particle would describe an elliptical trajectory with a velocity

$$\mathbf{v} = \hat{x}\omega a\sin(\psi - \omega t) + \hat{y}\omega b\sin(\psi - \omega t + \gamma). \quad (3)$$

Equations (2) and (3) yield

$$\mathbf{r}\times\mathbf{v}\cdot\hat{z} = \omega ab\sin\gamma, \quad (4)$$

which shows that the sign of $\sin\gamma$ defines the direction in which the particle describes its trajectory. For instance, if $\sin\gamma$ is positive, $\mathbf{r}\times\mathbf{v}$ and $\hat{z}$ have the same direction, and the paticle moves counterclockwise when viewed from above (see Fig. 1).

A similar result holds for the electromagnetic wave given by (1). In this case $\mathbf{r}$ and $\mathbf{v}$ are, respectively, replaced by $\mathbf{E}$ and $\partial\mathbf{E}/\partial t$.

The rotation direction of the electric field is related to the wave helicity,[1,2] which is said to be positive (negative) if, at a *fixed point in space*, the electric field rotates counterclockwise (clockwise), to an observer facing the approaching wave, i.e., looking in the direction opposite to that of wave propagation.

From (1) we get

$$\mathbf{E}\times\frac{\partial\mathbf{E}}{\partial t} = \hat{z}\omega AB\sin(\beta - \alpha), \quad (5)$$

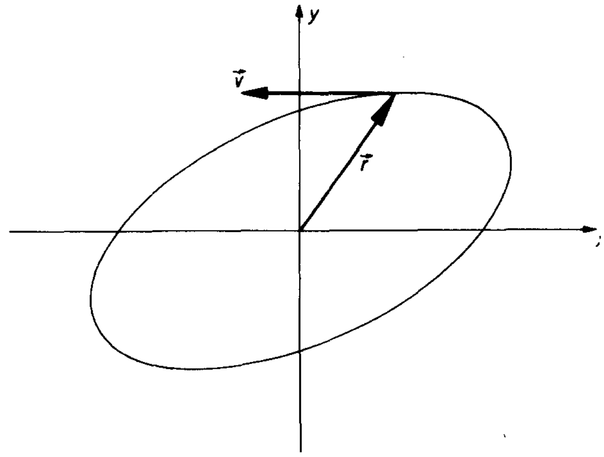

Fig. 1. Mechanical system ($\sin\gamma > 0$).