Undergraduate Lecture Notes in Topological Quantum Field Theory

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Abstract

These third–year lecture notes are designed for a 1–semester course in topological quantum field theory (TQFT). Assumed background in mathematics and physics are only standard second–year subjects: multivariable calculus, introduction to quantum mechanics and basic electromagnetism.

Keywords: quantum mechanics/field theory, path integral, Hodge decomposition, Chern–Simons and Yang–Mills gauge theories, Conformal field theory

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1 Introduction

There is a number of good textbooks in quantum field theory (QFT, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. However, they are all designed for the graduate-level study and we can only hope that undergraduate students can read some easy parts of them. Moreover, there are certainly no undergraduate-level textbooks for TQFT, so pure students are forced to *try to read* the original papers from its inventors, Ed Witten [11] and Michael Atiyah [12]. The goal of the present lecture notes is to try to fill in this gap, to give the talented undergraduates the very first glimpse of the mathematical physics of the XXI Century.

Throughout these lecture notes we will use the following *conventions*:
(i) natural units, in which (some or all of) the following definitions are used: $c = \hbar = m = 1$;

(ii) $\mathrm{i} = \sqrt{-1}$, $\dot{z} = dz/dt$, $\partial_z = \partial/\partial z$;
(iii) Einstein's summation convention over repeated indices, while nD means $n-$dimensional.

1.1 Basics of non-relativistic quantum mechanics

Recall that quantum theory was born with Max Planck's 1900 paper, in which he derived the correct shape of the black-body spectrum which now bears his name, eliminating the ultraviolet catastrophe – with the price of introducing a 'bizarre assumption' (today called *Planck's quantum hypothesis*) that energy was only emitted in certain finite chunks, or 'quanta'. In 1905, Albert Einstein took this bold idea one step further. Assuming that radiation could only transport energy in such chunks, 'photons', he was able to explain the so-called *photo-electric effect*. In 1913, Niels Bohr made a new breakthrough by postulating that the amount of angular momentum in an atom was quantized, so that the electrons were confined to a discrete set of orbits, each with a definite energy. If the electron jumped from one orbit to a lower one, the energy difference was sent off in the form of a photon. If the electron was in the innermost allowed orbit, there were no orbits with less energy to jump to, so the atom was stable. In addition, Bohr's theory successfully explained a slew of spectral lines that had been measured for Hydrogen. The famous *wave–particle duality of matter* was proposed by French prince Louis de Broglie in 1923 in his Ph.D. thesis: that electrons and other particles acted like standing waves. Such waves, like vibrations of a guitar string, can only occur with certain discrete (quantized) frequencies.[1] In November 1925, Erwin Schrödinger gave a seminar on de Broglie's work in Zurich. When he was finished, P. Debye said in effect, "You speak about waves. But where is the wave equation?" Schrödinger went on to produce and publish his famous wave equation, the master key for so much of modern physics. An equivalent formulation involving infinite matrices was provided by Werner Heisenberg, Max Born and Pasquale Jordan around the same time. With this new powerful mathematical underpinning, quantum theory made explosive progress. Within a few years, a host of hitherto unexplained measurements had been successfully explained, including spectra of more complicated atoms and various numbers describing properties of chemical reactions. For more details, see, e.g. [13].

1.1.1 Quantum states and operators

Non-relativistic quantum-mechanical systems have two modes of evolution in time [14, 15]. The first, governed by standard, *time–dependent Schrödinger equation*:

$$\mathrm{i}\hbar\,\partial_t\,|\psi\rangle = \hat{H}\,|\psi\rangle\ , \tag{1}$$

[1] The idea was so new that the examining committee went outside its circle for advice on the acceptability of the thesis. Einstein gave a favorable opinion and the thesis was accepted.

describes the time evolution of quantum systems when they are undisturbed by measurements. 'Measurements' are defined as *interactions* of the quantum system with its classical environment. As long as the system is sufficiently isolated from the environment, it follows Schrödinger equation. If an interaction with the environment takes place, i.e., a measurement is performed, the system abruptly *decoheres* i.e., collapses or reduces to one of its classically allowed states.

A *time–dependent state of a quantum system* is determined by a normalized, complex, *wave psi–function* $\psi = \psi(t)$. In Dirac's words [14], this is a unit *ket* vector $|\psi\rangle$, which is an

element of the *Hilbert space* $L^2(\psi) \equiv \mathcal{H}$, with a coordinate basis (q^i).[2] The state ket–vector $|\psi(t)\rangle$ is subject to action of the Hermitian operators, obtained by the procedure of *quantization* of classical biodynamic quantities, and whose real eigenvalues are being measured.

[2]The family of all possible states ($|\psi\rangle, |\phi\rangle$, etc.) of a quantum system confiture what is known as a *Hilbert space*. It is a *complex vector space*, which means that can perform the complex–number–weighted combinations that we considered before for quantum states. If $|\psi\rangle$ and $|\phi\rangle$ are both elements of the Hilbert space, then so also is $w|\psi\rangle + z|\phi\rangle$, for any pair of complex numbers w and z. Here, we even alow $w = z = 0$, to give the element $\mathbf{0}$ of the Hilbert space, which does not represent a possible physical state. We have the normal algebraic rules for a vector space:

$$
\begin{aligned}
&|\psi\rangle + |\phi\rangle = |\phi\rangle + |\psi\rangle,\\
&|\psi\rangle + (|\phi\rangle + |\chi\rangle) = (|\psi\rangle + |\phi\rangle) + |\chi\rangle,\\
&w(z|\psi\rangle) = (wz)|\psi\rangle,\\
&(w+z)|\psi\rangle = w|\psi\rangle + z|\psi\rangle,\\
&z(|\psi\rangle + |\phi\rangle) = z|\psi\rangle + z|\phi\rangle\\
&0|\psi\rangle = \mathbf{0}, \qquad z\mathbf{0} = \mathbf{0}.
\end{aligned}
$$

A Hilbert space can sometimes have a finite number of dimensions, as in the case of the spin states of a particle. For spin $\frac{1}{2}$, the Hilbert space is just 2D, its elements being the complex linear combinations of the two states $|\uparrow\rangle$ and $|\downarrow\rangle$. For spin $\frac{1}{2}n$, the Hilbert space is $(n+1)$D. However, sometimes the Hilbert space can have an infinite number of dimensions, as e.g., the states of position or momentum of a particle. Here, each alternative position (or momentum) that the particle might have counts as providing a separate dimension for the Hilbert space. The general state describing the quantum location (or momentum) of the particle is a complex–number superposition of all these different individual positions (or momenta), which is the wave $\psi-$function for the particle.

Another property of the Hilbert space, crucial for quantum mechanics, is the *Hermitian inner (scalar) product*, which can be applied to any pair of Hilbert–space vectors to produce a single complex number. To understand how important the Hermitian inner product is for quantum mechanics, recall that the Dirac's 'bra–ket' notation is formulated on the its basis. If we have the two quantum states (i.e., Hilbert–space vectors) are $|\psi\rangle$ and $|\phi\rangle$, then their Hermitian scalar product is denoted $\langle\psi|\phi\rangle$, and it satisfies a number of simple algebraic properties:

$$
\begin{aligned}
&\overline{\langle\psi|\phi\rangle} = \langle\phi|\psi\rangle, \qquad \text{(bar denotes complex–conjugate)}\\
&\langle\psi|(|\phi\rangle + |\chi\rangle) = \langle\psi|\phi\rangle + \langle\psi|\chi\rangle,\\
&(z\langle\psi|)|\phi\rangle = z\langle\psi|\phi\rangle,\\
&\langle\psi|\phi\rangle \geq 0, \qquad \langle\psi|\phi\rangle = 0 \quad \text{if} \quad |\psi\rangle = \mathbf{0}.
\end{aligned}
$$

For example, probability of finding a quantum particle at a given location is a *squared length* $|\psi|^2$ of a Hilbert–space position vector $|\psi\rangle$, which is the scalar product $\langle\psi|\psi\rangle$ of the vector $|\psi\rangle$ with itself. A *normalized state* is given by a Hilbert–space vector whose squared length is *unity*.

The second important thing that the Hermitian scalar product gives us is the notion of *orthogonality* between Hilbert–space vectors, which occurs when the scalar product of the two vectors is *zero*. In ordinary terms, orthogonal states are things that are independent of one another. The importance of this concept for quantum physics is that the different alternative outcomes of any measurement are always orthogonal to each other. For example, states $|\uparrow\rangle$ and $|\downarrow\rangle$ are mutually orthogonal. Also, orthogonal are *all* different possible *positions* that a quantum particle might be located in.

Quantum superposition is a generalization of the algebraic principle of linear combination of vectors. The Hilbert space has a set of states $|\varphi_i\rangle$ (where the index i runs over the degrees–of–freedom of the system) that form a basis and the most general state of such a system can be written as $|\psi\rangle = \sum_i c_i \, |\varphi_i\rangle$. The system is said to be in a state $|\psi(t)\rangle$, describing the motion of the *de Broglie waves*, which is a linear superposition of the basis states $|\varphi_i\rangle$ with weighting coefficients c_i that can in general be complex. At the microscopic or quantum level, the state of the system is described by the wave function $|\psi\rangle$, which in general appears as a linear superposition of all basis states. This can be interpreted as the system being in all these states at once. The coefficients c_i are called the *probability amplitudes* and $|c_i|^2$ gives the probability that $|\psi\rangle$ will collapse into state $|\varphi\rangle$ when it decoheres (interacts with the environment). By simple normalization we have the constraint that $\sum_i |c_i|^2 = 1$. This emphasizes the fact that the wavefunction describes a *real, physical system*, which must be in one of its allowable classical states and therefore by summing over all the possibilities, weighted by their corresponding probabilities, one must get unity. In other words, we have the *normalization condition* for the psi–function, determining the unit length of the state ket–vector

$$\langle \psi(t) | \psi(t) \rangle \; = \int \psi^* \psi \; dV = \int |\psi|^2 \, dV = 1,$$

where $\psi^* = \langle \psi(t)|$ denotes the *bra* vector, the complex–conjugate to the ket $\psi = |\psi(t)\rangle$, and $\langle \psi(t) | \psi(t) \rangle$ is their scalar product, i.e., Dirac *bracket*. For this reason the scene of quantum mechanics is the functional space of square–integrable complex psi–functions, i.e., the Hilbert space $L^2(\psi)$.

When the system is in the state $|\psi(t)\rangle$, the average value $\langle f \rangle$ of any physical observable f is equal to

$$\langle f \rangle \; = \; \langle \psi(t)| \, \hat{f} \, |\psi(t)\rangle,$$

where $\hat{f}$ is the Hermitian operator corresponding to f.

A quantum system is *coherent* if it is in a linear superposition of its basis states. If a measurement is performed on the system and this means that the system must somehow interact with its environment, the superposition is destroyed and the system is observed to be in only one basis state, as required classically. This process is called *reduction* or *collapse* of the wavefunction or simply *decoherence* and is governed by the form of the wavefunction $|\psi\rangle$.

Entanglement, on the other hand, is a purely quantum phenomenon and has no classical analogue. It accounts for the ability of quantum systems to exhibit correlations in counter-intuitive 'action–at–a–distance' ways. Entanglement is what makes all the difference in the operation of quantum computers versus classical ones. Entanglement gives 'special powers' to quantum computers because it gives quantum states the potential to exhibit and maintain correlations that cannot be accounted for classically. Correlations between bits are what make information encoding possible in classical computers. For instance, we can require two

bits to have the same value thus encoding a relationship. If we are to subsequently change the encoded information, we must change the correlated bits in tandem by explicitly accessing each bit. Since quantum bits exist as superpositions, *correlations* between them also exist in superposition. When the superposition is destroyed (e.g., one qubit is measured), the correct correlations are *instantaneously* 'communicated' between the qubits and this communication allows *many qubits* to be accessed *at once*, preserving their correlations, something that is absolutely impossible classically.

More precisely, the *first quantization* is a *linear representation* of all classical dynamical variables (like coordinate, momentum, energy, or angular momentum) by linear *Hermitian* operators acting on the associated Hilbert state–space $\mathcal{H}$, which has the following properties [14]:

1. *Linearity:* $\alpha f + \beta g \to \alpha\,\hat{f} + \beta\,\hat{g}$, (for all constants $\alpha, \beta \in \mathbb{C}$);

2. A 'dynamical' variable, equal to unity everywhere in the phase–space, corresponds to unit operator: $1 \to \hat{I}$; and

3. *Classical Poisson brackets*

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$$

   quantize to the corresponding *commutators*

$$\{f, g\} \to -\mathrm{i}\hbar[\hat{f}, \hat{g}], \qquad [\hat{f}, \hat{g}] = \hat{f}\hat{g} - \hat{g}\hat{f}.$$

Like Poisson bracket, commutator is bilinear and skew–symmetric operation, satisfying Jacobi identity. For Hermitian operators $\hat{f}, \hat{g}$ their commutator $[\hat{f}, \hat{g}]$ is anti–Hermitian; for this reason i is required in $\{f, g\} \to -\mathrm{i}\hbar[\hat{f}, \hat{g}]$.

Property (2) is introduced for the following reason. In Hamiltonian mechanics each dynamical variable f generates some transformations in the phase–space via Poisson brackets. In quantum mechanics it generates transformations in the state–space by direct application to a state, i.e.,

$$\dot{u} = \{u, f\}, \qquad \partial_t|\psi\rangle = \frac{\mathrm{i}}{\hbar}\hat{f}|\psi\rangle. \tag{2}$$

Exponent of anti–Hermitian operator is unitary. Due to this fact, transformations, generated by Hermitian operators

$$\hat{U} = \exp\frac{\mathrm{i}\hat{f}t}{\hbar},$$

are unitary. They are *motions* – scalar product preserving transformations in the Hilbert state–space $\mathcal{H}$. For this property i is needed in (2).

Due to property (2), the transformations, generated by classical variables and quantum operators, have the same algebra.

For example, the quantization of energy E gives:

$$E \to \hat{E} = \mathrm{i}\hbar\,\partial_t.$$

The relations between operators must be similar to the relations between the relevant physical quantities observed in classical mechanics.

For example, the quantization of the classical equation $E = H$, where

$$H = H(p_i, q^i) = T + U$$

denotes the Hamilton's function of the total system energy (the sum of the kinetic energy T and potential energy U), gives the Schrödinger equation of motion of the state ket–vector $|\psi(t)\rangle$ in the Hilbert state–space $\mathcal{H}$

$$\mathrm{i}\hbar\,\partial_t|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle.$$

In the simplest case of a single particle in the potential field U, the operator of the total system energy – Hamiltonian is given by:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U,$$

where m denotes the mass of the particle and ∇ is the classical gradient operator. So the first term on the r.h.s denotes the kinetic energy of the system, and therefore the momentum operator must be given by:

$$\hat{p} = -\mathrm{i}\hbar\nabla.$$

Now, for each pair of states $|\varphi\rangle, |\psi\rangle$ their scalar product $\langle\varphi|\psi\rangle$ is introduced, which is [15]:

1. Linear (for right multiplier):

$$\langle\varphi|\alpha_1\psi_1 + \alpha_2\psi_2\rangle = \alpha_1\langle\varphi|\psi_1\rangle + \alpha_2\langle\varphi|\psi_2\rangle;$$

2. In transposition transforms to complex conjugated:

$$\langle\varphi|\psi\rangle = \overline{\langle\psi|\varphi\rangle};$$

this implies that it is 'anti–linear' for left multiplier:

$$\langle\alpha_1\varphi_1 + \alpha_2\varphi_2\rangle = \bar{\alpha}_1\langle\varphi_1|\psi\rangle + \bar{\alpha}_2\langle\varphi_2|\psi\rangle);$$

3. Additionally it is often required, that the scalar product should be positively defined:

$$\text{for all } |\psi\rangle, \quad \langle\psi|\psi\rangle \geq 0 \qquad \text{and} \qquad \langle\psi|\psi\rangle = 0 \qquad \text{iff} \quad |\psi\rangle = 0.$$

Complex conjugation of classical variables is represented as Hermitian conjugation of operators.[3]

If the two Hermitian operators $\hat{f}$ and $\hat{g}$ commute, i.e., $[\hat{f}, \hat{g}] = 0$ (see Heisenberg picture below), than the corresponding quantities can simultaneously have definite values. If the two operators do not commute, i.e., $[\hat{f}, \hat{g}] \neq 0$, the quantities corresponding to these operators cannot have definite values simultaneously, i.e., the general *Heisenberg uncertainty relation* is valid:

$$(\Delta\hat{f})^2 \cdot (\Delta\hat{g})^2 \geq \frac{\hbar}{4}[\hat{f}, \hat{g}]^2,$$

where Δ denotes the deviation of an individual measurement from the mean value of the distribution. The well-known particular cases are ordinary uncertainty relations for coordinate–momentum $(q - p)$, and energy–time $(E - t)$:

$$\Delta q \cdot \Delta p_q \geq \frac{\hbar}{2}, \quad \text{and} \quad \Delta E \cdot \Delta t \geq \frac{\hbar}{2}.$$

For example, the rules of commutation, analogous to the classical ones written by the Poisson's brackets, are postulated for canonically–conjugate coordinate and momentum operators:

$$[\hat{q}^i, \hat{q}^j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}^i, \hat{p}_j] = \mathrm{i}\hbar\delta^i_j\hat{I},$$

[3]Two operators $\hat{f}, \hat{f}^+$ are called Hermitian conjugated (or adjoint), if

$$\langle\varphi|\hat{f}\psi\rangle = \langle\hat{f}^+\varphi|\psi\rangle \qquad (\text{for all } \varphi, \psi).$$

This scalar product is also denoted by $\langle\varphi|\hat{f}|\psi\rangle$ and called a matrix element of an operator.
– operator is Hermitian (self–adjoint) if $\hat{f}^+ = \hat{f}$ and anti–Hermitian if $\hat{f}^+ = -\hat{f}$;
– operator is unitary, if $\hat{U}^+ = \hat{U}^{-1}$; such operators preserve the scalar product:

$$\langle\hat{U}\varphi|\hat{U}\psi\rangle = \langle\varphi|\hat{U}^+\hat{U}|\psi\rangle = \langle\varphi|\psi\rangle.$$

Real classical variables should be represented by Hermitian operators; complex conjugated classical variables $(a, \bar{a})$ correspond to Hermitian conjugated operators $(\hat{a}, \hat{a}^+)$.

Multiplication of a state by complex numbers does not change the state physically.

Any Hermitian operator in Hilbert space has only real eigenvalues:

$$\hat{f}|\psi_i\rangle = f_i|\psi_i\rangle, \qquad (\text{for all } \ f_i \in \mathbb{R}).$$

Eigenvectors $|\psi_i\rangle$ form complete orthonormal basis (eigenvectors with different eigenvalues are automatically orthogonal; in the case of multiple eigenvalues one can form orthogonal combinations; then they can be normalized).

where δ^i_j is the Kronecker's symbol. By applying the commutation rules to the system Hamiltonian $\hat{H} = \hat{H}(\hat{p}_i, \hat{q}^i)$, the *quantum Hamilton's equations* are obtained:

$$\dot{\hat{p}}_i = -\partial_{\hat{q}^i}\hat{H} \qquad \text{and} \qquad \dot{\hat{q}}^i = \partial_{\hat{p}_i}\hat{H}.$$

A quantum state can be observed either in the *coordinate* $q-$*representation*, or in the *momentum* $p-$*representation*. In the $q-$representation, operators of coordinate and momentum have respective forms: $\hat{q} = q$, and $\hat{p}_q = -\mathrm{i}\hbar\frac{\partial}{\partial q}$, while in the $p-$representation, they have respective forms: $\hat{q} = \mathrm{i}\hbar\frac{\partial}{\partial p_q}$, and $\hat{p}_q = p_q$. The forms of the state vector $|\psi(t)\rangle$ in these two representations are mathematically related by a *Fourier–transform pair*.

1.1.2 Three quantum pictures

In the $q-$representation, there are three main pictures (see e.g., [15]):

1. *Schrödinger picture*,
2. *Heisenberg picture*, and
3. *Dirac interaction picture*.

These three pictures mutually differ in the time–dependence, i.e., time–evolution of the state–vector wavefunction $|\psi(t)\rangle$ and the Hilbert coordinate basis (q^i) together with the system operators.

1. In the *Schrödinger* (S) *picture*, under the action of the *evolution operator* $\hat{S}(t)$ the state–vector $|\psi(t)\rangle$ rotates:

$$|\psi(t)\rangle = \hat{S}(t)\,|\psi(0)\rangle,$$

and the coordinate basis (q^i) is fixed, so the operators are constant in time:

$$\hat{F}(t) = \hat{F}(0) = \hat{F},$$

and the system evolution is determined by the Schrödinger wave equation:

$$\mathrm{i}\hbar\,\partial_t|\psi^S(t)\rangle = \hat{H}^S\,|\psi^S(t)\rangle.$$

If the Hamiltonian does not explicitly depend on time, $\hat{H}(t) = \hat{H}$, which is the case with the absence of variables of macroscopic fields, the state vector $|\psi(t)\rangle$ can be presented in the form:

$$|\psi(t)\rangle = \exp\left(-\mathrm{i}\frac{E}{\hbar}t\right)|\psi\rangle,$$

satisfying the time–independent Schrödinger equation

$$\hat{H}\,|\psi\rangle \;= E\,|\psi\rangle,$$

which gives the eigenvalues E_m and eigenfunctions $|\psi_m\rangle$ of the Hamiltonian $\hat{H}$.

2. In the *Heisenberg* (H) *picture*, under the action of the evolution operator $\hat{S}(t)$, the coordinate basis (q^i) rotates, so the operators of physical variables evolve in time by the similarity transformation:

$$\hat{F}(t) = \hat{S}^{-1}(t)\,\hat{F}(0)\,\hat{S}(t),$$

while the state vector $|\psi(t)\rangle$ is constant in time:

$$|\psi(t)\rangle \;= |\psi(0)\rangle \;= |\psi\rangle,$$

and the system evolution is determined by the *Heisenberg equation of motion*:

$$\mathrm{i}\hbar\,\partial_t \hat{F}^H(t) = [\hat{F}^H(t), \hat{H}^H(t)],$$

where $\hat{F}(t)$ denotes arbitrary Hermitian operator of the system, while the commutator, i.e., Poisson quantum bracket, is given by:

$$[\hat{F}(t), \hat{H}(t)] = \hat{F}(t)\,\hat{H}(t) - \hat{H}(t)\,\hat{F}(t) = \hat{\imath}K.$$

In both Schrödinger and Heisenberg picture the evolution operator $\hat{S}(t)$ itself is determined by the Schrödinger–like equation:

$$\mathrm{i}\hbar\,\partial_t \hat{S}(t) = \hat{H}\,\hat{S}(t),$$

with the initial condition $\hat{S}(0) = \hat{I}$. It determines the Lie group of transformations of the Hilbert space $L^2(\psi)$ in itself, the Hamiltonian of the system being the generator of the group.

3. In the *Dirac interaction* (I) *picture* both the state vector $|\psi(t)\rangle$ and coordinate basis (q^i) rotate; therefore the system evolution is determined by both the Schrödinger wave equation and the Heisenberg equation of motion:

$$\mathrm{i}\hbar\,\partial_t|\psi^I(t)\rangle = \hat{H}^I\,|\psi^I(t)\rangle, \quad \text{and} \quad \mathrm{i}\hbar\,\partial_t \hat{F}^I(t) = [\hat{F}^I(t), \hat{H}^O(t)].$$

Here, $\hat{H} = \hat{H}^0 + \hat{H}^I$, where $\hat{H}^0$ corresponds to the Hamiltonian of the free fields and $\hat{H}^I$ corresponds to the Hamiltonian of the interaction.

In particular, the stationary (time-independent) Schrödinger equation,

$$\hat{H}\,\psi = \hat{E}\,\psi,$$

can be obtained from the condition for the minimum of the *quantum action*:

$$\delta S[\psi] = 0.$$

The quantum action is usually defined by the integral:

$$S[\psi] = \langle\psi(t)|\,\hat{H}\,|\psi(t)\rangle = \int \psi^* \hat{H}\psi \; dV,$$

with the additional normalization condition for the unit–probability of the psi–function:

$$\langle\psi(t)|\psi(t)\rangle = \int \psi^*\psi \; dV = 1.$$

When the functions ψ and ψ^* are considered to be formally independent and only one of them, say ψ^* is varied, we can write the condition for an extreme of the action:

$$\delta S[\psi] = \int \delta\psi^* \hat{H}\psi \; dV - E \int \delta\psi^*\psi \; dV = \int \delta\psi^* (\hat{H}\psi - E\psi)\, dV = 0,$$

where E is a Lagrangian multiplier. Owing to the arbitrariness of $\delta\psi^*$, the Schrödinger equation $\hat{H}\psi - \hat{E}\psi = 0$ must hold.

1.1.3 Dirac's probability amplitude and time–dependent perturbation

Most quantum–mechanical problems cannot be solved exactly. For such problems we can use *Dirac's perturbation method*, which consists in splitting up the time–dependent Hamiltonian $H = H(t)$ into two parts:

$$H(t) = H_0 + \epsilon H_1(t),$$

in which $H_0 = E$ must be simple, non-autonomous, energy function that can be dealt with exactly, while $\epsilon H_1(t) = V(t)$ is small time–dependent perturbation, which can be expanded as a power series in a small numerical factor ϵ. The first part, H_0, may then be considered as the Hamiltonian of a simplified, or unperturbed system that can be exactly solved, while the addition of the second part $\epsilon H_1(t)$ will require small corrections, of the nature of a power–series expanded perturbation in the solution for the unperturbed system. Provided the perturbation series in ϵ converges, the perturbation method will give the answer to our problem with any desired accuracy; even when the series does not converge, the first approximation obtained by means of it is usually fairly accurate [14].

Therefore, we do not consider any modification to be made in the states of the unperturbed system $E = H_0$, but we suppose that the perturbed system $H(t)$, instead of remaining permanently in one of its states, is continually changing from one state to another (or, making transmissions), under the influence of the perturbation $V(t) = \epsilon H_1(t)$.

We will work in the *Heisenberg's representation* for the unperturbed system E, assuming that we have a general set of linear Hermitian operators α's to label the representatives. Let us suppose that at initial time t_0 the system is in a state for which the α's certainly have the values α', so that the basic ket $|\alpha'\rangle$ would correspond to this state. This state would be

stationary if there were no perturbation, i.e., if $H(t) = E$. The perturbation $V(t)$ cause the E to change. At time t the ket corresponding to the state $|\alpha'\rangle$ in the *Schrödinger's picture* will be $T\,|\alpha'\rangle$, according to equation

$$\begin{aligned} |At\rangle &= T\,|At_0\rangle\,, \qquad && \text{as well as} \\ \mathrm{i}\frac{dT}{dt} &= H(t)T \qquad && \text{and} \qquad \mathrm{i}\frac{d\,|At\rangle}{dt} = H(t)\,|At\rangle\,, \end{aligned} \tag{3}$$

where T is a linear Hermitian operator independent of the ket $|At\rangle$ and depending only on time (t_0 and t). The probability of the α's having the values α'' is given by the absolute square of the *probability amplitude* (or, *transition amplitude*) $\langle\alpha''|\,T\,|\alpha'\rangle$ (for the system's transition from the state $|\alpha'\rangle$ to the state $|\alpha''\rangle$),

$$P(\alpha', \alpha'') = |\,\langle\alpha''|\,T\,|\alpha'\rangle\,|^2. \tag{4}$$

For $\alpha' \neq \alpha''$, $P(\alpha', \alpha'')$ is the probability of a transition taking place from state α' to state α'' during the time interval $[t_0, t]$; $P(\alpha', \alpha')$ is the probability of no transition taking place at all, while the sum of $P(\alpha', \alpha'')$ for all α'' is unity.

Let us now suppose more generally that initially the system is in one of the various states α' with the probability $P_{\alpha'}$ for each. To deal effectively with this problem, we introduce the *von Neumann's quantum density function* ρ, a quantum–mechanical analogue to the *Gibbs statistical density function* $\rho = \rho(t)$ of a *Gibbs ensemble* with the classical Hamiltonian $H(q, p)$, which evolves within the ensemble's $n-$dimensional phase–space $\mathcal{P} = \{(q^i, p_i)\,|\,i = 1, ..., n\}$ according to the *Poisson equation*

$$\begin{aligned} \partial_t \rho \quad &= \quad -[\rho, H(q,p)], \\ \text{with the normalizing condition} \quad &: \quad \iint_{\mathcal{P}} \rho\, dq^i dp_i = 1. \end{aligned}$$

The von Neumann's quantum density function ρ corresponding to the initial probability distribution $P_{\alpha'}$ is given by

$$\rho_0 = \sum_{\alpha'} |\alpha'\rangle\, P_{\alpha'}\, \langle\alpha'|\,.$$

At time t, each ket $|\alpha'\rangle$ will have changed to $T\,|\alpha'\rangle$ and each bra $\langle\alpha'|$ will change to $\langle\alpha'|\,\bar{T}$ (where $\bar{T}$ is complex–conjugate to T), so ρ_0 will have changed to

$$\rho(t) = \sum_{\alpha'} T\,|\alpha'\rangle\, P_{\alpha'}\, \langle\alpha'|\,\bar{T}.$$

The *probability amplitude* of α's then having the values of α'' will be (using (4))

$$\langle\alpha''|\,\rho(t)\,|\alpha'\rangle = \sum_{\alpha'} \langle\alpha''|\,T\,|\alpha'\rangle\, P_{\alpha'}\, \langle\alpha'|\,\bar{T}\,|\alpha''\rangle = P_{\alpha'} P(\alpha', \alpha'').$$

This result expresses that the probability of the system being in the state α'' at time t equals the sum of the probabilities of the system being initially in any state $\alpha' \neq \alpha''$, and making a transition from state α' to the final state α''. Thus, the various transition probabilities act independently of one another, according to the ordinary laws of probability.

The whole problem of calculating transitions thus reduces to the determination of the probability amplitudes $\langle \alpha'' | \, T \, | \alpha' \rangle$ [14]. These can be worked out from (3), or

$$\mathrm{i}\dot{T} = [H_0 + \epsilon H_1(t)]T = (E + V)T \qquad (\text{where} \quad \dot{T} = dT/dt). \tag{5}$$

This calculation can be simplified if instead of T and V operators, we are working with

$$T^* = \exp[\mathrm{i}E(t - t_0)]T \qquad \text{and} \qquad V^* = \exp[\mathrm{i}E(t - t_0)]V \exp[-\mathrm{i}E(t - t_0)], \tag{6}$$

where V^* is the result of applying a *unitary transformation* to V. Using (6) we obtain

$$\mathrm{i}\dot{T}^* = \exp[\mathrm{i}E(t - t_0)]VT = V^*T^*, \tag{7}$$

which is more convenient then (5) as it makes the time evolution of T^*depend only on the (unitary transformed) perturbation V^* and not on the unperturbed state E. From (6) we get the probability amplitude

$$\begin{aligned} \left\langle \alpha'' \right| T^* \left| \alpha' \right\rangle &= \exp[\mathrm{i}E(t - t_0)] \left\langle \alpha'' \right| T \left| \alpha' \right\rangle, \qquad \text{so that} \\ P(\alpha', \alpha'') &= |\left\langle \alpha'' \right| T^* \left| \alpha' \right\rangle|^2, \end{aligned}$$

which shows that T and T^* are equally good for determining transition probabilities.

So far, our work in this subsection has been exact. Now we assume the perturbation $V(t) = \epsilon H_1(t)$ is a small quantity of the first order in ϵ and express T^* in the form

$$T^* = 1 + T_1^* + T_2^* + ..., \tag{8}$$

where $T_1^* = T_1^*(\epsilon)$, $T_2^* = T_2^*(\epsilon^2)$, etc. Substituting (8) into (7) we get the expansion

$$\begin{aligned} \mathrm{i}\dot{T}_1^* &= V^*, \\ \mathrm{i}\dot{T}_2^* &= V^*T_1^*, \\ \mathrm{i}\dot{T}_3^* &= V^*T_2^*, \\ &\ldots \end{aligned}$$

From the first of these equations we obtain

$$T_1^* = -\mathrm{i}\int_{t_0}^{t} V^*(t')\, dt',$$

from the second we get

$$T_2^* = -\int_{t_0}^{t} V^*(t')\,dt' \int_{t_0}^{t} V^*(t'')\,dt'',$$

and so on.

Now, the perturbation form of the transition probability $P(\alpha', \alpha'') = |\langle \alpha''| T^* |\alpha'\rangle|^2$ is, if we retain only the first–order term T_1^* (which is sufficiently accurate for many practical problems), given by

$$P(\alpha', \alpha'') = \left| \int_{t_0}^{t} \left\langle \alpha'' \right| V^*(t') \left| \alpha' \right\rangle \, dt' \right|^2 .$$

If we retain the first two terms, T_1^* and T_2^*, the transition probability $P(\alpha', \alpha'')$ is given by

$$\begin{aligned} P(\alpha', \alpha'') &= \left| \int_{t_0}^{t} \left\langle \alpha'' \right| V^*(t') \left| \alpha' \right\rangle \, dt' \right. \\ &- \left. \mathrm{i} \sum_{\alpha''' \neq \alpha'', \alpha'} \int_{t_0}^{t} \left\langle \alpha'' \right| V^*(t') \left| \alpha''' \right\rangle \, dt' \int_{t_0}^{t} \left\langle \alpha''' \right| V^*(t'') \left| \alpha' \right\rangle \, dt'' \right|^2 , \end{aligned}$$

where α''' is the so–called intermediate state (between α' and α''). This shows the perturbative calculation of the transition probability $P(\alpha', \alpha'')$: we fist calculate the perturbative expansion of the transition amplitude $\langle \alpha''| T^* |\alpha'\rangle$, and then take its absolute square to obtain the overall transition probability. For more technical details, see [14].

Both Dirac's concepts introduced in this subsection, namely transition amplitude and time–dependent perturbation, will prove essential later in the development of the *Feynman path integral*, as well as the *Feynman diagrams* approach to *quantum field theory* (QFT).

1.1.4 State–space for n non-relativistic quantum particles

Classical state–space for the system of n particles is its $6N$D phase–space $\mathcal{P}$, including all position and momentum vectors, $\mathbf{r}_i = (x, y, z)_i$ and $\mathbf{p}_i = (p_x, p_y, p_z)_i$ respectively (for $i = 1, ..., n$). The *quantization* is performed as a *linear representation* of the real Lie algebra $\mathcal{L}_P$ of the phase–space $\mathcal{P}$, defined by the Poisson bracket $\{A, B\}$ of classical variables A, B – into the corresponding real Lie algebra $\mathcal{L}_H$ of the Hilbert space $\mathcal{H}$, defined by the commutator $[\hat{A}, \hat{B}]$ of skew–Hermitian operators $\hat{A}, \hat{B}$ [15].

We start with the *Hilbert space* $\mathcal{H}_x$ for a single 1D quantum particle, which is composed of all vectors $|\psi_x\rangle$ of the form

$$|\psi_x\rangle = \int_{-\infty}^{+\infty} \psi(x)\, |x\rangle \, dx,$$

where $\psi(x) = \langle x|\psi\rangle$ are square integrable Fourier coefficients,

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx < +\infty.$$

The position and momentum Hermitian operators, $\hat{x}$ and $\hat{p}$, respectively, act on the vectors $|\psi_x\rangle \in \mathcal{H}_x$ in the following way:

$$\begin{aligned} \hat{x}|\psi_x\rangle &= \int_{-\infty}^{+\infty} \hat{x}\,\psi(x)\,|x\rangle\,dx, \qquad \int_{-\infty}^{+\infty} |x\,\psi(x)|^{\,2}\,dx < +\infty, \\ \hat{p}|\psi_x\rangle &= \int_{-\infty}^{+\infty} -\mathrm{i}\hbar\partial_{\hat{x}}\psi(x)\,|x\rangle\,dx, \qquad \int_{-\infty}^{+\infty} |-\mathrm{i}\hbar\partial_x\psi(x)|^2\,dx < +\infty. \end{aligned}$$

The *orbit Hilbert space* $\mathcal{H}_1^o$ for a single 3D quantum particle with the full set of compatible observable $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z}), \hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$, is defined as

$$\mathcal{H}_1^o = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z,$$

where $\hat{\mathbf{r}}$ has the common generalized eigenvectors of the form

$$|\hat{\mathbf{r}}\rangle = |x\rangle \times |y\rangle \times |z\rangle\,.$$

$\mathcal{H}_1^o$ is composed of all vectors $|\psi_r\rangle$ of the form

$$|\psi_r\rangle = \int_{\mathcal{H}^o} \psi(\mathbf{r})\,|\mathbf{r}\rangle\,d\mathbf{r} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \psi(x,y,z)\,|x\rangle\times|y\rangle\times|z\rangle\,dxdydz,$$

where $\psi(\mathbf{r}) = \langle \mathbf{r}|\psi_r\rangle$ are square integrable Fourier coefficients,

$$\int_{-\infty}^{+\infty} |\psi(\mathbf{r})|^{\,2}\,d\mathbf{r} < +\infty.$$

The position and momentum operators, $\hat{\mathbf{r}}$ and $\hat{\mathbf{p}}$, respectively, act on the vectors $|\psi_r\rangle \in \mathcal{H}_1^o$ in the following way:

$$\begin{aligned} \hat{\mathbf{r}}|\psi_r\rangle &= \int_{\mathcal{H}_1^o} \hat{\mathbf{r}}\,\psi(\mathbf{r})\,|\mathbf{r}\rangle\,d\mathbf{r}, \qquad \int_{\mathcal{H}_1^o} |\mathbf{r}\,\psi(\mathbf{r})|^{\,2}\,d\mathbf{r} < +\infty, \\ \hat{\mathbf{p}}|\psi_r\rangle &= \int_{\mathcal{H}_1^o} -\mathrm{i}\hbar\partial_{\hat{\mathbf{r}}}\psi(\mathbf{r})\,|\mathbf{r}\rangle\,d\mathbf{r}, \qquad \int_{\mathcal{H}_1^o} |-\mathrm{i}\hbar\partial_{\mathbf{r}}\psi(\mathbf{r})|^2\,d\mathbf{r} < +\infty. \end{aligned}$$

Now, if we have a system of n 3D particles, let $\mathcal{H}_i^o$ denote the orbit Hilbert space of the ith particle. Then the composite orbit state–space $\mathcal{H}_n^o$ of the whole system is defined as a direct product

$$\mathcal{H}_n^o = \mathcal{H}_1^o \otimes \mathcal{H}_2^o \otimes ... \otimes \mathcal{H}_n^o.$$

$\mathcal{H}_n^o$ is composed of all vectors

$$|\psi_r^n\rangle = \int_{\mathcal{H}_n^o} \psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n)\,|\mathbf{r}_1\rangle\times|\mathbf{r}_2\rangle\times...\times|\mathbf{r}_n\rangle\,d\mathbf{r}_1 d\mathbf{r}_2 ... d\mathbf{r}_n$$

where $\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)=\langle\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n|\psi_r^n\rangle$ are square integrable Fourier coefficients

$$\int_{\mathcal{H}_n^o}\left|\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\right|^2\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n<+\infty.$$

The position and momentum operators $\hat{\mathbf{r}}_i$ and $\hat{\mathbf{p}}_i$ act on the vectors $|\psi_r^n\rangle\in\mathcal{H}_n^o$ in the following way:

$$\begin{aligned}\hat{\mathbf{r}}_i|\psi_r^n\rangle &= \int_{\mathcal{H}_n^o}\left\{\hat{\mathbf{r}}_i\right\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\,|\mathbf{r}_1\rangle\times|\mathbf{r}_2\rangle\times...\times|\mathbf{r}_n\rangle\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n,\\ \hat{\mathbf{p}}_i|\psi_r^n\rangle &= \int_{\mathcal{H}_n^o}\left\{-\mathrm{i}\hbar\partial_{\hat{\mathbf{r}}_i}\right\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\,|\mathbf{r}_1\rangle\times|\mathbf{r}_2\rangle\times...\times|\mathbf{r}_n\rangle\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n,\end{aligned}$$

with the square integrable Fourier coefficients

$$\begin{aligned}\int_{\mathcal{H}_n^o}\left|\left\{\hat{\mathbf{r}}_i\right\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\right|^2\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n &< +\infty,\\ \int_{\mathcal{H}_n^o}\left|\left\{-\mathrm{i}\hbar\partial_{\mathbf{r}_i}\right\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\right|^2\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n &< +\infty,\end{aligned}$$

respectively. In general, any set of vector Hermitian operators $\{\hat{\mathbf{A}}_i\}$ corresponding to all the particles, act on the vectors $|\psi_r^n\rangle\in\mathcal{H}_n^o$ in the following way:

$$\hat{\mathbf{A}}_i|\psi_r^n\rangle=\int_{\mathcal{H}_n^o}\{\hat{\mathbf{A}}_i\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\,|\mathbf{r}_1\rangle\times|\mathbf{r}_2\rangle\times...\times|\mathbf{r}_n\rangle\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n,$$

with the square integrable Fourier coefficients

$$\int_{\mathcal{H}_n^o}\left|\left\{\hat{\mathbf{A}}_i\right\}\psi\left(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_n\right)\right|^2\,d\mathbf{r}_1d\mathbf{r}_2...d\mathbf{r}_n<+\infty.$$

1.2 Transition to quantum fields

1.2.1 Amplitude, relativistic invariance and causality

We will see later that in QFT, the fundamental quantity is not any more Schrödinger's *wavefunction* but the rather the closely related, yet different, Dirac–Feynman's *amplitude*. To introduce the amplitude concept within the non-relativistic quantum mechanics, suppose that $\mathbf{x}=(x,y,z)$ and consider the amplitude for a free particle to propagate in time t from $\mathbf{x}_0$ to $\mathbf{x}=\mathbf{x}(t)$, which is given by

$$U(t)=\left\langle\mathbf{x}\right|\mathrm{e}^{-\mathrm{i}Ht}\left|\mathbf{x}_0\right\rangle.$$

As the kinetic energy of a free particle is $E = \mathbf{p}^2/2m$, we have

$$\begin{aligned} U(t) &= \left\langle \mathbf{x}\right| \mathrm{e}^{-\mathrm{i}(\mathbf{p}^2/2m)t} \left|\mathbf{x}_0\right\rangle = \int d^3p \left\langle \mathbf{x}\right| \mathrm{e}^{-\mathrm{i}(\mathbf{p}^2/2m)t} \left|\mathbf{p}\right\rangle \left\langle \mathbf{p}|\mathbf{x}_0\right\rangle \\ &= \frac{1}{(2\pi)^2} \int d^3p\, \mathrm{e}^{-\mathrm{i}(\mathbf{p}^2/2m)t} \cdot \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} = \left(\frac{m}{2\pi \mathrm{i}t}\right)^{3/2} \mathrm{e}^{\mathrm{i}m(\mathbf{x}-\mathbf{x}_0)^2/2t}. \end{aligned}$$

Later we will deal with the amplitude in the relativistic framework.

As we have seen in the previous section, in non-relativistic quantum mechanics observables are represented by self-adjoint operators that in the Heisenberg picture depend on time, so that for any *quantum observable* $\mathcal{O}$, the Heisenberg equation of motion holds (in normal units):

$$\mathrm{i}\partial_t \mathcal{O} = [\mathcal{O}, H]. \tag{9}$$

Therefore measurements are localized in time but are global in space. The situation is radically different in the relativistic case. Because no signal can propagate faster than the speed of light, measurements have to be localized both in time and space. Causality demands then that two measurements carried out in causally-disconnected regions of space–time cannot interfere with each other. In mathematical terms this means that if $\mathcal{O}_{R_1}$ and $\mathcal{O}_{R_2}$ are the observables associated with two measurements localized in two causally-disconnected regions R_1, R_2, they satisfy the *commutator* relation [4]

$$[\mathcal{O}_{R_1}, \mathcal{O}_{R_2}] = 0, \qquad \text{if } (x_1 - x_2)^2 < 0, \text{ for all } x_1 \in R_1,\, x_2 \in R_2. \tag{10}$$

Hence, in a relativistic theory, the basic operators in the Heisenberg picture must depend on the space-time position x^μ. Unlike the case in non-relativistic quantum mechanics, here the position x *is not* an observable, but just a label, similarly to the case of time in ordinary quantum mechanics. Causality is then imposed microscopically by requiring

$$[\mathcal{O}(x), \mathcal{O}(y)] = 0, \qquad \text{if } (x - y)^2 < 0. \tag{11}$$

A smeared operator $\mathcal{O}_R$ over a space-time region R can then be defined as

$$\mathcal{O}_R = \int d^4x\, \mathcal{O}(x)\, f_R(x),$$

where $f_R(x)$ is the *characteristic function* associated with R,

$$f_R(x) = \begin{cases} 1, & \text{for} \quad x \in R, \\ 0, & \text{for} \quad x \notin R. \end{cases}$$

Equation (10) follows now from the micro–causality condition (11).

Therefore, relativistic invariance forces the introduction of quantum fields. It is only when we insist in keeping a single-particle interpretation that we crash against causality violations.

To illustrate the point, let us consider a single particle wave function $\psi(t,x)$ that initially is localized in the position $x=0$

$$\psi(0,x)=\delta(x).$$

Evolving this wave function using the Hamiltonian $H=\sqrt{-\nabla^2+m^2}$ we find that the wave function can be written as

$$\psi(t,x)=\mathrm{e}^{-\mathrm{i}t\sqrt{-\nabla^2+m^2}}\delta(x)=\int\frac{d^3p}{(2\pi)^3}\mathrm{e}^{\mathrm{i}p\cdot x-\mathrm{i}t\sqrt{p^2+m^2}}.$$

Integrating over the angular variables, the wave function can be recast in the form

$$\psi(t,x)=\frac{1}{2\pi^2|x|}\int_{-\infty}^{\infty}p\,dk\,\mathrm{e}^{\mathrm{i}p|x|}\,\mathrm{e}^{-\mathrm{i}t\sqrt{p^2+m^2}}.$$

The resulting integral can be evaluated using the complex integration contour C. The result is that, for any $t>0$, one finds that $\psi(t,x)\neq 0$ for any x. If we insist in interpreting the wave function $\psi(t,x)$ as the probability density of finding the particle at the location x in the time t we find that the probability leaks out of the light cone, thus violating causality.

In the Heisenberg picture, the amplitude for a particle to propagate from point y to point x in the *field* ϕ is defined as

$$D(x-y)=\langle 0|\phi(x)\phi(y)|0\rangle=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E}\,\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}.$$

In this picture we can make time-dependent the *field operator* $\phi=\phi(x)$ and its cannonically–conjugate *momentum operator* $\pi=\pi(x)$, as

$$\phi(x)=\phi(\mathbf{x},t)=\,\mathrm{e}^{\mathrm{i}Ht}\phi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}Ht},\qquad\pi(x)=\pi(\mathbf{x},t)=\,\mathrm{e}^{\mathrm{i}Ht}\pi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}Ht}.$$

Using (9) we can compute the time dependence of ϕ and π as

$$\mathrm{i}\dot{\phi}(\mathbf{x},t)=\mathrm{i}\pi(\mathbf{x},t),\qquad\mathrm{i}\dot{\pi}(\mathbf{x},t)=-\mathrm{i}(-\nabla^2+m^2)\phi(\mathbf{x},t).$$

Combining the two results we get the *Klein–Gordon equation*

$$\ddot{\phi}=(\nabla^2-m^2)\phi.$$

1.2.2 Gauge theories

Recall that a *gauge theory* is a theory that admits a symmetry with a local parameter. For example, in every quantum theory the global phase of the wave $\psi-$function is arbitrary and does not represent something physical. Consequently, the theory is invariant under a global change of phases (adding a constant to the phase of all wave functions, everywhere); this is

a global symmetry. In quantum electrodynamics, the theory is also invariant under a local change of phase, that is, one may shift the phase of all wave functions so that the shift may be different at every point in space-time. This is a local symmetry. However, in order for a well–defined derivative operator to exist, one must introduce a new field, the *gauge field*, which also transforms in order for the local change of variables (the phase in our example) not to affect the derivative. In quantum electrodynamics this gauge field is the electromagnetic potential 1–form A (see Appendix), in components within the nD coframe $\{dx^\mu\}$ on a smooth manifold M (dual to the frame, i.e., basis of tangent vectors $\{\partial_\mu = \partial/\partial x^\mu\}$, given by

$$A = A_\mu dx^\mu, \quad \text{such that} \quad A_{\text{new}} = A_{\text{old}} + df, \qquad (f \text{ is any scalar function})$$

– leaves the electromagnetic field 2–form $F = dA$ unchanged. This change df of local gauge of variable A is termed *gauge transformation*. In quantum field theory the excitations of fields represent particles. The particle associated with excitations of the gauge field is the *gauge boson*. All the fundamental interactions in nature are described by gauge theories. In particular, in quantum electrodynamics, whose gauge transformation is a local change of phase, the gauge group is the circle group $U(1)$ (consisting of all complex numbers with absolute value 1), and the gauge boson is the photon (see e.g., [16]).

The *gauge field* of classical electrodynamics, given in local coframe $\{dx^\mu\}$ on M as an electromagnetic potential 1–form

$$A = A_\mu dx^\mu = A_\mu dx^\mu + df, \qquad (f = \text{ arbitrary scalar field}),$$

is globally a *connection* on a $U(1)-$bundle of M.[4] The corresponding electromagnetic field,

[4]Recall that in the 19th Century, Maxwell unified Faraday's electric and magnetic fields. Maxwell's theory led to Einstein's special relativity where this unification becomes a spin-off of the unification of space end time in the form of the *Faraday tensor* [17]

$$F = E \wedge dt + B,$$

where F is electromagnetic $2-$form on space-time, E is electric $1-$form on space, and B is magnetic $2-$form on space. Gauge theory considers F as secondary object to a connection–potential $1-$form A. This makes half of the Maxwell equations into tautologies [18], i.e.,

$$F = dA \quad \Longrightarrow \quad dF = 0 \quad : \quad \text{Bianchi identity},$$

but does not imply the second half of Maxwell's equations,

$$\delta F = -4\pi J \quad : \quad \text{dual Bianchi identity}.$$

To understand the deeper meaning of the connection–potential $1-$form A, we can integrate it along a path γ in space-time, $x \xrightarrow{\gamma} y$. Classically, the integral $\int_\gamma A$ represents an *action* for a charged point particle to move along the path γ. Quantum–mechanically, $\exp\left(\mathrm{i}\int_\gamma A\right)$ represents a *phase* (within the unitary Lie group $U(1)$) by which the particle's wave–function changes as it moves along the path γ, so A is a $U(1)-$connection.

In other words, Maxwell's equations can be formulated using complex line bundles, or principal bundles with fibre $U(1)$. The connection ∇ on the line bundle has a curvature $F = \nabla^2$ which is a 2–form that automatically

locally the 2–form on M,

$$\begin{aligned} F &= dA, \qquad \text{in components given by} \\ F &= \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu, \qquad \text{with} \quad F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu \end{aligned}$$

is globally the *curvature* of the connection A[5] under the gauge–covariant derivative,

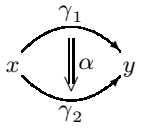
$$D_\mu = \partial_\mu - \mathrm{i}eA_\mu, \tag{12}$$

where e is the charge coupling constant.[6] In particular, in 4D space-time electrodynamics, the 1–form *electric current density* J has the components $J_\mu = (\rho, \mathbf{j}) = (\rho, j_x, j_y, j_z)$ (where ρ is the charge density), the 2–form *Faraday* F is given in components of electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \text{with} \qquad F_{\nu\mu} = -F_{\mu\nu},$$

satisfies $dF = 0$ and can be interpreted as a field–strength. If the line bundle is trivial with flat reference connection d, we can write $\nabla = d + A$ and $F = dA$ with A the 1–form composed of the electric potential and the magnetic vector potential.

[5]The only thing that matters here is the *difference* α *between two paths* γ_1 and γ_2 *in the action* $\int_\gamma A$ [18], which is a 2–morphism (see e.g., [19, 20])



[6]If a gauge transformation is given by

$$\psi \mapsto \mathrm{e}^{i\Lambda}\psi$$

and for the gauge potential

$$A_\mu \mapsto A_\mu + \frac{1}{e}(\partial_\mu \Lambda),$$

then the gauge–covariant derivative,

$$D_\mu = \partial_\mu - ieA_\mu$$

transforms as

$$D_\mu \mapsto \partial_\mu - \mathrm{i}eA_\mu - i(\partial_\mu \Lambda)$$

and $D_\mu \psi$ transforms as

$$D_\mu \mapsto \partial_\mu - ieA_\mu - i(\partial_\mu \Lambda).$$

while its dual 2–form *Maxwell* $\star F$ has the following components

$$\star F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & B_x & 0 \end{pmatrix}, \qquad \text{with} \qquad \star F_{\nu\mu} = - \star F_{\mu\nu},$$

so that classical electrodynamics is governed by the *Maxwell equations*, which in modern exterior formulation read

$$\begin{aligned} dF &= 0, \qquad \delta F = -4\pi J, \qquad \text{or in components,} \\ F_{[\mu\nu,\eta]} &= 0, \qquad F_{\mu\nu},^{\mu} = -4\pi J_{\mu}, \end{aligned}$$

where $\star$ is the Hodge star operator and δ is the Hodge codiferential (see section 3.2 below), comma denotes the partial derivative and the 1–form of electric current $J = J_{\mu} dx^{\mu}$ is conserved, by the electrical *continuity equation*,

$$\delta J = 0, \qquad \text{or in components,} \qquad J_{\mu},^{\mu} = 0.$$

The first, sourceless Maxwell equation, $dF = 0$, gives vector magnetostatics and magnetodynamics,

$$\begin{aligned} \text{Magnetic Gauss' law} &: \operatorname{div} \mathbf{B} = 0, \\ \text{Faraday's law} &: \partial_t \mathbf{B} + \operatorname{curl} \mathbf{E} = 0. \end{aligned}$$

The second Maxwell equation with source, $\delta F = J$, gives vector electrostatics and electrodynamics,

$$\begin{aligned} \text{Electric Gauss' law} &: \operatorname{div} \mathbf{E} = 4\pi\rho, \\ \text{Ampère's law} &: \partial_t \mathbf{E} - \operatorname{curl} \mathbf{B} = -4\pi \mathbf{j}. \end{aligned}$$

The standard *Lagrangian* for the free electromagnetic field, $F = dA$, is given by [19, 20, 15]

$$\mathcal{L}(A) = \frac{1}{2}(F \wedge \star F),$$

with the corresponding *action functional*

$$S(A) = \frac{1}{2} \int F \wedge \star F.$$

Maxwell's equations are generally applied to macroscopic averages of the fields, which vary wildly on a microscopic scale in the vicinity of individual atoms, where they undergo quantum effects as well (see below).

1.2.3 Free and interacting field theories

A generic *gauge–covariant derivative* with Lorentz index μ is denoted by D_μ. For a Maxwell field, D_μ is given by (12).

$$\begin{aligned} \text{Dirac slash notation} &: \not{\partial} \stackrel{\text{def}}{=} \gamma^\mu \partial_\mu, \qquad \not{D} \stackrel{\text{def}}{=} \gamma^\mu D_\mu, \\ \text{Dirac algebra} &: \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{n\times n}. \end{aligned}$$

Standard free theories are Klein–Gordon and Dirac fields:

$$\begin{aligned} \text{Klein–Gordon equation} &: (\partial^2 + m^2)\psi = 0, \\ \text{Dirac equation} &: (\mathrm{i}\not{\partial} - m)\psi = 0. \end{aligned}$$

Two main examples of interacting theories are ϕ^4-theory and QED:

1. ϕ^4-theory:

$$\begin{aligned} \text{Lagrangian} &: \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \\ \text{Equation of motion} &: (\partial^2 + m^2)\phi = -\frac{\lambda}{3!}\phi^3. \end{aligned}$$

2. QED:

$$\begin{aligned} \text{Lagrangian} &: \mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{int}} \\ &= -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(\mathrm{i}\not{D} - m)\psi. \\ \text{Gauge invariance} &: \psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x) \Longrightarrow A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x). \\ \text{Equation of motion} &: (\mathrm{i}\not{D} - m)\psi = 0. \end{aligned}$$

1.2.4 Dirac QED

The *Dirac equation* for a particle with mass m (in natural units) reads (see, e.g., [15])

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi = 0, \qquad (\mu = 0, 1, 2, 3) \tag{13}$$

where $\psi(x)$ is a 4–component spinor[7] wave–function, the so–called Dirac spinor, while γ^μ are 4×4 *Dirac* $\gamma-$*matrices*,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

They obey the *anticommutation relations*

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu},$$

where $g_{\mu\nu}$ is the metric tensor.

Dirac's $\gamma-$matrices are conventionally derived as

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \qquad (k = 1, 2, 3)$$

where σ^k are *Pauli* $\sigma-$*matrices*[8] (a set of 2×2 complex Hermitian and unitary matrices), defined as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

obeying both the commutation and anticommutation relations

$$[\sigma_i, \sigma_j] = 2\mathrm{i}\, \varepsilon_{ijk}\, \sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij} \cdot I,$$

where ε_{ijk} is the Levi–Civita symbol, δ_{ij} is the Kronecker delta, and I is the identity matrix.

Now, the Lorentz–invariant form of the Dirac equation (13) for an electron with a charge e and mass m_e, moving with a 4–momentum 1–form $p = p_\mu dx^\mu$ in a classical electromagnetic field defined by 1–form $A = A_\mu dx^\mu$, reads (see, e.g., [15]):

$$\{\mathrm{i}\gamma^\mu [p_\mu - eA_\mu] - m_\mathrm{e}\}\, \psi(x) = 0, \tag{14}$$

[7]The most convenient definitions for the 2–spinors, like the Dirac spinor, are:
$\phi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\phi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\chi^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\chi^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

[8]In quantum mechanics, each Pauli matrix represents an observable describing the spin of a spin $\frac{1}{2}$ particle in the three spatial directions. Also, $i\sigma_j$ are the generators of rotation acting on non-relativistic particles with spin $\frac{1}{2}$. The state of the particles are represented as two–component spinors.

In *quantum information*, single–qubit quantum gates are 2×2 unitary matrices. The Pauli matrices are some of the most important single–qubit operations.

and is called the *covariant Dirac equation*.

The formal QED Lagrangian (density) includes three terms,

$$\mathcal{L}(x) = \mathcal{L}_{\rm em}(x) + \mathcal{L}_{\rm int}(x) + \mathcal{L}_{\rm e-p}(x), \tag{15}$$

related respectively to the free electromagnetic field 2–form $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$, the electron–positron field (in the presence of the external vector potential 1–form $A^{\rm ext}_\mu$), and the interaction field (dependent on the charge–current 1–form $J = J_\mu dx^\mu$). The free electromagnetic field Lagrangian in (15) has the standard electrodynamic form

$$\mathcal{L}_{\rm em}(x) = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where the electromagnetic fields are expressible in terms of components of the potential 1–form $A = A_\mu dx^\mu$ by

$$F_{\mu\nu} = \partial_\mu A^{\rm tot}_\nu - \partial_\nu A^{\rm tot}_\mu, \quad \text{with} \quad A^{\rm tot}_\mu = A^{\rm ext}_\mu + A_\mu.$$

The electron-positron field Lagrangian is given by Dirac's equation (14) as

$$\mathcal{L}_{\rm e-p}(x) = \bar{\psi}(x) \left\{ {\rm i}\gamma^\mu \left[p_\mu - e A^{\rm ext}_\mu \right] - m_{\rm e} \right\} \psi(x),$$

where $\bar{\psi}(x)$ is the Dirac adjoint spinor wave function.

The interaction field Lagrangian

$$\mathcal{L}_{\rm int}(x) = -J^\mu A_\mu,$$

accounts for the interaction between the uncoupled electrons and the radiation field.

The field equations deduced from (15) read

$$\begin{aligned} \left\{ {\rm i}\gamma^\mu \left[p_\mu - e A^{\rm ext}_\mu \right] - m_{\rm e} \right\} \psi(x) &= \gamma^\mu \psi(x) A_\mu, \\ \partial^\mu F_{\mu\nu} &= J_\nu. \end{aligned} \tag{16}$$

The formal QED requires the solution of the system (16) when $A^\mu(x)$, $\psi(x)$ and $\bar{\psi}(x)$ are quantized fields.

1.2.5 Abelian Higgs Model

The Abelian[9] Higgs model is an example of gauge theory used in particle and condensed matter physics. Besides the electromagnetic field it contains a self-interacting scalar field,

[9] An *Abelian* (or, commutative) *group* (even better, Lie group, see Appendix), is such a group G that satisfies the condition: $a \cdot b = b \cdot a$ for all $a, b \in G$. In other words, its *commutator*, $[a, b] := a^{-1} b^{-1} a b$ equals the identity element.

the so-called *Higgs field*, minimally coupled to electromagnetism. From the conceptual point of view, it is advantageous to consider this field theory in $(2+1)$D space-time and to extend it subsequently to $(3+1)$D for applications. The Abelian Higgs Lagrangian reads [38]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi),$$

which contains the complex (charged), self-interacting scalar field ϕ. The Higgs potential is the *Mexican hat* function of the real and imaginary part of the Higgs field,

$$V(\phi) = \frac{1}{4}\lambda(|\phi|^2 - a^2)^2.$$

By construction, this Higgs potential is minimal along a circle $|\phi| = a$ in the complex ϕ plane. The constant λ controls the strength of the self–interaction of the Higgs field and, for stability reasons, is assumed to be positive, $\lambda \geq 0$. The Higgs field is minimally coupled to the radiation field A_μ, i.e., the partial derivative ∂_μ is replaced by the covariant derivative, $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$. Gauge fields and field strengths are related by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{\mathrm{i}e}\left[D_\mu, D_\nu\right].$$

The inhomogeneous Maxwell equations are obtained from the least action principle,

$$\delta S = 0, \qquad \text{with} \quad S = \int \mathcal{L} d^4x = 0,$$

by variation of the action S with respect to the gauge fields A_μ (and their derivatives $\partial_\mu A_\mu$). With

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} &= -F^{\mu\nu}, \qquad \frac{\delta\mathcal{L}}{\delta A_\nu} = -j^\nu\,, \qquad \text{we get} \\ \partial_\mu F^{\mu\nu} &= j^\nu, \qquad j_\nu = \mathrm{i}e(\phi^\star\partial_\nu\phi - \phi\partial_\nu\phi^\star) - 2e^2\phi^*\phi A_\nu. \end{aligned}$$

We remark here that the homogeneous Maxwell equations are not dynamical equations of motion. They are integrability conditions and guarantee that the field strength can be expressed in terms of the gauge fields. The homogeneous equations follow from the *Jacobi identity* of the covariant derivative

$$[D_\mu, [D_\nu, D_\sigma]] + [D_\sigma, [D_\mu, D_\nu]] + [D_\nu, [D_\sigma, D_\mu]] = 0.$$

Multiplication with the totally antisymmetric 4–tensor $\epsilon^{\mu\nu\rho\sigma}$, yields the homogeneous equations for the dual field strength $\tilde{F}^{\mu\nu}$

$$\left[D_\mu, \tilde{F}^{\mu\nu}\right] = 0, \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}.$$

The transition: $F \longrightarrow \tilde{F}$ corresponds to the following duality relation of electric and magnetic fields, $\mathbf{E} \longrightarrow \mathbf{B}$, $\mathbf{B} \longrightarrow -\mathbf{E}$.

Variation with respect to the charged matter field yields the equation of motion:

$$D_\mu D^\mu \phi + \frac{\delta V}{\delta \phi^*} = 0.$$

Gauge theories contain redundant variables. This redundancy manifests itself in the presence of local symmetry transformations, or gauge transformations, $U(x) = e^{\mathrm{i}e\alpha(x)}$, which rotate the phase of the matter field and shift the value of the gauge field in a space-time dependent manner

$$\phi \longrightarrow \phi^{[U]} = U(x)\phi(x)\,, \qquad A_\mu \longrightarrow A_\mu^{[U]} = A_\mu + U(x)\,\frac{1}{\mathrm{i}e}\,\partial_\mu\, U^\dagger(x)\,. \tag{17}$$

The covariant derivative D_μ has been defined such that $D_\mu\phi$ transforms covariantly, i.e., like the matter field ϕ itself.

$$D_\mu\phi(x) \longrightarrow U(x)\, D_\mu\phi(x).$$

This transformation property together with the invariance of $F_{\mu\nu}$ guarantees invariance of $\mathcal{L}$ and of the equations of motion. A gauge field which is gauge equivalent to $A_\mu = 0$ is called a pure gauge. According to (17) a pure gauge satisfies

$$A_\mu^{pg}(x) = U(x)\frac{1}{\mathrm{i}e}\,\partial_\mu\, U^\dagger(x) = -\partial_\mu\,\alpha(x),$$

and the corresponding field strength vanishes.

Note that the non-Abelian Higgs model has the action:

$$S(\phi, A) = \frac{1}{4}\int \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + |D\phi|^2 + V(|\phi|),$$

where now the non-Abelian field A is contained both in the covariant derivative D and in the components $F^{\mu\nu}$ and $F_{\mu\nu}$ (see Yang–Mills theory below).

2 Feynman Path Integral

2.1 The action–amplitude formalism

The 'driving engine' of quantum field theory is the Feynman path integral. Very briefly, there are three basic forms of the path integral (see, e.g., [21, 15]):

1. *Sum–over–histories,* developed in Feynman's version of quantum mechanics (QM)[10] [22];

2. *Sum–over–fields,* started in Feynman's version of quantum electrodynamics (QED) [23]
and later improved by Fadeev–Popov [24];

3. *Sum–over–geometries/topologies* in quantum gravity (QG), initiated by S. Hawking and properly developed in the form of causal dynamical triangulations (see [25]; for a 'softer' review, see [26]).

In all three versions, Feynman's *action–amplitude formalism* includes two components:

[10]Feynman's *amplitude* is a space-time version of the Schrödinger's *wavefunction* ψ, which describes how the (non-relativistic) quantum state of a physical system changes in space and time, i.e.,

$$\langle \mathrm{Out}_{t_{fin}} | \mathrm{In}_{t_{ini}} \rangle = \psi(\mathbf{x}, t), \qquad (\text{for } \ \mathbf{x} \in [\mathrm{In}, \mathrm{Out}], \ t \in [t_{ini}, t_{fin}]).$$

In particular, quantum wavefunction ψ is a complex–valued function of real space variables $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, which means that its domain is in $\mathbb{R}^n$ and its range is in the complex plane, formally $\psi(\mathbf{x}) : \mathbb{R}^n \to \mathbb{C}$. For example, the one–dimensional *stationary plane wave* with wave number k is defined as

$$\psi(x) = \mathrm{e}^{\mathrm{i}kx}, \qquad (\text{for } \ x \in \mathbb{R}),$$

where the real number k describes the wavelength, $\lambda = 2\pi/k$. In n dimensions, this becomes

$$\psi(x) = \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}},$$

where the momentum vector $\mathbf{p} = \mathbf{k}$ is the vector of the wave numbers $\mathbf{k}$ in natural units (in which $\hbar = m = 1$).

More generally, quantum wavefunction is also time dependent, $\psi = \psi(\mathbf{x}, \mathbf{t})$. The time–dependent plane wave is defined by

$$\psi(\mathbf{x}, \mathbf{t}) = \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}p^2 t/2}. \tag{18}$$

In general, $\psi(\mathbf{x}, \mathbf{t})$ is governed by the Schrödinger equation [27, 15] (in natural units $\hbar = m = 0$)

$$\mathrm{i}\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = -\frac{1}{2}\Delta\psi(\mathbf{x}, t), \tag{19}$$

where Δ is the $n-$dimensional Laplacian. The solution of (19) is given by the integral of the time–dependent plane wave (18),

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x} - \mathrm{i}p^2 t/2} \hat{\psi}_0(\mathbf{p}) d^n p,$$

which means that $\psi(\mathbf{x}, t)$ is the inverse Fourier transform of the function

$$\hat{\psi}(\mathbf{p}, t) = \mathrm{e}^{-\mathrm{i}p^2 t/2} \hat{\psi}_0(\mathbf{p}),$$

where $\hat{\psi}_0(\mathbf{p})$ has to be calculated for each initial wavefunction. For example, if initial wavefunction is Gaussian,

$$f(x) = \exp(-a\frac{x^2}{2}), \qquad \text{with the Fourier transform} \qquad \hat{f}(p) = \frac{1}{\sqrt{a}} \exp(-\frac{p^2}{2a}).$$

$$\text{then} \qquad \hat{\psi}_0(p) = \tfrac{1}{\sqrt{a}} \exp(-\tfrac{p^2}{2a}).$$

1. A real–valued, classical, *Hamilton's action functional,*

$$S[\Phi] \;:=\; \int_{t_{ini}}^{t_{fin}} L[\Phi]\, dt, \tag{20}$$

with the Lagrangian energy function defined over the Lagrangian density $\mathcal{L}$,

$$L[\Phi] = \int d^n x\, \mathcal{L}(\Phi, \partial_\mu \Phi), \qquad (\partial_\mu \equiv \partial/\partial x^\mu),$$

while Φ is a common symbol denoting all three things to be summed upon (histories, fields and geometries). The action functional $S[\Phi]$ obeys the *Hamilton's least action principle,*

$\delta S[\Phi] = 0,$ and gives, using standard variational methods,[11] the Euler–Lagrangian equations, which define the shortest path, the extreme field, and the geometry of minimal curvature (and without holes).

[11] In Lagrangian field theory, the fundamental quantity is the action

$$S[\Phi] = \int_{t_{in}}^{t_{out}} L\, dt = \int_{\mathbb{R}^4} d^n x\, \mathcal{L}(\Phi, \partial_\mu \Phi)\,,$$

so that the least action principle, $\delta S[\Phi] = 0$, gives

$$\begin{aligned} 0 &= \int_{\mathbb{R}^4} d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta(\partial_\mu \Phi) \right\} \\ &= \int_{\mathbb{R}^4} d^n x \left\{ \frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) \delta\Phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi \right) \right\}. \end{aligned}$$

The last term can be turned into a surface integral over the boundary of the $\mathbb{R}^4$ (4D space-time region of integration). Since the initial and final field configurations are assumed given, $\delta\Phi = 0$ at the temporal beginning t_{in} and end t_{out} of this region, which implies that the surface term is zero. Factoring out the $\delta\Phi$ from the first two terms, and since the integral must vanish for arbitrary $\delta\Phi$, we arrive at the Euler-lagrange equation of motion for a field,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) - \frac{\partial \mathcal{L}}{\partial \Phi} = 0.$$

If the Lagrangian (density) $\mathcal{L}$ contains more fields, there is one such equation for each. The momentum density $\pi(x)$ of a field, conjugate to $\Phi(x)$ is defined as: $\pi(x) = \frac{\partial \mathcal{L}}{\partial_\mu \Phi(x)}$.

For example, the standard electromagnetic action

$$S = -\frac{1}{4} \int_{\mathbb{R}^4} d^4 x\, F_{\mu\nu} F^{\mu\nu}, \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

gives the sourceless Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\sigma\eta} \partial_\nu F_{\sigma\eta} = 0,$$

where the field strength tensor $F_{\mu\nu}$ and the Maxwell equations are invariant under the *gauge transformations,*

$$A_\mu \longrightarrow A_\mu + \partial_\mu \epsilon.$$

The equations of motion of charged particles are given by the Lorentz–force equation,

$$m \frac{du^\mu}{d\tau} = e F^{\mu\nu} u_\nu,$$

where e is the charge of the particle and $u^\mu(\tau)$ its four-velocity as a function of the proper time.

2. A complex–valued, quantum *transition amplitude*,[12]

$$\langle \mathrm{Out}_{t_{fin}} | \mathrm{In}_{t_{ini}} \rangle \ := \int_{\Omega} \mathcal{D}[\Phi]\, \mathrm{e}^{\mathrm{i}S[\Phi]}, \tag{23}$$

[12]The transition amplitude is closely related to *partition function* Z, which is a quantity that encodes the statistical properties of a system in thermodynamic equilibrium. It is a function of temperature and other parameters, such as the volume enclosing a gas. Other thermodynamic variables of the system, such as the total energy, free energy, entropy, and pressure, can be expressed in terms of the partition function or its derivatives. In particular, the partition function of a *canonical ensemble* is defined as a sum $Z(\beta) = \sum_j \mathrm{e}^{-\beta E_j}$, where $\beta = 1/(k_B T)$ is the 'inverse temperature', where T is an ordinary temperature and k_B is the Boltzmann's constant. However, as the position x^i and momentum p_i variables of an ith particle in a system can vary continuously, the set of microstates is actually uncountable. In this case, some form of *coarse–graining* procedure must be carried out, which essentially amounts to treating two mechanical states as the same microstate if the differences in their position and momentum variables are 'small enough'. The partition function then takes the form of an integral. For instance, the partition function of a gas consisting of N molecules is proportional to the $6N-$dimensional phase–space integral,

$$Z(\beta) \sim \int_{\mathbb{R}^{6N}} d^3p_i\, d^3x^i \exp[-\beta H(p_i, x^i)],$$

where $H = H(p_i, x^i)$, $(i = 1, ..., N)$ is the classical Hamiltonian (total energy) function.

Given a set of random variables X_i taking on values x^i, and purely potential Hamiltonian function $H(x^i)$, the partition function is defined as

$$Z(\beta) = \sum_{x^i} \exp\left[-\beta H(x^i)\right].$$

The function H is understood to be a real-valued function on the space of states $\{X_1, X_2, \cdots\}$ while β is a real-valued free parameter (conventionally, the inverse temperature). The sum over the x^i is understood to be a sum over all possible values that the random variable X_i may take. Thus, the sum is to be replaced by an integral when the X_i are continuous, rather than discrete. Thus, one writes

$$Z(\beta) = \int dx^i \exp\left[-\beta H(x^i)\right],$$

for the case of continuously-varying random variables X_i.

Now, the number of variables X_i need not be countable, in which case the set of coordinates $\{x^i\}$ becomes a field $\phi = \phi(x)$, so the sum is to be replaced by the *Euclidean path integral* (that is a Wick–rotated Feynman transition amplitude (24) in imaginary time), as

$$Z(\phi) = \int \mathcal{D}[\phi] \exp\left[-H(\phi)\right].$$

More generally, in quantum field theory, instead of the field Hamiltonian $H(\phi)$ we have the action $S(\phi)$ of the theory. Both Euclidean path integral,

$$Z(\phi) = \int \mathcal{D}[\phi] \exp\left[-S(\phi)\right], \qquad \text{real path integral in imaginary time,} \tag{21}$$

and Lorentzian one,

$$Z(\phi) = \int \mathcal{D}[\phi] \exp\left[\mathrm{i}S(\phi)\right], \qquad \text{complex path integral in real time,} \tag{22}$$

are usually called 'partition functions'. While the Lorentzian path integral (22) represents a quantum-field theory-generalization of the Schrödinger equation, the Euclidean path integral (21) represents a statistical-field-theory generalization of the Fokker–Planck equation.

where $\mathcal{D}[\Phi]$ is 'an appropriate' Lebesgue–type measure,

$$\mathcal{D}[\Phi] = \lim_{N\to\infty} \prod_{s=1}^{N} \Phi_s^i, \qquad (i = 1, ..., n),$$

so that we can 'safely integrate over a continuous spectrum and sum over a discrete spectrum of our problem domain Ω', of which the absolute square is the real–valued probability density function,

$$P \;:=\; |\langle \mathrm{Out}_{t_{fin}} | \mathrm{In}_{t_{ini}} \rangle\rangle|^2.$$

This procedure can be redefined in a mathematically cleaner way if we Wick–rotate the time variable t to imaginary values, $t \mapsto \tau = t$, thereby making all integrals real:

$$\int \mathcal{D}[\Phi]\, \mathrm{e}^{\mathrm{i}S[\Phi]} \;\xrightarrow{Wick}\; \int \mathcal{D}[\Phi]\, \mathrm{e}^{-S[\Phi]}. \tag{24}$$

For example, in non-relativistic quantum mechanics, the propagation amplitude from x_a to x_b is given by the *configuration path integral*

$$U(x_a, x_b; T) = \langle x_b | x_a \rangle = \left\langle x_b \right| \mathrm{e}^{-\mathrm{i}HT} \left| x_a \right\rangle = \int \mathcal{D}[x(t)]\, \mathrm{e}^{\mathrm{i}S[x(t)]},$$

which satisfies the Schrödinger equation (in natural units)

$$i\frac{\partial}{\partial T} U(x_a, x_b; T) = \hat{H} U(x_a, x_b; T), \qquad \text{where} \qquad \hat{H} = -\frac{1}{2}\frac{\partial^2}{\partial x_b^2} + V(x_b).$$

The *phase–space path integral* (without peculiar constants in the functional measure) reads

$$U(q_a, q_b; T) = \left(\prod_i \int \mathcal{D}[q(t)]\mathcal{D}[p(t)] \right) \exp\left[\mathrm{i} \int_0^T \left(p_i \dot{q}^i - H(q,p) \right)\, dt \right],$$

where the functions $q(t)$ (space coordinates) are constrained at the endpoints, but the functions $p(t)$ (canonically–conjugated momenta) are not. The functional measure is just the product of the standard integral over phase space at each point in time

$$\mathcal{D}[q(t)]\mathcal{D}[p(t)] = \prod_i \frac{1}{2\pi} \int dq^i dp_i.$$

Applied to a non-relativistic real scalar field $\phi(x,t)$, this path integral becomes

$$\left\langle \phi_b(x,t) \right| \mathrm{e}^{-\mathrm{i}HT} \left| \phi_a(x,t) \right\rangle = \int \mathcal{D}[\phi] \exp\left[\mathrm{i} \int_0^T \mathcal{L}(\phi)\, d^4x \right], \quad \text{with} \quad \mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi).$$

2.2 Correlation functions and generating functional

If we have two fields in the interacting theory, the corresponding two–point correlation function, or two–point Green's function, is denoted by $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, where the notation $|\Omega\rangle$ is introduced to denote the ground state of the interacting theory, which is generally different from $|0\rangle$, the ground state of the free theory. The correlation function can be interpreted physically as the amplitude for propagation of a particle or excitation between y and x. In the free theory, it is simply the Feynman propagator

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle_{\text{free}} = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}\,\mathrm{e}^{-ip\cdot(x-y)}}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

We would like to know how this expression changes in the interacting theory. Once we have analyzed the two–point correlation functions, it will be easy to generalize our results to higher correlation functions in which more than two field operators appear.

In general we have:

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\left\langle 0|T\{\phi_I(x)\phi_I(y)\exp[-\mathrm{i}\int_{-T}^{T} dt\, H_I(t)]\}|0\right\rangle}{\left\langle 0|T\{\exp[-\mathrm{i}\int_{-T}^{T} dt\, H_I(t)]\}|0\right\rangle},$$

$$\left\langle 0|T\{\phi_I(x)\phi_I(y)\exp[-\mathrm{i}\int_{-T}^{T} dt\, H_I(t)]\}|0\right\rangle = \left(\begin{array}{c}\text{sum of all possible Feynman diagrams}\\ \text{with two external points}\end{array}\right),$$

where each diagram is built out of Feynman propagators, vertices and external points.

The virtue of considering the time–ordered product is clear: It allows us to put everything inside one large $T-$operator. A similar formula holds for higher correlation functions of arbitrarily many fields; for each extra factor of ϕ on the left, put an extra factor of ϕ_I on the right.

In the interacting theory, the corresponding two–point correlation function is given by

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \left(\begin{array}{c}\text{sum of all connected diagrams}\\ \text{with two external points}\end{array}\right).$$

This is generalized to higher correlation functions as

$$\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle = \left(\begin{array}{c}\text{sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{array}\right).$$

In a scalar field theory, the generating functional of correlation functions is defined as

$$Z[J] = \int \mathcal{D}[\phi] \exp\left[\mathrm{i}\int d^4x\,[\mathcal{L} + J(x)\phi(x)]\right] = \langle\Omega|\,\mathrm{e}^{-\mathrm{i}HT}|\Omega\rangle = \,\mathrm{e}^{-iE[J]};$$

this is a functional integral over $\phi(x)$ in which we have added a source term $J(x)\phi(x)$ to $\mathcal{L} = \mathcal{L}(\phi)$.

For example, the generating functional of the free Klein–Gordon theory is simply

$$Z[J] = Z_0 \exp\left[-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right].$$

2.3 Quantization of the electromagnetic field

Consider the path integral

$$\begin{aligned} Z[A] &= \int \mathcal{D}[A]\, \mathrm{e}^{\mathrm{i}S[A]}, \qquad \text{where the action for the free e.-m. field is} \\ S[A] &= \int d^4x \left[-\frac{1}{4}(F_{\mu\nu})^2\right] = \frac{1}{2}\int d^4x\, A_\mu(x)\left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu(x). \end{aligned}$$

$Z[A]$ is the path integral over each of the four spacetime components:

$$\mathcal{D}[A] = \mathcal{D}[A]^0 \mathcal{D}[A]^1 \mathcal{D}[A]^2 \mathcal{D}[A]^3.$$

This functional integral is badly divergent, due to gauge invariance. Recall that $F_{\mu\nu}$, and hence L, is invariant under a general gauge transformation of the form

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x).$$

The troublesome modes are those for which $A_\mu(x) = \partial_\mu \alpha(x)$, that is, those that are gauge–equivalent to $A_\mu(x) = 0$. The path integral is badly defined because we are redundantly integrating over a continuous infinity of physically equivalent field configurations. To fix this problem, we would like to isolate the interesting part of the path integral, which counts each physical configuration only once. This can be accomplished using the *Faddeev–Popov trick*, which effectively adds a term to the system Lagrangian and after which we get

$$Z[A] = \int \mathcal{D}[A] \exp\left[\mathrm{i}\int_{-T}^{T} d^4x \, \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right],$$

where ξ is any finite constant.

This procedure needs also to be applied to the formula for the two–point correlation function

$$\langle\Omega|T\,\mathcal{O}(A)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}[A]\,\mathcal{O}(A) \exp\left[\mathrm{i}\int_{-T}^{T} d^4x\,\mathcal{L}\right]}{\int \mathcal{D}[A] \exp\left[\mathrm{i}\int_{-T}^{T} d^4x\,\mathcal{L}\right]},$$

which after Faddeev–Popov procedure becomes

$$\langle\Omega|T\,\mathcal{O}(A)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}[A]\,\mathcal{O}(A) \exp\left[\mathrm{i}\int_{-T}^{T} d^4x\, \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right]}{\int \mathcal{D}[A] \exp\left[\mathrm{i}\int_{-T}^{T} d^4x\, \left[\mathcal{L} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\right]\right]}.$$

3 Path–Integral TQFT

3.1 Schwarz–type and Witten–type theories

Consider a set of fields $\{\phi_i\}$ on a Riemannian $n-$manifold M (with a metric $g_{\mu\nu}$) and real functional of these fields, $S[\phi_i]$, which is the action of the theory. Also consider a set of operators $\mathcal{O}_\alpha(\phi_i)$ (labeled by some set of indices α), which are arbitrary functionals of the fields $\{\phi_i\}$. The *vacuum expectation value* (VEV) of a product of these operators is defined as the path integral (see [37])

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = \int \mathcal{D}[\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\exp\left(-S[\phi_i]\right).$$

A quantum field theory is considered *topological* if it possesses the following property:

$$\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = 0, \tag{25}$$

i.e., if the VEVs of some set of selected operators remain invariant under variations of the metric $g_{\mu\nu}$ on M. In this case, the operators $\mathcal{O}_\alpha(\phi_i)$ are called *observables*.

There are two ways to formally guarantee that condition (25) is satisfied. The first one corresponds to the situation in which both, the action, S, as well as the operators $\mathcal{O}_\alpha$, are metric independent. These TQFTs are called *Schwarz–type*. In the case of Schwarz–type theories one must first construct an action which is independent of the metric $g_{\mu\nu}$. The method is best illustrated by considering an example. Let us take into consideration the most interesting case of this type of theories: *Chern–Simons gauge theory*. The data in Chern–Simons gauge theory are the following: a differentiable compact 3–manifold M, a gauge group G, which will be taken simple and compact, and an integer parameter k. The action is the integral of the *Chern–Simons form* associated to a *gauge connection* A corresponding to the group G,

$$S_{\rm CS}[A] = \int_M \mathrm{Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A). \tag{26}$$

Observables are constructed out of operators which do not contain the metric $g_{\mu\nu}$. In gauge invariant theories, as it is the case, one must also demand for these operators invariance under gauge transformations. An important set of observables in Chern–Simons gauge theory is constituted by the trace of the holonomy of the gauge connection A in some representation

R along a 1–cycle γ, that is the *Wilson loop*,[13]

$$\mathrm{Tr}_R\left(\mathrm{Hol}_\gamma(A)\right) = \mathrm{Tr}_R \mathrm{P}\exp\int_\gamma A. \tag{27}$$

The VEVs are labeled by representations R_i and embeddings γ_i of S^1 into M [37]

$$\langle \mathrm{Tr}_{R_1}\mathrm{Pe}^{\int_{\gamma_1} A}\ldots\mathrm{Tr}_{R_n}\mathrm{Pe}^{\int_{\gamma_n} A}\rangle = \int [DA]\mathrm{Tr}_{R_1}\mathrm{Pe}^{\int_{\gamma_1} A}\ldots\mathrm{Tr}_{R_n}\mathrm{Pe}^{\int_{\gamma_n} A}\mathrm{e}^{\frac{\mathrm{i}k}{4\pi}S_{\mathrm{CS}}(A)}.$$

A non-perturbative analysis of Chern–Simons gauge theory shows that the invariants associated to the observables $\mathcal{O}_\alpha(\phi_i)$ are knot and link invariants of polynomial type as the Jones polynomial and its generalizations. The perturbative analysis has also led to this result and has shown to provide a very useful framework to study Vassiliev invariants.

The second way to guarantee (25) corresponds to the case in which there exists a symmetry, whose infinitesimal form is denoted by δ, satisfying the following properties:

$$\delta\mathcal{O}_\alpha(\phi_i) = 0, \qquad T_{\mu\nu}(\phi_i) = \delta G_{\mu\nu}(\phi_i), \tag{28}$$

where $T_{\mu\nu}(\phi_i)$ is the energy–momentum tensor of the theory, given by

$$T_{\mu\nu}(\phi_i) = \frac{\delta}{\delta g^{\mu\nu}} S[\phi_i], \tag{29}$$

while $G_{\mu\nu}(\phi_i)$ is some tensor.

The fact that δ in (28) is a symmetry of the theory means that the transformations $\delta\phi_i$ of the fields are such that $\delta S[\phi_i] = 0$ and $\delta\mathcal{O}_\alpha(\phi_i) = 0$. Conditions (28) lead formally to the following relation for VEVs:

$$\begin{aligned} &\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = \\ &-\int \mathcal{D}[\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)T_{\mu\nu}\exp\left(-S[\phi_i]\right) \\ = \quad &-\int \mathcal{D}[\phi_i]\delta\left(\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)G_{\mu\nu}\exp\left(-S[\phi_i]\right)\right) = 0, \end{aligned} \tag{30}$$

which implies that the quantum field theory can be regarded as topological. In (30) it has been assumed that the action and the measure $\mathcal{D}[\phi_i]$ are invariant under the symmetry δ. We

[13] A *holonomy* on a smooth manifold is a general geometrical consequence of the curvature of the manifold connection, measuring the extent to which parallel transport around closed loops fails to preserve the geometrical data being transported. Related to holonomy is a *Wilson loop*, which is a gauge–invariant observable obtained from the holonomy of the gauge connection around a given loop. More precisely, a Wilson loop is a quantity defined by the trace of a path–ordered exponential of a gauge field A_μ transported along a closed curve (loop) γ, $W_\gamma = \mathrm{Tr}\left(P\exp[\mathrm{i}\oint_\gamma A_\mu dx^\mu]\right)$, where P is the path-ordering operator.

have assumed also in (30) that the observables are metric–independent. This is a common situation in this type of theories, but it does not have to be necessarily so. In fact, in view of (30), it would be possible to consider a wider class of operators satisfying:

$$\frac{\delta}{\delta g_{\mu\nu}}\mathcal{O}_\alpha(\phi_i) = \delta O_\alpha^{\mu\nu}(\phi_i), \tag{31}$$

where $O_\alpha^{\mu\nu}(\phi_i)$ is a certain functional of the fields of the theory.

This second type of TQFTs are called *cohomological* of *Witten–type*. One of its main representatives is *Donaldson–Witten theory*, which can be regarded as a certain *twisted* version of $N = 2$ supersymmetric Yang–Mills theory. It is important to remark that the symmetry δ must be a scalar symmetry. The reason is that, being a global symmetry, the corresponding parameter must be covariantly constant and for arbitrary manifolds this property, if it is satisfied at all, implies strong restrictions unless the parameter is a scalar.

Most of the TQFTs of cohomological type satisfy the relation: $S[\phi_i] = \delta\Lambda(\phi_i)$, for some functional $\Lambda(\phi_i)$. This means that the topological observables of the theory (in particular the partition function itself) are independent of the value of the coupling constant. For example, consider the VEV [37]

$$\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle = \int \mathcal{D}[\phi_i]\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\exp\left(-\frac{1}{g^2}S[\phi_i]\right). \tag{32}$$

Under a change in the coupling constant, $1/g^2 \to 1/g^2 - \Delta$, one has (assuming that the observables do not depend on the coupling), up to first order in Δ:

$$\begin{aligned}
&\langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle \longrightarrow \langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle \\
&\Delta\int \mathcal{D}[\phi_i]\delta\left[\mathcal{O}_{\alpha_1}(\phi_i)\mathcal{O}_{\alpha_2}(\phi_i)\cdots\mathcal{O}_{\alpha_p}(\phi_i)\Lambda(\phi_i)\exp\left(-\frac{1}{g^2}S[\phi_i]\right)\right] \\
&= \langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\cdots\mathcal{O}_{\alpha_p}\rangle.
\end{aligned}$$

Hence, observables can be computed either in the weak coupling limit, $g \to 0$, or in the strong coupling limit, $g \to \infty$.

3.2 Hodge decomposition theorem

The *Hodge star* operator $\star : \Omega^p(M) \to \Omega^{n-p}(M)$, which maps any exterior $p-$form $\alpha \in \Omega^p(M)$ into its *dual* $(n-p)-$form $\star\alpha \in \Omega^{n-p}(M)$ on a smooth $n-$manifold M, is defined as (see, e.g. [28, 29])

$$\alpha\wedge\star\beta = \beta\wedge\star\alpha = \langle\alpha,\beta\rangle\,\mu, \qquad \star\star\,\alpha = (-1)^{p(n-p)}\alpha, \qquad (\text{for } \alpha,\beta\in\Omega^p(M)),$$

The $\star$ operator depends on the Riemannian metric $g = g_{ij}$ on M and also on the orientation (reversing orientation will change the sign) [19, 20]. Using the star operator, for any two

$p-$forms $\alpha, \beta \in \Omega^p(M)$ with compact support on M we define bilinear and positive–definite Hodge L^2-inner product as

$$\langle \alpha, \beta \rangle := \int_M \alpha \wedge \star \beta. \tag{33}$$

where $\alpha \wedge \star \beta$ is an $n-$form.

Given the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ on a smooth manifold M (see Appendix), its Hodge dual (or, formal adjoint) is the *codifferential* δ, a linear map $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$, which is a generalization of the divergence, defined by [28, 29]

$$\delta = (-1)^{n(p+1)+1} \star d \star \qquad \text{so that} \qquad d = (-1)^{np} \star \delta \star .$$

That is, if the dimension n of the manifold M is even, then $\delta = - \star d \star$.

Applied to any $p-$form $\omega \in \Omega^p(M)$, the codifferential δ gives

$$\delta\omega = (-1)^{n(p+1)+1} \star d \star \omega, \qquad \delta d\omega = (-1)^{np+1} \star d \star d\omega.$$

If $\omega = f$ is a $0-$form, or function, then $\delta f = 0$. If a $p-$form α is a codifferential of a $(p+1)-$form β, that is $\alpha = \delta\beta$, then β is called the *coexact* form. A $p-$form α is *coclosed* if $\delta\alpha = 0$; then $\star\alpha$ is closed (i.e., $d \star \alpha = 0$) and conversely.

The Hodge codifferential δ satisfies the following set of rules:

- $\delta\delta = \delta^2 = 0$, the same as $dd = d^2 = 0$;
- $\delta\star = (-1)^{p+1} \star d$; $\star\delta = (-1)^p \star d$;
- $d\delta\star = \star\delta d$; $\star d\delta = \delta d\star$.

The codifferential δ can be coupled with the exterior derivative d to construct the *Hodge Laplacian* $\Delta : \Omega^p(M) \to \Omega^p(M)$, a harmonic generalization of the Laplace–Beltrami differential operator, given by

$$\Delta = \delta d + d\delta = (d + \delta)^2.$$

Δ satisfies the following set of rules:

$$\delta\,\Delta = \Delta\,\delta = \delta d\delta; \qquad d\,\Delta = \Delta\,d = d\delta d; \qquad \star\,\Delta = \Delta \star .$$

A $p-$form α is called *harmonic* iff

$$\Delta\alpha = 0 \iff d\alpha = \delta\alpha = 0.$$

Thus, α is harmonic in a compact domain $D \subset M$ iff it is both closed and coclosed in D. Informally, every harmonic form is both closed and coclosed. As a proof, we have:

$$0 = \langle \alpha,\ \Delta\alpha \rangle = \langle \alpha, d\delta\alpha \rangle + \langle \alpha, \delta d\alpha \rangle = \langle \delta\alpha, \delta\alpha \rangle + \langle d\alpha, d\alpha \rangle .$$

Since $\langle \beta,\ \beta\rangle \geq 0$ for any form β, $\langle \delta\alpha, \delta\alpha\rangle$ and $\langle d\alpha, d\alpha\rangle$ must vanish separately. Thus, $d\alpha = 0$ and $\delta\alpha = 0$. All harmonic $p-$forms on a smooth manifold M form the vector space $H^p_\Delta(M)$.

Now, the celebrated *Hodge decomposition theorem* (HDT) states that, on a compact orientable smooth $n-$manifold M (with $n \geq p$), any exterior $p-$form can be written as a unique sum of an *exact* form, a *coexact* form, and a *harmonic* form. More precisely, for any form $\omega \in \Omega^p(M)$ there are unique forms $\alpha \in \Omega^{p-1}(M)$, $\beta \in \Omega^{p+1}(M)$ and a harmonic form $\gamma \in \Omega^p(M)$, such that

$$\text{HDT}: \quad \overset{\text{any form}}{\omega} = \overset{\text{exact}}{d\alpha} + \overset{\text{coexact}}{\delta\beta} + \overset{\text{harmonic}}{\gamma}$$

For the proof, see [28, 29].

In physics community, the exact form $d\alpha$ is called *longitudinal*, while the coexact form $\delta\beta$ is called *transversal*, so that they are mutually orthogonal. Thus any form can be orthogonally decomposed into a harmonic, a longitudinal and transversal form. For example, in fluid dynamics, any vector-field v can be decomposed into the sum of two vector-fields, one of which is divergence–free, and the other is curl–free.

Since γ is harmonic, $d\gamma = 0$. Also, by Poincaré lemma, $d(d\alpha) = 0$. In case ω is a closed $p-$form, $d\omega = 0$, then the term $\delta\beta$ in HDT is absent, so we have the *short Hodge decomposition*,

$$\omega = d\alpha + \gamma, \tag{34}$$

thus ω and γ differ by $d\alpha$. In topological terminology, ω and γ belong to the same *cohomology class* $[\omega] \in H^p(M)$. Now, by the de Rham theorems it follows that if C is any $p-$cycle, then

$$\int_C \omega = \int_C \gamma,$$

that is, γ and ω have the same periods. More precisely, if ω is any closed $p-$form, then there exists a unique harmonic $p-$form γ with the same periods as those of ω (see [28, 30]).

The *Hodge–Weyl theorem* [28, 29] states that every de Rham cohomology class has a unique harmonic representative. In other words, the space $H^p_\Delta(M)$ of harmonic $p-$forms on a smooth manifold M is isomorphic to the pth de Rham cohomology group,

$$H^p_{DR}(M) := \frac{Z^p(M)}{B^pM} = \frac{\text{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right)}{\text{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right)}, \tag{35}$$

or, $H^p_\Delta(M) \cong H^p_{DR}(M)$. That is, the harmonic part γ of HDT depends only on the global structure, i.e., the topology of M.

For example, in $(2+1)$D electrodynamics, $p-$form Maxwell equations in the Fourier domain Σ are written as [31]

$$\begin{aligned} dE &= \mathrm{i}\omega B, & dB &= 0, \\ dH &= -\mathrm{i}\omega D + J, & dD &= Q, \end{aligned}$$

where H and ω are 0–forms (magnetizing field and field frequency), D (electric displacement field), J (electric current density) and E (electric field) are 1–forms, while B (magnetic field) and Q (electric charge density) are 2–forms. From $d^2 = 0$ it follows that the J and the Q satisfy the *continuity equation*

$$dJ = \mathrm{i}\omega Q.$$

Constitutive equations, which include all metric information in this framework, are written in terms of Hodge star operators (that fix an isomorphism between p forms and $(2-p)$ forms in the $(2+1)$ case)

$$D = \star E, \qquad B = \star H.$$

Applying HDT to the electric field intensity 1–form E, we get [32]

$$E = d\phi + \delta A + \chi,$$

where ϕ is a 0–form (a scalar field) and A is a 2–form; $d\phi$ represents the static field and δA represents the dynamic field, and χ represents the harmonic field component. If domain Σ is contractible, χ is identically zero and we have the short Hodge decomposition,

$$E = d\phi + \delta A.$$

3.3 Hodge decomposition and gauge path integral

3.3.1 Functional measure on the space of differential forms

The Hodge inner product (33) leads to a natural (metric–dependent) functional measure $\mathcal{D}\mu[\omega]$ on $\Omega^p(M)$, which normalizes the *Gaussian functional integral*

$$\int \mathcal{D}\mu[\omega]\, \mathrm{e}^{\mathrm{i}\langle\omega|\omega\rangle} = 1. \tag{36}$$

One can use the invariance of (36) to determine how the functional measure transforms under the Hodge decomposition. Using HDT and its orthogonality with respect to the inner product (33), it was shown in [34] that

$$\langle\omega,\omega\rangle = \langle\gamma,\gamma\rangle + \langle d\alpha, d\alpha\rangle + \langle\delta\beta,\delta\beta\rangle = \langle\gamma,\gamma\rangle + \langle\alpha,\delta d\alpha\rangle + \langle\beta, d\delta\beta\rangle\,, \tag{37}$$

where the following differential/conferential identities were used [33]

$$\langle d\alpha, d\alpha\rangle = \langle\alpha,\delta d\alpha\rangle \qquad \text{and} \qquad \langle\delta\beta,\delta\beta\rangle = \langle\beta, d\delta\beta\rangle.$$

Since, for any linear operator O, one has

$$\int \mathcal{D}\mu[\omega] \exp \mathrm{i}\langle\omega|O\omega\rangle = \det{}^{-1/2}(O),$$

(36) and (37) imply that

$$\mathcal{D}\mu[\omega] = \mathcal{D}\mu[\gamma]\mathcal{D}\mu[\alpha]\mathcal{D}\mu[\beta] \det{}^{1/2}(\delta d)\det{}^{1/2}(d\delta).$$

3.3.2 Abelian Chern–Simons theory

Recall that the classical action for an Abelian Chern–Simons theory,

$$S = \int_M A \wedge dA\,,$$

is invariant (up to a total divergence) under the gauge transformation:

$$A \longmapsto A + d\varphi. \tag{38}$$

We wish to compute the *partition function* for the theory

$$Z := \int \frac{1}{V_G} \mathcal{D}\mu[A]\, \mathrm{e}^{\mathrm{i}S[A]},$$

where V_G denotes the volume of the group of gauge transformations in (38), which must be factored out of the partition function in order to guarantee that the integration is performed only over physically distinct gauge fields. We can handle this by using the Hodge decomposition to parametrize the potential A in terms of its gauge invariant, and gauge dependent parts, so that the volume of the group of gauge transformations can be explicitly factored out, leaving a functional integral over gauge invariant modes only [34].

We now transform the integration variables:

$$A \longmapsto \alpha, \beta, \gamma,$$

where α, β, γ parameterize respectively the exact, coexact, and harmonic parts of the connection A. Using the Jacobian (37) as well as the following identity on 0–forms $\Delta = \delta d$, we get [34]

$$Z = \int \frac{1}{V_G} \mathcal{D}\mu[\alpha]\mathcal{D}\mu[\beta]\mathcal{D}\mu[\gamma] \det{}^{1/2}(\Delta) \det{}^{1/2}(d\delta)\, \mathrm{e}^{\mathrm{i}S},$$

from which it follows that

$$V_G = \int \mathcal{D}\mu[\alpha], \tag{39}$$

while the classical action functional becomes, after integrating by parts, using the harmonic properties of γ and the nilpotency of the exterior derivative operators, and dropping surface terms:

$$S = -\langle \beta, \star\delta d\delta\beta \rangle \ .$$

Note that S depends only the coexact (transverse) part of A. Using (39) and integrating over β yields:

$$Z = \int \mathcal{D}\mu[\gamma] \det{}^{-1/2} (\star\delta d\delta) \det{}^{1/2} (\Delta) \det{}^{1/2} (d\delta) \, .$$

Also, it was proven in [34] that

$$\det(\star\delta d\delta) = \det{}^{1/2}((d\delta d)(\delta d\delta)) = \det{}^{\frac{3}{2}}(d\delta).$$

As a consequence of Hodge duality we have the identity

$$\det(\delta d) = \det(d\delta),$$

from which it follows that

$$Z = \int \mathcal{D}\mu[\gamma] \det{}^{-3/4} \left(\Delta^T_{(1)} \right) \det{}^{1/2} (\Delta) \det{}^{1/2} \left(\Delta^T_{(1)} \right) \ .$$

The operator $\Delta^T_{(1)}$ is the transverse part of the Hodge Laplacian acting on $1-$forms:

$$\Delta^T_{(1)} := (\delta d)_{(1)}.$$

Applying identity for the Hodge Laplacian $\Delta_{(p)}$ [34]

$$\det \left(\Delta_{(p)} \right) = \det \left((\delta d)_{(p)} \right) \det \left((\delta d)_{(p-1)} \right) ,$$

we get

$$\det \left(\Delta^T_{(1)} \right) = \det \left(\Delta_{(1)} \right) / \det (\Delta)$$

and hence

$$Z = \int \mathcal{D}\mu[\gamma] \det{}^{-1/4} \left(\Delta_{(1)} \right) \det{}^{3/4} (\Delta) \, .$$

The space of harmonic forms γ (of any order) is a finite set. Hence, the integration over harmonic forms (3.3.2) is a simple sum.

4 Non-Abelian Gauge Theories

4.1 Intro to non-Abelian theories

QED is the simplest example of a gauge theory coupled to matter based in the Abelian gauge symmetry of local U(1) phase rotations. However, it is possible also to construct gauge theories based on non-Abelian groups. Actually, our knowledge of the strong and weak interactions is based on the use of such non-Abelian generalizations of QED.

Let us consider a gauge group G (see Appendix) with generators T^a, $(a = 1, \ldots, \dim G)$ satisfying the Lie algebra

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c.$$

A gauge field taking values on the Lie algebra of $\mathcal{G}$ can be introduced $A_\mu \equiv A^a_\mu T^a$, which transforms under a gauge transformations as

$$A_\mu \longrightarrow \frac{1}{\mathrm{i}g} U \partial_\mu U^{-1} + U A_\mu U^{-1}, \qquad U = \mathrm{e}^{\mathrm{i}\chi^a(x) T^a},$$

where g is the coupling constant. The associated field strength is defined as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu.$$

Notice that this definition of the $F^a_{\mu\nu}$ reduces to the one used in QED in the Abelian case when $f^{abc} = 0$. In general, however, unlike the case of QED the field strength is not gauge invariant. In terms of $F_{\mu\nu} = F^a_{\mu\nu} T^a$ it transforms as

$$F_{\mu\nu} \longrightarrow U F_{\mu\nu} U^{-1}.$$

The coupling of matter to a non-Abelian gauge field is done by introducing again the *covariant derivative*. For a field $\Phi \longrightarrow U\Phi$ in a representation of $\mathcal{G}$, the covariant derivative is given by

$$D_\mu \Phi = \partial_\mu \Phi - \mathrm{i} g A^a_\mu T^a \Phi.$$

With the help of this we can write a generic Lagrangian for a non-Abelian gauge field coupled to scalars ϕ and spinors ψ as

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \mathrm{i}\overline{\psi} \not{D} \psi + \overline{D_\mu \phi} D^\mu \phi - \overline{\psi} \left[M_1(\phi) + \mathrm{i}\gamma_5 M_2(\phi) \right] \psi - V(\phi).$$

4.2 Yang–Mills theory

In non-Abelian gauge theories, gauge fields are matrix-valued functions of space-time. In SU(N) gauge theories they can be represented by the generators of the corresponding Lie algebra, i.e., gauge fields and their color components are related by

$$A_\mu(x) = A^a_\mu(x) \frac{\lambda^a}{2}, \tag{40}$$

where the color sum runs over the N^2-1 generators. The generators are hermitian, traceless $N \times N$ matrices whose commutation relations are specified by the structure constants f^{abc} [38]

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = \mathrm{i} f^{abc} \frac{\lambda^c}{2}\,.$$

The normalization is chosen as

$$\mathrm{Tr}\left(\frac{\lambda^a}{2} \cdot \frac{\lambda^b}{2}\right) = \frac{1}{2}\delta_{ab}.$$

Most of our applications will be concerned with $SU(2)$ gauge theories; in this case the generators are the *Pauli matrices*,

$$\lambda^a = \tau^a, \qquad \text{with structure constants} \qquad f^{abc} = \epsilon^{abc}.$$

Covariant derivative, field strength tensor, and its color components are respectively defined by

$$D_\mu = \partial_\mu + \mathrm{i}gA_\mu, \tag{41}$$

$$F^{\mu\nu} = \frac{1}{\mathrm{i}g}[D_\mu, D_\nu], \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu. \tag{42}$$

The definition of electric and magnetic fields in terms of the field strength tensor is the same as in electrodynamics

$$E^{ia}(x) = -F^{0ia}(x)\,, \qquad B^{ia}(x) = -\frac{1}{2}\epsilon^{ijk} F^{jka}(x)\ . \tag{43}$$

The dimensions of gauge field and field strength in 4D space-time are

$$[A] = \ell^{-1}, \qquad [F] = \ell^{-2},$$

and therefore in absence of a scale, $A^a_\mu \sim M^a_{\mu\nu}\frac{x^\nu}{x^2}$, with arbitrary constants $M^a_{\mu\nu}$. In general, the action associated with these fields exhibits infrared and ultraviolet logarithmic divergencies. In the following we will discuss

- *Yang–Mills Theories:* Only gauge fields are present. The Yang–Mills Lagrangian is

$$\mathcal{L}_{YM} = -\frac{1}{4}F^{\mu\nu a}F^a_{\mu\nu} = -\frac{1}{2}\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu}) = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2). \tag{44}$$

- *Quantum Chromodynamics:* QCD contains besides the gauge fields (gluons), fermion fields (quarks). Quarks are in the fundamental representation, i.e., in SU(2) they are represented by 2-component color spinors. The QCD Lagrangian is (flavor dependences suppressed)

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_m, \qquad \mathcal{L}_m = \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m)\,\psi, \tag{45}$$

  with the action of the covariant derivative on the quarks given by

$$(D_\mu\psi)^i = (\partial_\mu\delta^{ij} + \mathrm{i}gA_\mu^{ij})\,\psi^j, \qquad (i,j = 1\ldots N\,).$$

- *Georgi–Glashow Model:* In the Georgi–Glashow model [39] (non-Abelian Higgs model), the gluons are coupled to a scalar, self-interacting ($V(\phi)$) (Higgs) field in the adjoint representation. The Higgs field has the same representation in terms of the generators as the gauge field (40) and can be thought of as a 3-component color vector in $SU(2)$. Lagrangian and action of the covariant derivative are respectively

$$\mathcal{L}_{GG} = \mathcal{L}_{YM} + \mathcal{L}_m, \qquad \mathcal{L}_m = \frac{1}{2}D_\mu\phi D^\mu\phi - V(\phi), \tag{46}$$

$$(D_\mu\phi)^a = [D_\mu,\phi\,]^{\,a} = (\partial_\mu\delta^{ac} - gf^{abc}A_\mu^b)\phi^c\,. \tag{47}$$

4.2.1 Yang–Mills action

The general principle of least action,

$$\delta S = 0, \qquad \text{with} \quad S = \int \mathcal{L}\,d^4x,$$

applied to the gauge fields,

$$\begin{aligned}\delta S_{YM} &= -\int d^4x\,\mathrm{Tr}\,(F_{\mu\nu}\delta F^{\mu\nu}) = -\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}\frac{2}{\mathrm{i}g}\,[D^\mu,\delta A^\nu]\right)\\ &= 2\int d^4x\,\mathrm{Tr}\,(\delta A^\nu\,[D^\mu,F_{\mu\nu}])\end{aligned}$$

gives the inhomogeneous field equations [38]

$$[D_\mu, F^{\mu\nu}] = j^\nu, \tag{48}$$

with j^ν the color current associated with the matter fields

$$j^{a\nu} = \frac{\delta\mathcal{L}_m}{\delta A_\nu^a}. \tag{49}$$

For QCD and the Georgi–Glashow model, these currents are given respectively by

$$j^{a\nu} = g\bar{\psi}\gamma^{\nu}\frac{\tau^{a}}{2}\psi, \qquad j^{a\nu} = gf^{abc}\phi^{b}(D^{\nu}\phi)^{c}\,. \tag{50}$$

As in electrodynamics, the homogeneous field equations for the Yang–Mills field strength

$$\left[D_{\mu}, \tilde{F}^{\mu\nu}\right] = 0,$$

with the dual field strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\,\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho},$$

are obtained as the *Jacobi identities* of the covariant derivative,

$$[D_{\mu}, [D_{\nu}, D_{\rho}]] + [D_{\nu}, [D_{\rho}, D_{\mu}]] + [D_{\rho}, [D_{\nu}, D_{\mu}]] = 0.$$

4.2.2 Gauge transformations

Gauge transformations change the color orientation of the matter fields locally, i.e., in a space-time dependent manner, and are defined as

$$U(x) = \exp\{\mathrm{i}g\alpha(x)\} = \exp\left\{\mathrm{i}g\alpha^{a}(x)\,\frac{\tau^{a}}{2}\right\},$$

with the arbitrary gauge function $\alpha^{a}(x)$. Matter fields transform covariantly with U

$$\psi v U\psi, \qquad \phi \longrightarrow U\phi U^{\dagger}. \tag{51}$$

The transformation property of A is chosen such that the covariant derivatives of the matter fields $D_{\mu}\psi$ and $D_{\mu}\phi$ transform as the matter fields ψ and ϕ respectively. As in electrodynamics, this requirement makes the gauge fields transform inhomogeneously [38]

$$A_{\mu}(x) \longrightarrow U(x)\left(A_{\mu}(x) + \frac{1}{\mathrm{i}g}\partial_{\mu}\right)U^{\dagger}(x) = A_{\mu}^{[U]}(x) \tag{52}$$

resulting in a covariant transformation law for the field strength

$$F_{\mu\nu} \longrightarrow UF_{\mu\nu}U^{\dagger}. \tag{53}$$

Under infinitesimal gauge transformations ($|g\alpha^{a}(x)| \ll 1$)

$$A_{\mu}^{a}(x) \longrightarrow A_{\mu}^{a}(x) - \partial_{\mu}\alpha^{a}(x) - gf^{abc}\alpha^{b}(x)\,A_{\mu}^{c}(x)\,. \tag{54}$$

As in electrodynamics, gauge fields which are gauge transforms of $A_\mu = 0$ are called pure gauges and are, according to (52), given by

$$A_\mu^{pg}(x) = U(x)\,\frac{1}{\mathrm{i}g}\,\partial_\mu U^\dagger(x)\ . \tag{55}$$

Physical observables must be independent of the choice of gauge (coordinate system in color space). Local quantities such as the Yang–Mills action density $\mathrm{Tr}\,(F^{\mu\nu}(x)F_{\mu\nu}(x))$ or matter field bilinears like $\bar\psi(x)\psi(x), \phi^a(x)\phi^a(x)$ are gauge invariant, i.e., their value does not change under local gauge transformations. One also introduces non-local quantities which, in generalization of the transformation law (53) for the field strength, change homogeneously under gauge transformations. In this construction a basic building block is the path ordered integral

$$\Omega(x,y,\mathcal{C}) = P\exp\left\{-\mathrm{i}g\int_{s_0}^{s} d\sigma \frac{dx^\mu}{d\sigma} A_\mu(x(\sigma))\right\} = P\exp\left\{-\mathrm{i}g\int_{\mathcal{C}} dx^\mu A_\mu\right\}. \tag{56}$$

It describes a gauge string between the space-time points $x = x(s_0)$ and $y = x(s)$. Ω satisfies the differential equation

$$\frac{d\Omega}{ds} = -\mathrm{i}g\frac{dx^\mu}{ds}A_\mu\Omega. \tag{57}$$

Gauge transforming this differential equation yields the transformation property of Ω

$$\Omega(x,y,\mathcal{C}) \longrightarrow U(x)\,\Omega(x,y,\mathcal{C})\,U^\dagger(y)\,. \tag{58}$$

With the help of Ω, non-local, gauge invariant quantities like

$$\mathrm{Tr}\,(F^{\mu\nu}(x)\Omega(x,y,\mathcal{C})\,F_{\mu\nu}(y))\,, \qquad \bar\psi(x)\Omega(x,y,\mathcal{C})\,\psi(y),$$

or closed gauge strings, the following SU(N)–Wilson loops

$$W_{\mathcal{C}} = \frac{1}{N}\mathrm{Tr}\ (\Omega(x,x,\mathcal{C})) \tag{59}$$

can be constructed. For pure gauges (55), the differential equation (57) is solved by

$$\Omega^{pg}(x,y,\mathcal{C}) = U(x)\,U^\dagger(y). \tag{60}$$

While $\bar\psi(x)\Omega(x,y,\mathcal{C})\,\psi(y)$ is an operator which connects the vacuum with meson states for $SU(2)$ and $SU(3)$, fermionic baryons appear only in $SU(3)$ in which gauge invariant states containing an odd number of fermions can be constructed. In SU(3) a point-like gauge invariant baryonic state is obtained by creating three quarks in a color antisymmetric state at the same space-time point

$$\psi(x) \sim \epsilon^{abc}\psi^a(x)\psi^b(x)\psi^c(x).$$

Under gauge transformations,

$$\begin{aligned}\psi(x) &\longrightarrow \epsilon^{abc}U_{a\alpha}(x)\psi^{\alpha}(x)U_{b\beta}(x)\psi^{\beta}(x)U_{c\gamma}(x)\psi^{\gamma}(x)\\ &= \det\left(U(x)\right)\epsilon^{abc}\psi^{a}(x)\psi^{b}(x)\psi^{c}(x)\,.\end{aligned}$$

Operators that create finite size baryonic states must contain appropriate gauge strings as given by the following expression

$$\psi(x,y,z)\sim\epsilon^{abc}[\Omega(u,x,\mathcal{C}_1)\psi(x)]^{a}\,[\Omega(u,y,\mathcal{C}_2)\psi(y)]^{b}\,[\Omega(u,z,\mathcal{C}_3)\psi(z)]^{c}\,.$$

The presence of these gauge strings makes ψ gauge invariant as is easily verified with the help of the transformation property (58). Thus, gauge invariance is enforced by color exchange processes taking place between the quarks.

4.3 Quantization of Yang–Mills theory

Gauge theories are formulated in terms of redundant variables. Only in this way, a covariant, local representation of the dynamics of gauge degrees of freedom is possible. For quantization of the theory both canonically or in the path integral, redundant variables have to be eliminated. This procedure is called gauge fixing. It is not unique and the implications of a particular choice are generally not well understood. In the path integral one performs a sum over all field configurations. In gauge theories this procedure has to be modified by making use of the decomposition of the space of gauge fields into equivalence classes, the gauge orbits. Instead of summing in the path integral over formally different but physically equivalent fields, the integration is performed over the equivalence classes of such fields, i.e., over the corresponding gauge orbits. The value of the action is gauge invariant, i.e., the same for all members of a given gauge orbit. Therefore, the action is seen to be a functional defined on classes (gauge orbits) [38]. Also the integration measure

$$\mathcal{D}\left[A\right]=\prod_{x,\mu,a}dA_{\mu}^{a}\left(x\right).$$

is gauge invariant since shifts and rotations of an integration variable do not change the value of an integral. Therefore, in the naive path integral

$$Z[A]=\int\mathcal{D}\left[A\right]\mathrm{e}^{\mathrm{i}S[A]}\propto\int\prod_{x}dU\left(x\right).$$

a 'volume' associated with the gauge transformations $\prod_x dU\left(x\right)$ can be factorized and thereby the integration be performed over the gauge orbits. To turn this property into a working algorithm, redundant variables are eliminated by imposing a *gauge condition,* $f[A]=0$, which is supposed to eliminate all gauge copies of a certain field configuration A.

In other words, the functional f has to be chosen such that, for arbitrary field configurations, the equation, $f[A^{[U]}] = 0$, determines uniquely the gauge transformation U. If successful, the set of all gauge equivalent fields, the gauge orbit, is represented by exactly one representative. In order to write down an integral over gauge orbits, we insert into the integral the gauge–fixing $\delta-$functional

$$\delta\left[f\left(A\right)\right] = \prod_{x}\prod_{a=1}^{N^2-1}\delta\left[f^a\left(A\left(x\right)\right)\right].$$

This modification of the integral however changes the value depending on the representative chosen, as the following elementary identity shows

$$\delta\left(g\left(x\right)\right) = \frac{\delta\left(x-a\right)}{\left|g'\left(a\right)\right|}, \qquad g\left(a\right)=0.$$

This difficulty is circumvented with the help of the Faddeev–Popov determinant $\Delta_f\left[A\right]$ defined implicitly by

$$\Delta_f\left[A\right]\int\mathcal{D}\left[U\right]\delta\left[f\left(A^{[U]}\right)\right] = 1.$$

Multiplication of the path integral $Z[A]$ with the above "1" and taking into account the gauge invariance of the various factors yields

$$\begin{aligned} Z[A] &= \int\mathcal{D}\left[U\right]\int\mathcal{D}\left[A\right]\mathrm{e}^{\mathrm{i}S[A]}\Delta_f\left[A\right]\delta\left[f\left(A^{[U]}\right)\right] \\ &= \int\mathcal{D}\left[U\right]\int\mathcal{D}\left[A\right]\mathrm{e}^{\mathrm{i}S\left[A^{[U]}\right]}\Delta_f\left[A^{[U]}\right]\delta\left[f\left(A^{[U]}\right)\right]. \end{aligned}$$

The gauge volume has been factorized and, being independent of the dynamics, can be dropped. In summary, the final definition of the generating functional for gauge theories is given in terms of a *sum over gauge orbits,*

$$Z\left[J\right] = \int\mathcal{D}\left[A\right]\Delta_f\left[A\right]\delta\left(f\left[A\right]\right)\mathrm{e}^{\mathrm{i}S[A]+\mathrm{i}\int d^4xJ^\mu A_\mu}.$$

4.3.1 Faddeev–Popov determinant

For the calculation of $\Delta_f\left[A\right]$, we first consider the change of the gauge condition $f^a\left[A\right]$ under infinitesimal gauge transformations. Taylor expansion

$$\begin{aligned} f_x^a\left[A^{[U]}\right] &\approx f_x^a\left[A\right] + \int d^4y\sum_{b,\mu}\frac{\delta f_x^a\left[A\right]}{\delta A_\mu^b\left(y\right)}\delta A_\mu^b\left(y\right) \\ &= f_x^a\left[A\right] + \int d^4y\sum_{b}M\left(x,y;a,b\right)\alpha^b\left(y\right), \end{aligned}$$

with δA^a_μ given by infinitesimal gauge transformations,

$$\begin{aligned} A^a_\mu(x) &\rightarrow A^a_\mu(x) - \partial_\mu \alpha^a(x) - g f^{abc} \alpha^b(x) A^c_\mu(x), \qquad \text{yields} \\ M(x,y;a,b) &= \left(\partial_\mu \delta^{b,c} + g f^{bcd} A^d_\mu(y)\right) \frac{\delta f^a_x[A]}{\delta A^c_\mu(y)}. \end{aligned}$$

In the second step, we compute the integral

$$\Delta_f^{-1}[A] = \int \mathcal{D}[U]\, \delta\left[f\left(A^{[U]}\right)\right],$$

by expressing the integration $\mathcal{D}[U]$ as an integration over the gauge functions α. We finally change to the variables $\beta = M\alpha$,

$$\Delta_f^{-1}[A] = |\det M|^{-1} \int \mathcal{D}[\beta]\, \delta[f(A) - \beta],$$

and arrive at the final expression for the Faddeev–Popov determinant [38]

$$\Delta_f[A] = |\det M|.$$

Examples:

- Lorentz gauge

$$\begin{aligned} f^a_x(A) &= \partial^\mu A^a_\mu(x) - \chi^a(x), \\ M(x,y;a,b) &= -\left(\delta^{ab}\Box - g f^{abc} A^c_\mu(y)\, \partial^\mu_y\right) \delta^{(4)}(x-y). \end{aligned}$$

- Coulomb gauge

$$\begin{aligned} f^a_x(A) &= \mathrm{div}\mathbf{A}^a(x) - \chi^a(x), \\ M(x,y;a,b) &= \left(\delta^{ab}\Delta + g f^{abc} \mathbf{A}^c(y)\, \nabla_y\right) \delta^{(4)}(x-y). \end{aligned}$$

- Axial gauge

$$\begin{aligned} f^a_x(A) &= n^\mu A^a_\mu(x) - \chi^a(x), \\ M(x,y;a,b) &= -\delta^{ab} n_\mu \partial^\mu_y \delta^{(4)}(x-y). \end{aligned}$$

4.4 Basics of Conformal field theory

A conformal field theory (CFT) is a quantum field theory (or, a statistical mechanics model at the critical point) that is invariant under *conformal transformations*. Conformal field theory is often studied in 2D where there is an infinite-dimensional group of local conformal transformations, described by the holomorphic functions. CFT has important applications in string theory, statistical mechanics, and condensed matter physics. For a good introduction to CFT see [BPZ84, DMS97]. We consider here only chiral CFTs in 2D (see [40]), where 'chiral'[14] means that all of our fields will be functions of a complex number $z = x + iy$ only and not functions of its conjugate $\bar{z}$.

To formally describe a 2D CFT we give its 'conformal data', including a set of primary fields, each with a conformal dimension Δ, a table of fusion rules of these fields and a central charge c. Data for three CFTs are given in Table 1.

The *operator product expansion* (OPE) describes what happens to two fields when their positions approach each other. We write the OPE for two arbitrary fields ϕ_i and ϕ_j as

$$\lim_{z\to w} \phi_i(z)\phi_j(w) = \sum_k C_{ij}^k (z-w)^{\Delta_k-\Delta_i-\Delta_j}\, \phi_k(w),$$

where the *structure constants* C_{ij}^k are only nonzero as indicated by the fusion table. Note that the OPE works *inside* a correlator. For example, in the $\mathbb{Z}_3$ para-fermion CFT (see Table 1), since $\sigma_1 \times \psi_1 = \epsilon$, for arbitrary fields ϕ_i we have [40]

$$\lim_{z\to w} \langle\, \phi_1(z_1)\dots\phi_M(z_M)\,\sigma_1(z)\psi_1(w)\,\rangle \sim$$
$$(z-w)^{2/5-1/15-2/3}\langle\, \phi_1(z_1)\dots\phi_M(z_M)\epsilon(w)\,\rangle.$$

In addition to the OPE, there is also an important 'neutrality' condition: a correlator is zero unless all of the fields can fuse together to form the identity field $\mathbf{1}$. For example, in the $\mathbb{Z}_3$ para-fermion field theory $\langle\psi_2\psi_1\rangle \neq 0$ since $\psi_2 \times \psi_1 = \mathbf{1}$, but $\langle\psi_1\psi_1\rangle = 0$ since $\psi_1 \times \psi_1 = \psi_2 \neq \mathbf{1}$.

Let us look at what happens when a fusion has more than one possible result. For example, in the Ising CFT, $\sigma \times \sigma = \mathbf{1} + \psi$. Using the OPE, we have

$$\lim_{w_1\to w_2} \sigma(w_1)\sigma(w_2) \sim \frac{\mathbf{1}}{(w_1-w_2)^{1/8}} + (w_1-w_2)^{3/8}\,\psi, \tag{61}$$

where we have neglected the constants C_{ij}^k. If we consider $\langle\sigma\sigma\rangle$, the neutrality condition picks out only the first term in (61) where the two σ's fuse to form $\mathbf{1}$. Similarly, $\langle\sigma\sigma\psi\rangle$ results in

[14] In general, a chiral field is a holomorphic field $W(z)$ which transforms as

$$L_n W(z) = -z^{n+1}\frac{\partial}{\partial z}W(z) - (n+1)\Delta z^n W(z), \qquad \text{with} \qquad \bar{L}_n W(z) = 0,$$

and similarly for an anti-chiral field. Here, Δ is the conformal weight of the chiral field W.

Chiral Bose Vertex: ($c = 1$)

	Δ
$e^{i\alpha\phi}$	$\alpha^2/2$

$\times$	$e^{i\alpha\phi}$
$e^{i\beta\phi}$	$e^{i(\alpha+\beta)\phi}$

Ising CFT: ($c = 1/2$)

	Δ
ψ	$1/2$
σ	$1/16$

$\times$	ψ	σ
ψ	$\mathbf{1}$	
σ	σ	$\mathbf{1}+\psi$

$\mathbb{Z}_3$ Parafermion CFT: ($c = 4/5$)

	Δ
ψ_1	$2/3$
ψ_2	$2/3$
σ_1	$1/15$
σ_2	$1/15$
ϵ	$2/5$

$\times$	ψ_1	ψ_2	σ_1	σ_2	ϵ
ψ_1	ψ_2				
ψ_2	$\mathbf{1}$	ψ_1			
σ_1	ϵ	σ_2	$\sigma_2+\psi_1$		
σ_2	σ_1	ϵ	$\mathbf{1}+\epsilon$	$\sigma_1+\psi_2$	
ϵ	σ_2	σ_1	$\sigma_1+\psi_2$	$\sigma_2+\psi_1$	$\mathbf{1}+\epsilon$

Table 1: Conformal data for three CFTs. Given is the list of primary fields in the CFT with their conformal dimension Δ, as well as the fusion table. In addition, every CFT has an identity field $\mathbf{1}$ with dimension $\Delta = 0$ which fuses trivially with any field ($\mathbf{1} \times \phi_i = \phi_i$ for any ϕ_i). Note that fusion tables are symmetric so only the lower part is given. In the Ising CFT the field ψ is frequently notated as ϵ. This fusion table indicates the nonzero elements of the fusion matrix N_{ab}^c. For example in the $\mathbb{Z}_3$ CFT, since $\sigma_1 \times \sigma_2 = \mathbf{1} + \epsilon$, $N_{\sigma_1\sigma_2}^{1} = N_{\sigma_1\sigma_2}^{\epsilon} = 1$ and $N_{\sigma_1\sigma_2}^{c} = 0$ for all c not equal to $\mathbf{1}$ or ϵ.

the second term of (61) where the two σ's fuse to form ψ which then fuses with the additional ψ to make $\mathbf{1}$.

Fields may also fuse to form the identity in more than one way. For example, in the correlator $\langle\sigma(w_1)\sigma(w_2)\sigma(w_3)\sigma(w_4)\rangle$ of the Ising CFT, the identity is obtained via two possible fusion paths — resulting in two different so-called 'conformal blocks'. On the one hand, one can fuse $\sigma(w_1)$ and $\sigma(w_2)$ to form $\mathbf{1}$ and similarly fuse $\sigma(w_3)$ and $\sigma(w_4)$ to form $\mathbf{1}$. Alternately, one can fuse $\sigma(w_1)$ and $\sigma(w_2)$ to form ψ and fuse $\sigma(w_3)$ and $\sigma(w_4)$ to form ψ then fuse the two resulting ψ fields together to form $\mathbf{1}$. The correlator generally gives a linear combination of the possible resulting conformal blocks. We should thus think of such a correlator as living in a vector space rather than having a single value. (If we instead choose to fuse 1 with 3, and 2 with 4, we would obtain two blocks which are linear combinations of the ones found by fusing 1 with 2 and 3 with 4. The resulting vectors space, however, is independent of the order of fusion). Crucially, transporting the coordinates w_i around each other makes a rotation within this vector space.

To be clear about the notion of conformal blocks, let us look at the explicit form of the Ising CFT correlator [40]

$$\lim_{w\to\infty}\langle\sigma(0)\sigma(z)\sigma(1)\sigma(w)\rangle = a_+\,F_+ + a_-\,F_-,$$
$$F_\pm(z) \sim (wz(1-z))^{-1/8}\sqrt{1\pm\sqrt{1-z}},$$

where a_+ and a_- are arbitrary coefficients. When $z\to 0$ we have $F_+\sim z^{-1/8}$ whereas $F_-\sim z^{3/8}$. Comparing to (61) we conclude that F_+ is the result of fusing $\sigma(0)\times\sigma(z)\to\mathbf{1}$ whereas F_- is the result of fusing $\sigma(0)\times\sigma(z)\to\psi$. As z is taken in a clockwise circle around the point $z=1$, the inner square-root changes sign, switching F_+ and F_-. Thus, this 'braiding' (or 'monodromy') operation transforms

$$\binom{a_+}{a_-} \to \mathrm{e}^{2\pi\mathrm{i}/8}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\binom{a_+}{a_-}$$

Having a multiple valued correlator (I.e., multiple conformal blocks) is a result of having such branch cuts. Braiding the coordinates (w's) around each other results in the correlator changing values within its allowable vector space.

A useful technique for counting conformal blocks is the *Bratteli diagram*. In Figure 1 we give the Bratteli diagram for the fusion of multiple σ fields in the Ising CFT. Starting with $\mathbf{1}$ at the lower left, at each step moving from the left to the right, we fuse with one more σ field. At the first step, the arrow points from $\mathbf{1}$ to $\boldsymbol{\sigma}$ since $\mathbf{1}\times\sigma=\sigma$. At the next step σ fuses with σ to produce either ψ or $\mathbf{1}$ and so forth. Each conformal block is associated with a path through the diagram. Thus to determine the number of blocks in $\langle\sigma\sigma\sigma\sigma\rangle$ we count the number of paths of four steps in the diagram starting at the lower left and ending at $\mathbf{1}$.

A particularly important CFT is obtained from a free Bose field theory in (1+1)D by keeping only the left moving modes. The free chiral Bose field $\phi(z)$, which is a sum of left

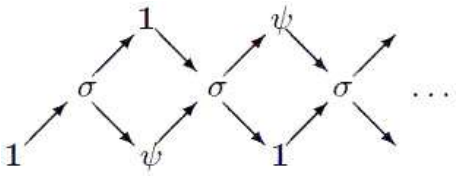


Figure 1: Bratteli diagram for fusion of multiple σ_1 fields in the $\mathbb{Z}_3$ para-fermion CFT (modified and adapted from [40]).

moving creation and annihilation operators, has a correlator $\langle\phi(z)\phi(z')\rangle = -\log(z-z')$. We then define the normal ordered 'chiral vertex operator', $\mathrm{e}^{\mathrm{i}\alpha\phi(z)}$, which is a conformal field. Since ϕ is a free field, Wick's theorem can be used to obtain [DMS97]

$$\left\langle \mathrm{e}^{\mathrm{i}\alpha_1\phi}(z_1)\dots\mathrm{e}^{\mathrm{i}\alpha_N\phi}(z_N)\right\rangle = \mathrm{e}^{-\sum_{i<j}\alpha_i\alpha_j\langle\phi(z_i)\phi(z_j)\rangle} = \prod_{i<j}\ (z_i-z_j)^{\alpha_i\alpha_j}.$$

5 Appendix

5.1 Manifolds and bundles

Geometrically, a manifold is a nonlinear (i.e., curved) space which is locally homeomorphic (i.e., topologically equivalent) to a linear (i.e., flat) Euclidean space $\mathbb{R}^n$; e.g., in a magnifying glass, each local patch of the apple surface looks like a plane, although globally (as a whole) the apple surface is totally different from the plane. Physically, a configuration manifold is a set of all degrees of freedom of a dynamical system.

More precisely, consider Consider a set M (see Figure 2) which is a *candidate* for a manifold. Any point $x \in M$[15] has its *Euclidean chart*, given by a 1–1 and *onto* map $\varphi_i : M \to \mathbb{R}^n$, with its *Euclidean image* $V_i = \varphi_i(U_i)$. More precisely, a chart φ_i is defined by

$$\varphi_i : M \supset U_i \ni x \mapsto \varphi_i(x) \in V_i \subset \mathbb{R}^n,$$

where $U_i \subset M$ and $V_i \subset \mathbb{R}^n$ are open sets.

Clearly, any point $x \in M$ can have several different charts (see Figure 2). Consider a case of two charts, $\varphi_i, \varphi_j : M \to \mathbb{R}^n$, having in their images two open sets, $V_{ij} = \varphi_i(U_i \cap U_j)$ and $V_{ji} = \varphi_j(U_i \cap U_j)$. Then we have *transition functions* φ_{ij} between them,

$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : V_{ij} \to V_{ji}, \qquad \text{locally given by} \qquad \varphi_{ij}(x) = \varphi_j(\varphi_i^{-1}(x)).$$

If transition functions φ_{ij} exist, then we say that two charts, φ_i and φ_j are *compatible*. Transition functions represent a general (nonlinear) *transformations of coordinates*, which are the core of classical *tensor calculus*.

[15] Note that sometimes we will denote the point in a manifold M by m, and sometimes by x (thus implicitly assuming the existence of coordinates $x = (x^i)$).

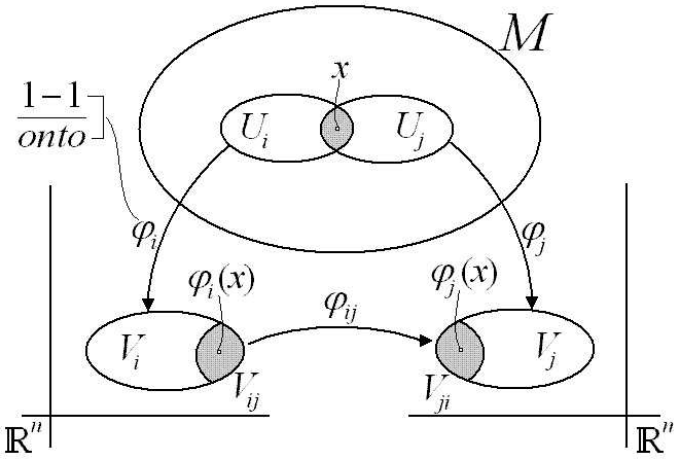


Figure 2: Geometric picture of the manifold concept.

A set of compatible charts $\varphi_i : M \to \mathbb{R}^n$, such that each point $x \in M$ has its Euclidean image in at least one chart, is called an *atlas*. Two atlases are *equivalent* iff all their charts are compatible (i.e., transition functions exist between them), so their union is also an atlas. A *manifold structure* is a class of equivalent atlases.

Finally, as charts $\varphi_i : M \to \mathbb{R}^n$ were supposed to be 1-1 and onto maps, they can be either *homeomorphisms*, in which case we have a *topological* (C^0) manifold, or *diffeomorphisms*, in which case we have a *smooth* (C^k) manifold.

On the other hand, tangent and cotangent bundles, TM and T^*M, respectively, of a smooth manifold M, are special cases of a more general geometrical object called *fibre bundle*, where the word *fiber* V of a map $\pi : Y \to X$ denotes the *preimage* $\pi^{-1}(x)$ of an element $x \in X$. It is a space which *locally* looks like a product of two spaces (similarly as a manifold locally looks like Euclidean space), but may possess a different *global* structure. To get a visual intuition behind this fundamental geometrical concept, we can say that a fibre bundle Y is a *homeomorphic generalization* of a *product space* $X \times V$ (see Figure 3), where X and V are called the *base* and the *fibre*, respectively. $\pi : Y \to X$ is called the *projection*, $Y_x = \pi^{-1}(x)$ denotes a fibre over a point x of the base X, while the map $f = \pi^{-1} : X \to Y$ defines the *cross–section*, producing the *graph* $(x, f(x))$ in the bundle Y (e.g., in case of a tangent bundle, $f = \dot{x}$ represents a velocity vector–field, so that the graph in a the bundle Y reads $(x, \dot{x})$).

A principal $G-$bundle is a bundle $\pi : Y \longrightarrow X$ generated by a Lie group G (see below) such that the group G preserves the fibers of the bundle Y.

The main reason why we need to study fibre bundles is that *all dynamical objects* (including vectors, tensors, differential forms and gauge potentials) are their *cross–sections*, representing *generalizations of graphs of continuous functions*. For more technical details,

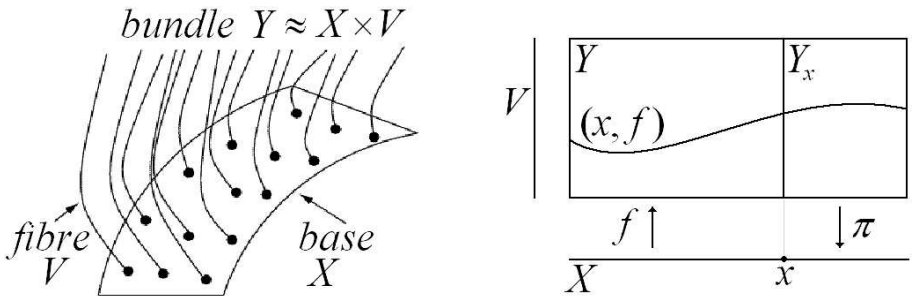


Figure 3: A sketch of a fibre bundle $Y \approx X \times V$ as a generalization of a product space $X \times V$; left – main components; right – a few details (see text for explanation).

see [19, 20].

5.2 Lie groups

A Lie group is a both a group and a manifold. More precisely, a *Lie group* is a smooth manifold M that has at the same time a group $G-$structure consistent with its manifold $M-$structure in the sense that *group multiplication* $\mu : G \times G \to G, \quad (g, h) \mapsto gh$ and the*group inversion* $\nu : G \to G, \quad g \mapsto g^{-1}$ are smooth functions. A point $e \in G$ is called the *group identity element*. For any Lie group G in a neighborhood of its identity element e it can be expressed in terms of a set of generators T^a $(a = 1, \ldots, \dim G)$ as

$$D(g) = \exp[-\mathrm{i}\alpha_a T^a] \equiv \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \alpha_{a_1} \ldots \alpha_{a_n} T^{a_1} \ldots T^{a_n},$$

where $\alpha_a \in \mathbb{C}$ are a set of coordinates of M in a neighborhood of e. Because of the general *Baker–Campbell–Haussdorf formula*, the multiplication of two group elements is encoded in the value of the commutator of two generators, that in general has the form

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c,$$

where $f^{abc} \in \mathbb{C}$ are called the structure constants. The set of generators with the commutator operation form the Lie algebra associated with the Lie group. Hence, given a representation of the Lie algebra of generators we can construct a representation of the group by exponentiation (at least locally near the identity).

In particular, for $SU(2)-$group, each group element is labeled by three real numbers α_k, $(k = 1, 2, 3)$. We have two basic representations: one is the fundamental representation (or spin $\frac{1}{2}$) defined by

$$D_{\frac{1}{2}}(\alpha_k) = \mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha_k \sigma^k},$$

with σ^{i} the Pauli matrices. The second one is the adjoint (or spin 1) representation which can be written as

$$D_1(\alpha_k) = \mathrm{e}^{-\mathrm{i}\alpha_k J^k}, \qquad \text{where}$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad J^3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Actually, J^k generate rotations around the x, y and z axis respectively.

Let M be a smooth manifold. An *action of a Lie group* G (with the unit element e) on M is a smooth map $\phi : G \times M \to M$, such that for all $x \in M$ and $g, h \in G$, (i) $\phi(e, x) = x$ and (ii) $\phi\left(g, \phi(h, x)\right) = \phi(gh, x)$. For more technical details, see [19, 20].

5.3 Differential forms and Stokes theorem

Given the space of exterior differential $p-$forms $\Omega^p(M)$ on a smooth manifold M, we have the *exterior derivative operator* $d : \Omega(M) \to \Omega^{p+1}(M)$ which generalizes ordinary vector differential operators (*grad*, *div* and *curl* see [28, 30, 19]) and transforms $p-$forms ω into $(p+1)-$forms $d\omega$, with the main property: $dd = d^2 = 0$. Given a $p-$form $\alpha \in \Omega^p(M)$ and a $q-$form $\beta \in \Omega^q(M)$, their exterior product is a $(p+q)-$form $\alpha \wedge \beta \in \Omega^{p+q}(M)$, where $\wedge$ is their anti-commutative exterior (or, 'wedge') product.

As differential forms are meant for integration, we have a generalization of all integral theorems from vector calculus in the form of the Stokes theorem: for the $p-$form ω, in an oriented nD domain C, which is a $p-$chain with a $(p-1)-$boundary ∂C,

$$\int_{\partial C} \omega = \int_C d\omega. \tag{62}$$

For any $p-$chain on a manifold M, *the boundary of a boundary is zero* [17], that is, $\partial\partial C = \partial^2 = 0$.

A $p-$form β is called *closed* if its exterior derivative $d = \partial_i dx^i$ is equal to zero, $d\beta = 0$.From this condition one can see that the closed form (the *kernel* of the exterior derivative operator d) is conserved quantity. Therefore, closed $p-$forms possess certain invariant properties, physically corresponding to the conservation laws (see e.g., [35, 20]).

Also, a $p-$form β that is an exterior derivative of some $(p-1)-$form α, $\beta = d\alpha$,is called *exact* (the *image* of the exterior derivative operator d). By Poincaré lemma, exact forms prove to be closed automatically, $d\beta = d(d\alpha) = 0$.

Since $d^2 = 0$, *every exact form is closed.* The converse is only partially true, by Poincaré lemma: every closed form is *locally exact*. In particular, there is a Poincaré lemma for contractible manifolds: Any closed form on a smoothly contractible manifold is exact. The Poincaré lemma is a generalization and unification of two well–known facts in vector calculus: (i) If $\operatorname{curl} F = 0$, then locally $F = \operatorname{grad} f$; and (ii) If $\operatorname{div} F = 0$, then locally $F = \operatorname{curl} G$.

A *cycle* is a $p-$chain, (or, an oriented $p-$domain) $C \in \mathcal{C}_p(M)$ such that $\partial C = 0$. A *boundary* is a chain C such that $C = \partial B$, for any other chain $B \in \mathcal{C}_p(M)$. Similarly, a *cocycle* (i.e., a *closed form*) is a cochain ω such that $d\omega = 0$. A *coboundary* (i.e., an *exact form*) is a cochain ω such that $\omega = d\theta$, for any other cochain θ. All exact forms are closed ($\omega = d\theta \Rightarrow d\omega = 0$) and all boundaries are cycles ($C = \partial B \Rightarrow \partial C = 0$). Converse is true only for smooth contractible manifolds, by Poincaré lemma.

Integration on a smooth manifold M should be thought of as a nondegenerate bilinear pairing $(,)$ between $p-$forms and $p-$chains (spanning a finite domain on M). Duality of $p-$forms and $p-$chains on M is based on the de Rham's 'period', defined as [28, 33]

$$\text{Period} := \int_C \omega := (C, \omega)\,,$$

where C is a cycle, ω is a cocycle, while $\langle C, \omega \rangle = \omega(C)$ is their inner product $(C, \omega) : \Omega^p(M) \times \mathcal{C}_p(M) \to \mathbb{R}$. From the Poincaré lemma, a closed $p-$form ω is exact iff $(C, \omega) = 0$.

The fundamental topological duality is based on the Stokes theorem (62), which can be re written as

$$(\partial C, \omega) = (C, d\omega)\,,$$

where ∂C is the boundary of the $p-$chain C oriented coherently with C on M. While the *boundary operator* ∂ is a global operator, the coboundary operator d is local, and thus more suitable for applications. The main property of the exterior differential,

$$d \circ d \equiv d^2 = 0 \quad \Longrightarrow \quad \partial \circ \partial \equiv \partial^2 = 0, \qquad \text{(and converse)},$$

can be easily proved using the Stokes' theorem as

$$0 = \left(\partial^2 C, \omega\right) = (\partial C, d\omega) = \left(C, d^2\omega\right) = 0.$$

5.4 De Rham cohomology

In the Euclidean 3D space $\mathbb{R}^3$ we have the following de Rham *cochain complex*

$$0 \to \Omega^0(\mathbb{R}^3) \xrightarrow[\text{grad}]{d} \Omega^1(\mathbb{R}^3) \xrightarrow[\text{curl}]{d} \Omega^2(\mathbb{R}^3) \xrightarrow[\text{div}]{d} \Omega^3(\mathbb{R}^3) \to 0.$$

Using the *closure property* for the exterior differential in $\mathbb{R}^3$, $d \circ d \equiv d^2 = 0$, we get the standard identities from vector calculus

$$\text{curl} \cdot \text{grad} = 0 \qquad \text{and} \qquad \text{div} \cdot \text{curl} = 0.$$

As a duality, in $\mathbb{R}^3$ we have the following *chain complex*

$$0 \leftarrow \mathcal{C}_0(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_1(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_2(\mathbb{R}^3) \xleftarrow{\partial} \mathcal{C}_3(\mathbb{R}^3) \leftarrow 0,$$

(with the closure property $\partial \circ \partial \equiv \partial^2 = 0$) which implies the following three boundaries:

$$C_1 \overset{\partial}{\mapsto} C_0 = \partial(C_1), \qquad C_2 \overset{\partial}{\mapsto} C_1 = \partial(C_2), \qquad C_3 \overset{\partial}{\mapsto} C_2 = \partial(C_3),$$

where $C_0 \in \mathcal{C}_0$ is a 0–boundary (or, a point), $C_1 \in \mathcal{C}_1$ is a 1–boundary (or, a line), $C_2 \in \mathcal{C}_2$ is a 2–boundary (or, a surface), and $C_3 \in \mathcal{C}_3$ is a 3–boundary (or, a hypersurface). Similarly, the de Rham complex implies the following three coboundaries:

$$C^0 \overset{d}{\mapsto} C^1 = d(C^0), \qquad C^1 \overset{d}{\mapsto} C^2 = d(C^1), \qquad C^2 \overset{d}{\mapsto} C^3 = d(C^2),$$

where $C^0 \in \Omega^0$ is 0–form (or, a function), $C^1 \in \Omega^1$ is a 1–form, $C^2 \in \Omega^2$ is a 2–form, and $C^3 \in \Omega^3$ is a 3–form.

In general, on a smooth nD manifold M we have the following de Rham cochain complex [28]

$$0 \to \Omega^0(M) \overset{d}{\longrightarrow} \Omega^1(M) \overset{d}{\longrightarrow} \Omega^2(M) \overset{d}{\longrightarrow} \Omega^3(M) \overset{d}{\longrightarrow} \cdots \overset{d}{\longrightarrow} \Omega^n(M) \to 0,$$

satisfying the closure property on M, $d \circ d \equiv d^2 = 0$.

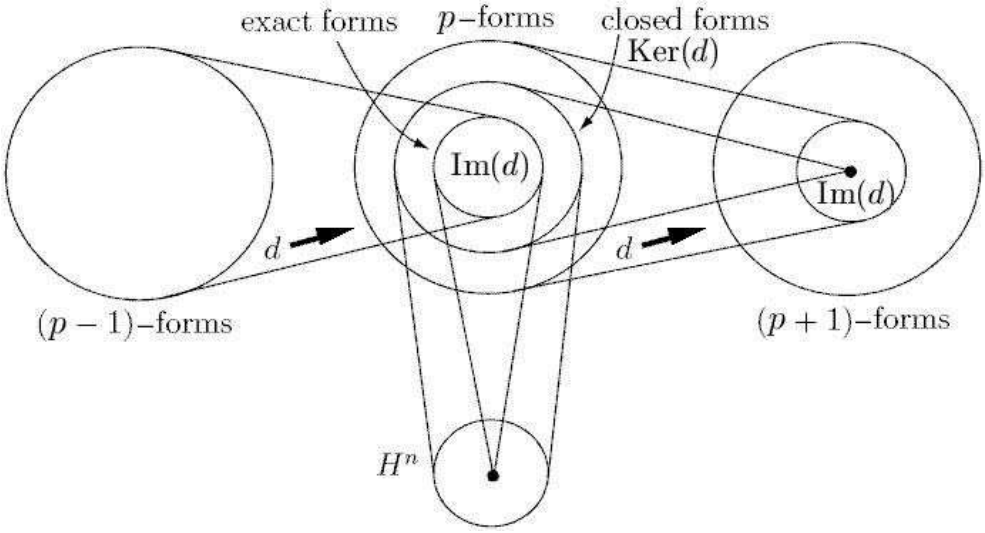


Figure 4: A small portion of the de Rham cochain complex, showing a homomorphism of cohomology groups.

Informally, the de Rham cohomology is the (functional) space of closed differential $p-$forms modulo exact ones on a smooth manifold.

More precisely, the subspace of all closed $p-$forms (cocycles) on a smooth manifold M is the kernel Ker(d) of the de Rham $d-$homomorphism (see Figure 4), denoted by $Z^p(M) \subset$

$\Omega^p(M)$, and the sub-subspace of all exact $p-$forms (coboundaries) on M is the image $\mathrm{Im}(d)$ of the de Rham homomorphism denoted by $B^p(M) \subset Z^p(M)$. The *quotient space*

$$H^p_{DR}(M) := \frac{Z^p(M)}{B^p M} = \frac{\mathrm{Ker}\left(d : \Omega^p(M) \to \Omega^{p+1}(M)\right)}{\mathrm{Im}\left(d : \Omega^{p-1}(M) \to \Omega^p(M)\right)},$$

is called the pth de Rham *cohomology group* of a manifold M. It is a topological invariant of a manifold. Two $p-$cocycles α,$\beta \in \Omega^p(M)$ are *cohomologous*, or belong to the same *cohomology class* $[\alpha] \in H^p(M)$, if they differ by a $(p-1)-$coboundary $\alpha - \beta = d\theta \in \Omega^{p-1}(M)$. The dimension $b_p = \dim H^p(M)$ of the de Rham cohomology group $H^p_{DR}(M)$ of the manifold M is called the Betti number b_p.

Similarly, the subspace of all $p-$cycles on a smooth manifold M is the kernel $\mathrm{Ker}(\partial)$ of the $\partial-$homomorphism, denoted by $Z_p(M) \subset \mathcal{C}_p(M)$, and the sub-subspace of all $p-$boundaries on M is the image $\mathrm{Im}(\partial)$ of the $\partial-$homomorphism, denoted by $B_p(M) \subset \mathcal{C}_p(M)$. Two $p-$cycles C_1,$C_2 \in \mathcal{C}_p$ are *homologous*, if they differ by a $(p-1)-$boundary $C_1 - C_2 = \partial B \in \mathcal{C}_{p-1}(M)$. Then C_1 and C_2 belong to the same *homology class* $[C] \in H_p(M)$, where $H_p(M)$ is the homology group of the manifold M, defined as

$$H_p(M) := \frac{Z_p(M)}{B_p(M)} = \frac{\mathrm{Ker}(\partial : \mathcal{C}_p(M) \to \mathcal{C}_{p-1}(M))}{\mathrm{Im}(\partial : \mathcal{C}_{p+1}(M) \to \mathcal{C}_p(M))},$$

where Z_p is the vector space of cycles and $B_p \subset Z_p$ is the vector space of boundaries on M. The dimension $b_p = \dim H_p(M)$ of the homology group $H_p(M)$ is, by the de Rham theorem, the same Betti number b_p.

If we know the Betti numbers for all (co)homology groups of the manifold M, we can calculate the *Euler–Poincaré characteristic* of M as

$$\chi(M) = \sum_{p=1}^{n} (-1)^p b_p.$$

For example, consider a small portion of the de Rham cochain complex of Figure 4 spanning a space-time 4–manifold M,

$$\Omega^{p-1}(M) \xrightarrow{d_{p-1}} \Omega^p(M) \xrightarrow{d_p} \Omega^{p+1}(M)$$

As we have seen above, cohomology classifies topological spaces by comparing two subspaces of Ω^p: (i) the space of $p-$cocycles, $Z^p(M) = \mathrm{Ker}\, d_p$, and (ii) the space of $p-$coboundaries, $B^p(M) = \mathrm{Im}\, d_{p-1}$. Thus, for the cochain complex of any space-time 4–manifold we have,

$$d^2 = 0 \quad \Rightarrow \quad B^p(M) \subset Z^p(M),$$

that is, every $p-$coboundary is a $p-$cocycle. Whether the converse of this statement is true, according to Poincaré lemma, depends on the particular topology of a space-time 4–manifold.

If every $p-$cocycle is a $p-$coboundary, so that B^p and Z^p are equal, then the cochain complex is exact at $\Omega^p(M)$. In topologically interesting regions of a space-time manifold M, exactness may fail [36], and we measure the failure of exactness by taking the pth cohomology group

$$H^p(M) = Z^p(M)/B^p(M).$$

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