LECTURE 5 – HADRON STATES & CONSEQUENCES OF GROUP THEORY

Contents

- Hadron States in $SU(3)_{flavour}$
- Hadron States in $SU(6)_{flavour \otimes spin}$
- Consequences of Group Theory
- The Standard Model
- Beyond the Standard Model

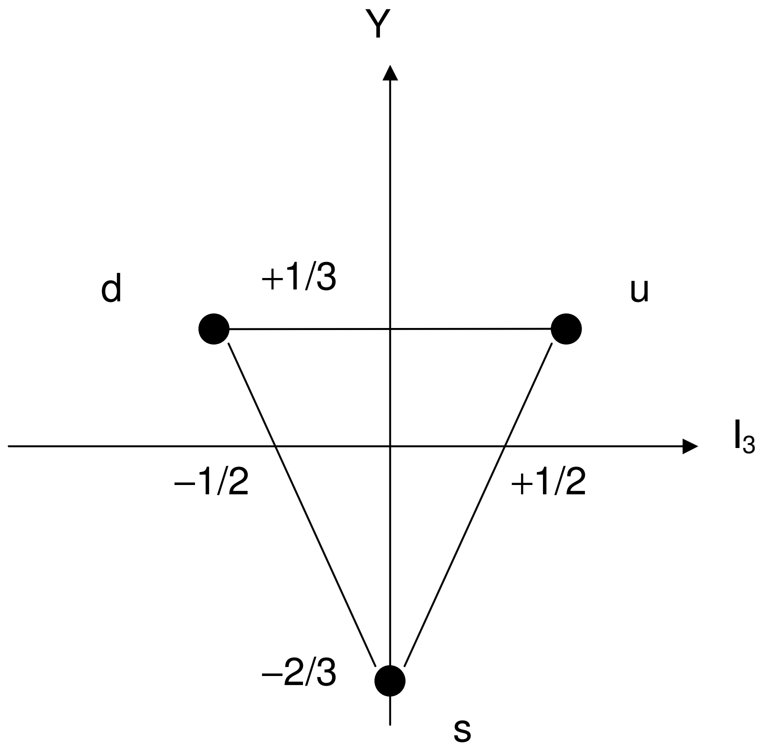
Messages

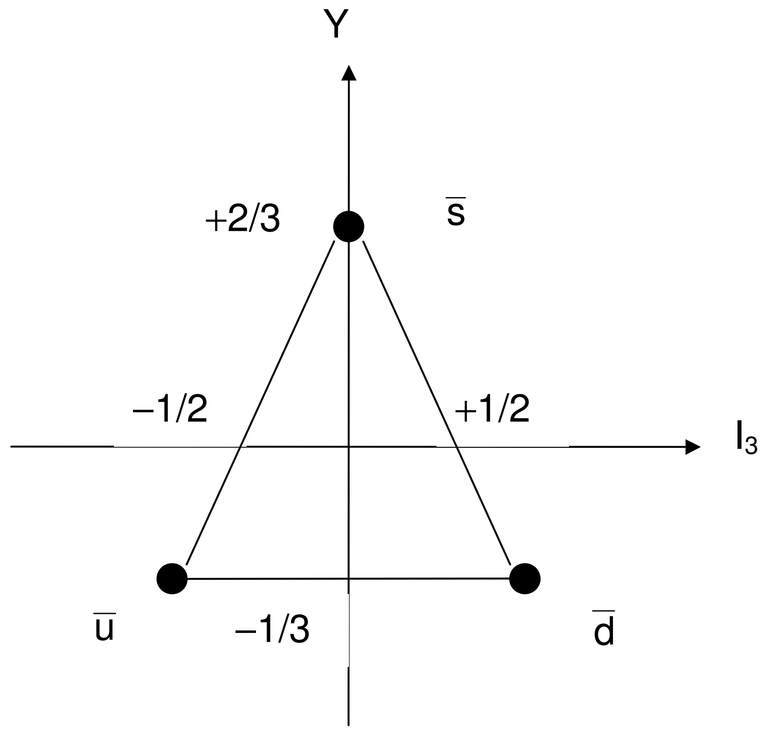
Group Theory provides

- A description of the hadron multiplets
- The framework for the Standard Model
- Indications for a unified description of the Standard Model and new physics

Hadron States in $SU(3)_{flavour}$ [11]

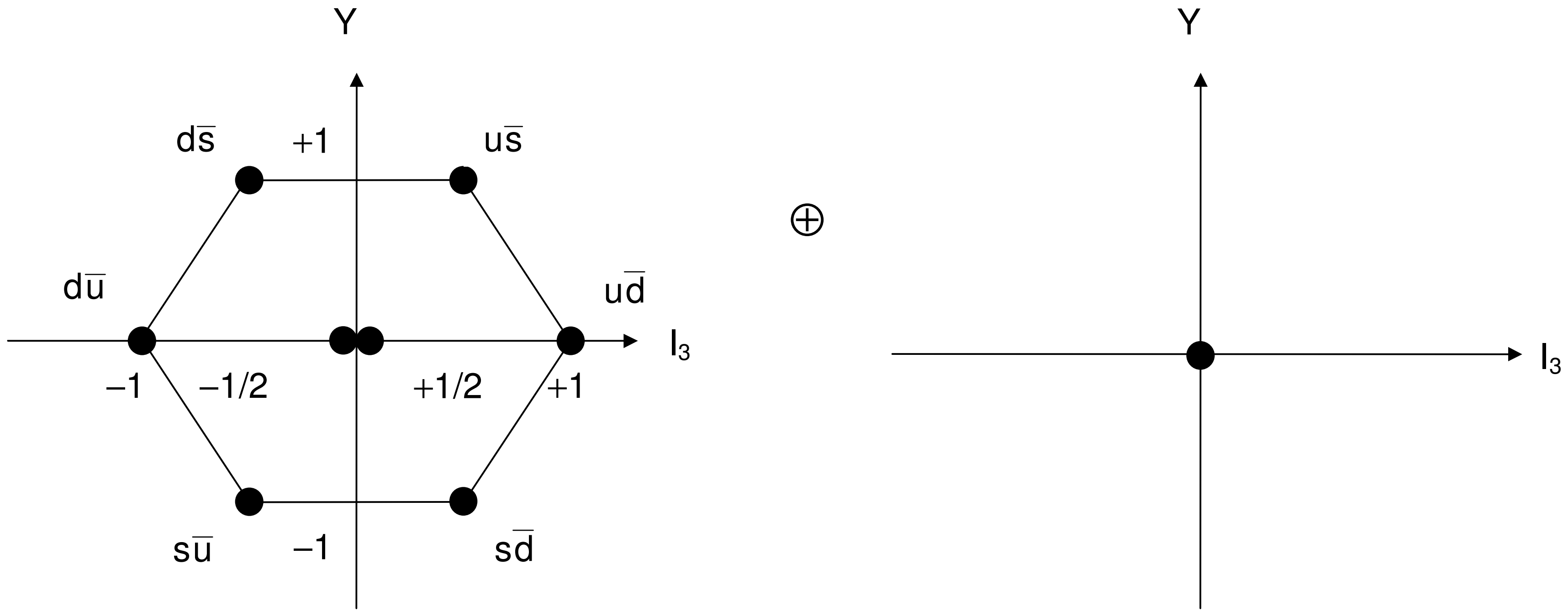
	I	I_3	S	B	$Y = S+B$	$Q = I_3+Y/2$
u	1/2	+1/2	0	1/3	+1/3	+2/3
d	1/2	–1/2	0	1/3	+1/3	–1/3
s	0	0	–1	1/3	–2/3	–1/3



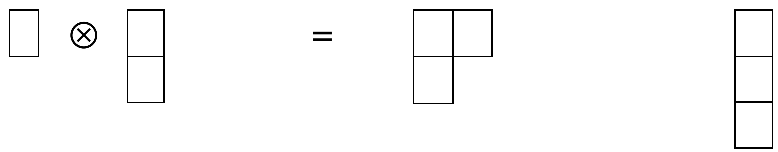


SU(3) symmetry implies nothing can distinguish between different quark states.

Mesons

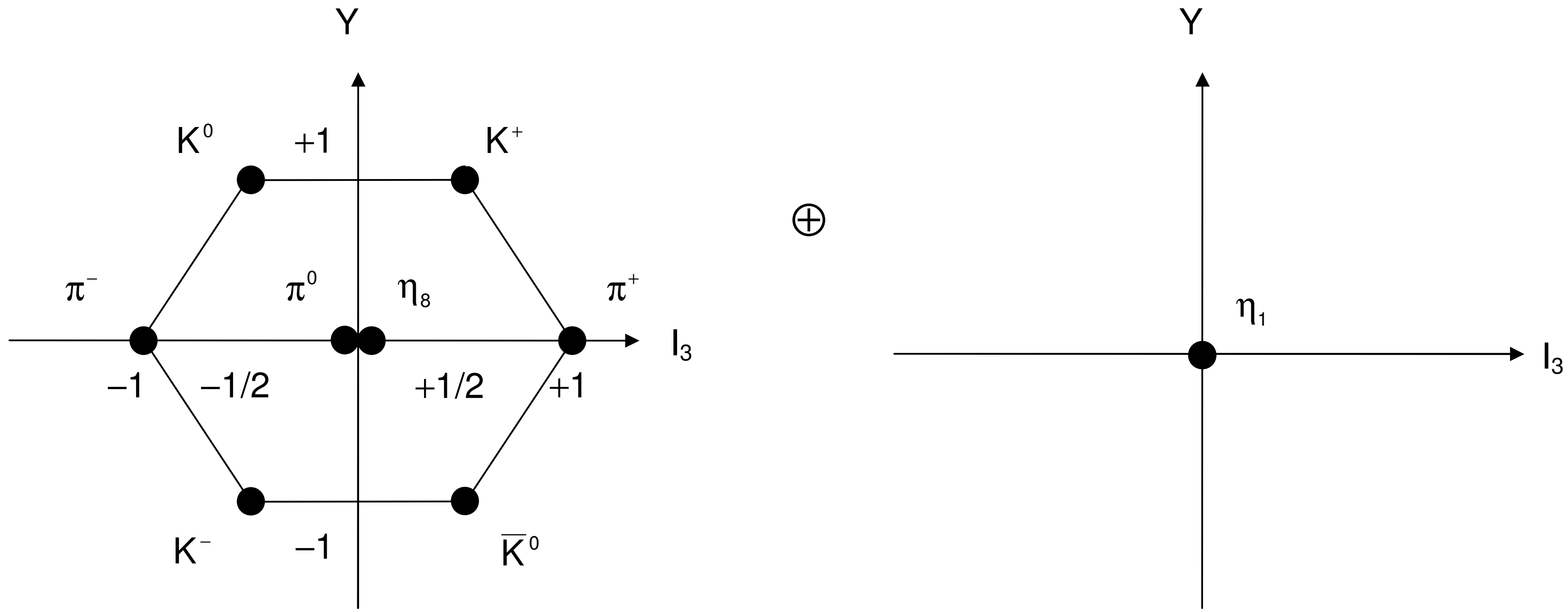


SU(3)



$3 \otimes \bar{3}$ = 8 1

Mixed Symmetry	Totally Antisymmetric



	S = −1	S = 0	S = +1
Octet	$K^- = s\bar{u}$ $\overline{K}^0 = s\bar{d}$	$\pi^- = -d\bar{u}$ $\pi^0 = \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$ $\pi^+ = u\bar{d}$ $\eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$	$K^0 = d\bar{s}$ $K^+ = u\bar{s}$
Singlet		$\eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$	

Octet – c.f. Gluon octet
Singlet – c.f. colourless meson state

The two η mesons have the same quantum numbers: $I = 0$, $I_3 = 0$, $S = 0$, $B = 0$, $Y = 0$, $Q = 0$ albeit that they have different forms under $SU(3)_{flavour}$ – one transforms as an **8**, the other as a **1**.

Because the $SU(3)_{flavour}$ is not exact, the observed mass eigenstates of the complete Hamiltonian are mixtures of η_8 and η_1:

$$\eta = \cos\theta_p \eta_8 - \sin\theta_p \eta_1$$
$$\eta' = \sin\theta_p \eta_8 + \cos\theta_p \eta_1$$

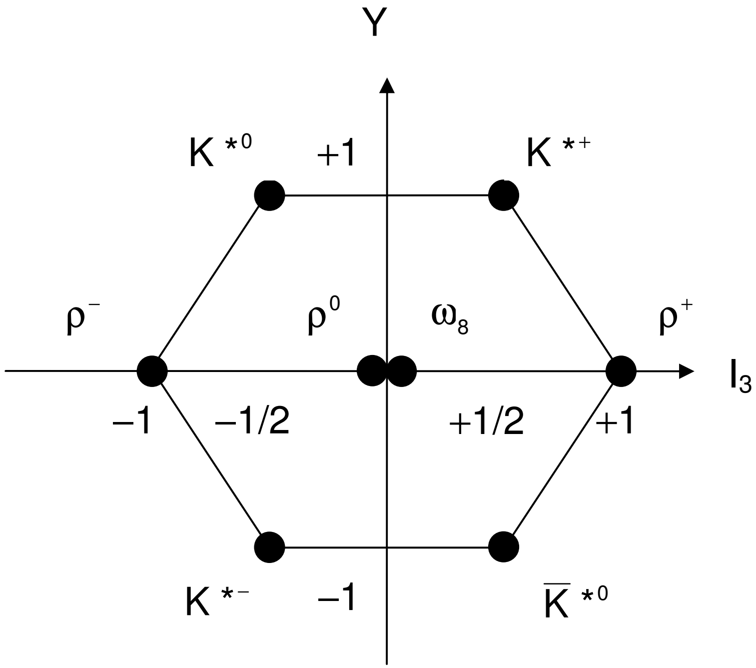
Experimentally, mixing angle $\theta_p \approx -20^o$

π^0 does not mix since it has $I = 1$.

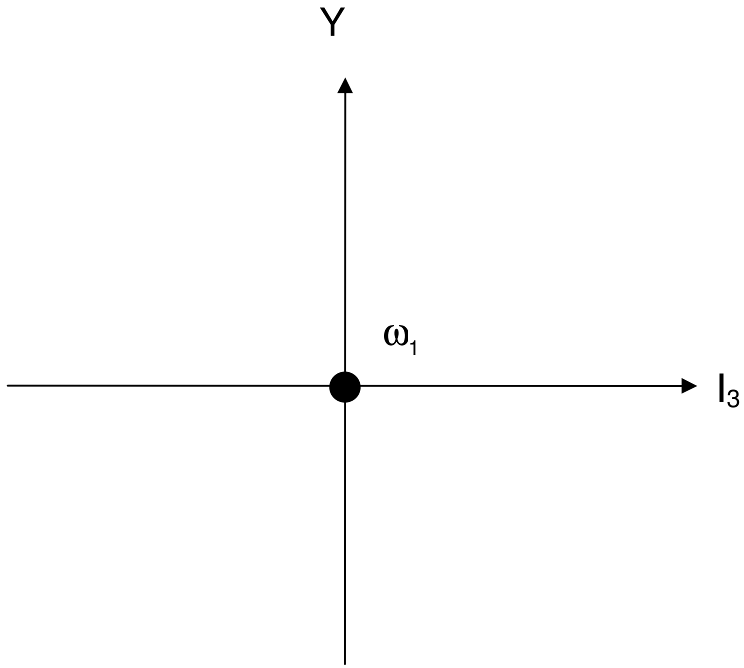
Meson Spin

The particles identified so far are the **pseudoscalar mesons** with J = 0, corresponding to a spin state $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$.

There is a set of heavier particles, the so-called **vector mesons** with J = 1:



$\oplus$



As with the η mesons, the ω mesons mix:

$$\omega = \cos\theta_v \omega_8 - \sin\theta_v \omega_1$$
$$\phi = \sin\theta_v \omega_8 + \cos\theta_v \omega_1$$

The mixing is such that ϕ is almost pure $s\bar{s}$:

$$\omega \approx \tfrac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$
$$\phi \approx s\bar{s}$$

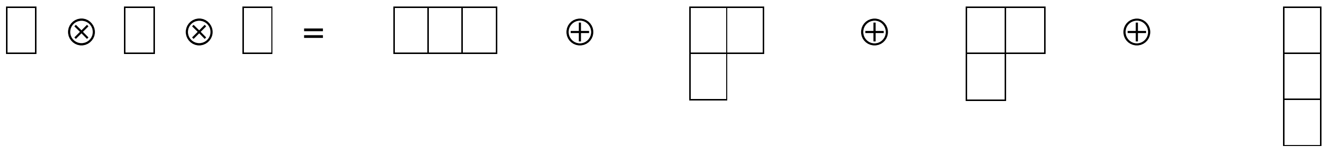
Sometimes, it can be helpful to express the meson wavefunctions in a symmetrised form – the flavour symmetry is described by the **G-parity**.

E.g.

$$\pi^+ = \tfrac{1}{\sqrt{2}}(u\bar{d} + \bar{d}u) \cdot \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \quad G = -1$$

$$\rho^+ = \tfrac{1}{\sqrt{2}}(u\bar{d} - \bar{d}u) \cdot \left\{ \begin{array}{c} \uparrow\uparrow \\ \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{array} \right\} \quad G = +1$$

Baryons

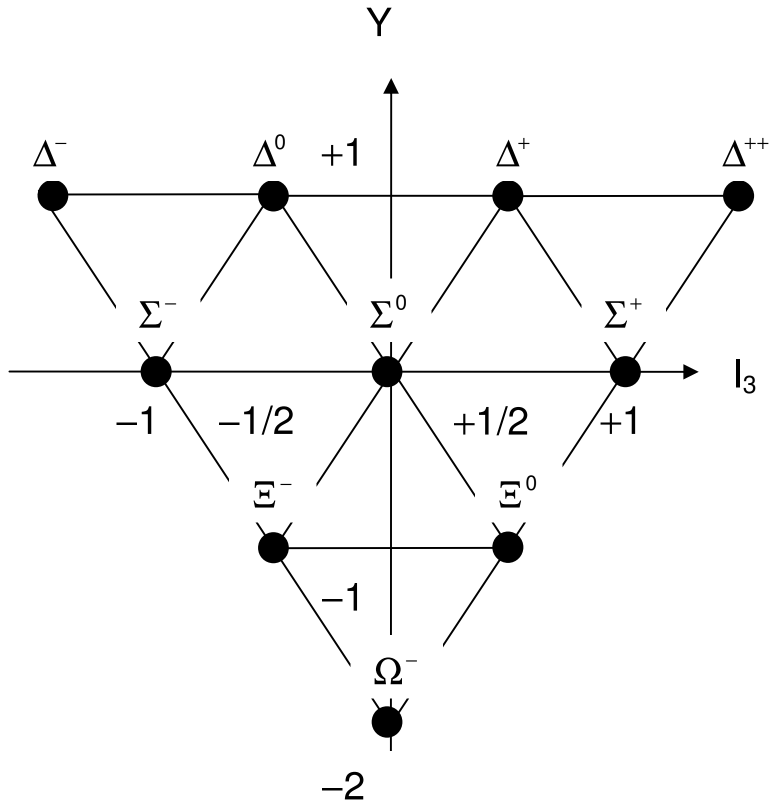


Totally Symmetric	Mixed Symmetry	Mixed Symmetry	Totally Antisymmetric

$$SU(3)\quad 3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

What are these states ?
A decuplet and one octet are seen, but not a singlet.

Decuplet



Note these are the heavier Σ and Ξ states.

By observation, these states have J = 3/2.

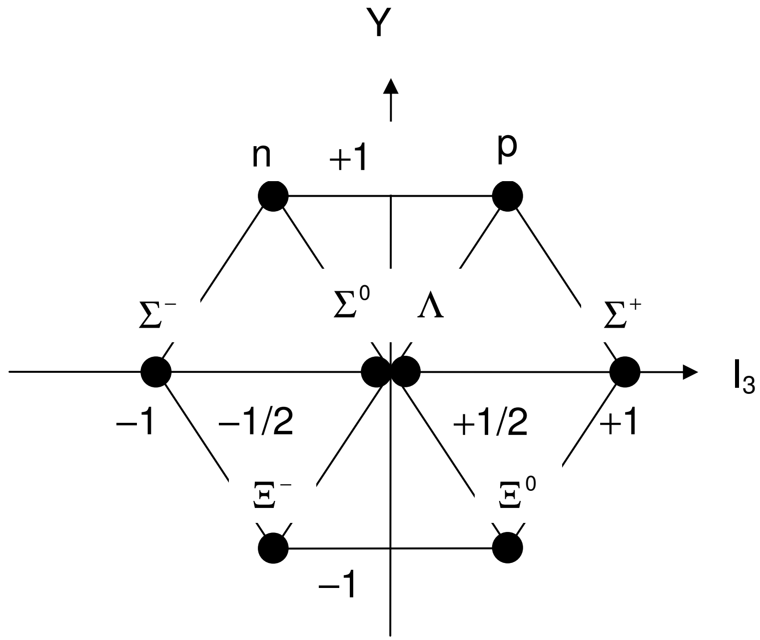
These states are symmetric.

u | u | u	$\sim uuu$
u | u | d	$\sim \frac{1}{\sqrt{3}}(uud + udu + duu)$

But the state $\Delta^{++} \sim uuu{\cdot}\uparrow\uparrow\uparrow$ – which is a symmetric combination of fermions.
The fermionic symmetry is restored by adding the antisymmetric colour wave-function:

$$\frac{1}{\sqrt{6}}(rbg + grb + bgr - rgb - brg - gbr)$$

Quark content	**Baryon**	**I**	**I_3**	**S**
u | u | u	Δ^{++}	3/2	+3/2	0
u | u | d	Δ^{+}	3/2	+1/2	0
u | d | d	Δ^{0}	3/2	–1/2	0
d | d | d	Δ^{-}	3/2	–3/2	0
u | u | s	Σ^{+}	1	+1	–1
u | d | s	Σ^{0}	1	0	–1
d | d | s	Σ^{-}	1	–1	–1
u | s | s	Ξ^{0}	1/2	+1/2	–2
d | s | s	Ξ^{-}	1/2	–1/2	–2
s | s | s	Ω^{-}	0	0	–3



Note these are the lighter Σ and Ξ states.

By observation, these states have J = 1/2

How are the octets derived ?

$$\begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \\ \hline\end{array} = \left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \oplus \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \right\} \otimes \begin{array}{|c|}\hline \\ \hline\end{array}$$

$$= \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{} & \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|c|}\hline 1 & \\ \hline \multicolumn{1}{|c|}{2} & \\ \cline{1-1}\end{array} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$$

Totally Symmetric	Mixed Symmetry Symmetric in $1\leftrightarrow 2$	Mixed Symmetry Antisymmetric in $1\leftrightarrow 2$	Totally Antisymmetric

Consider a proton ~ uud:

$$\begin{array}{|c|c|}\hline u_1 & u_2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline d_3 \\ \hline\end{array}$$

Symmetric in TL$\leftrightarrow$TR; antisymmetric in TL$\leftrightarrow$BL:

$\sim (u_1u_2 + u_2u_1) \otimes d_3 = u_1u_2d_3 - d_3u_2u_1 + u_2u_1d_3 - d_3u_1u_2$

$\sim$ uud – duu

$$\begin{array}{|c|}\hline u_1 \\ \hline d_2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline u_3 \\ \hline\end{array}$$

Antisymmetric in TL$\leftrightarrow$TR; symmetric in TL$\leftrightarrow$BL:

$\sim (u_1d_2 - d_2u_1) \otimes u_3 = (u_1d_2u_3 + u_3d_2u_1) - (d_2u_1u_3 + u_3u_1d_2)$

$\sim$ 2udu – duu – uud

Usually the symmetry is expressed with respect to the first two particles:

$\frac{1}{\sqrt{2}}(ud - du)u$ or $\frac{1}{\sqrt{6}}\{(ud + du)u - 2uud\}$

So which is the wavefunction for a proton ?

$$\tfrac{1}{\sqrt{2}}(ud - du)u \quad \text{or} \quad \tfrac{1}{\sqrt{6}}\{(ud + du)u - 2uud\}$$

There appears to be an ambiguity – which will be resolved when we look at SU(6).

For the octet states with two identical quark flavours, we will find analogous wavefunctions to the above (with the same ambiguity).
However, there will be different combinations for the neutral combinations of (u,d,s) corresponding to Λ and Σ^0.

Let us count the states of 3 quarks:

<u>All same flavour</u> e.g. uuu
3 ways of selecting quarks; 1 permutation of each selection $\Rightarrow$ 3 states
Members of **10**.

<u>Two same flavour</u> e.g. uud
6 ways of selecting quarks; 3 permutation of each selection $\Rightarrow$ 18 states
Members of **10**, $\mathbf{8_S}$, $\mathbf{8_A}$.

<u>All different flavours</u> e.g. uds
1 way of selecting quarks; 6 permutation of each selection $\Rightarrow$ 6 states
Members of **10**, $\mathbf{8_S}$, $\mathbf{8_A}$, **1**

A total of 27 = 3×3×3 states.

Quark content	Baryon	I	I_3	S
u u d	p	1/2	+1/2	0
u d d	n	1/2	–1/2	0
u u s	Σ^+	1	+1	–1
u d s	Σ^0	1	0	–1
d d s	Σ^-	1	–1	–1
u d s	Λ	0	0	–1
u s s	Ξ^0	1/2	+1/2	–2
d s s	Ξ^-	1/2	–1/2	–2

Hadron States in $SU(6)_{flavour\otimes spin}$ [15]

In $SU(3)_{flavour}$, for the baryons, it is not obvious

a. how to assign the octet wavefunctions
b. why there is no flavour singlet

Rather than $SU(3)_{flavour} \otimes SU(2)_{spin}$, consisting of $\{u,d,s\}\otimes\{\uparrow,\downarrow\}$, we consider $SU(6)_{flavour\otimes spin}$, consisting of the states $\{u\uparrow,d\uparrow,s\uparrow,u\downarrow,d\downarrow,s\downarrow\}$ – all of which are considered indistinguishable.

The flavour-spin states will be combined with an $SU(3)_{colour}$ singlet:

$$\tfrac{1}{\sqrt{6}}(rbg + grb + bgr - rgb - brg - gbr)$$

This is antisymmetric – so the flavour⊗spin states will need to be symmetric if they are to describe identical fermions (in particular for the uuu, ddd and sss states).

$SU(3)_{flavour}$	$3 \otimes 3 \otimes 3 =$	10	$\oplus$	8	$\oplus$	8	$\oplus$	1
		ϕ_S		$\phi_{M,S}$		$\phi_{M,A}$		ϕ_A
$SU(2)_{spin}$	$2 \otimes 2 \otimes 2 =$	4	$\oplus$	2	$\oplus$	2		
		χ_S		$\chi_{M,S}$		$\chi_{M,A}$		

ϕ_S is symmetric under exchange of all 3 quarks.
$\phi_{M,S}$ is symmetric under exchange of only 2 quarks, etc.

We can combine the flavour and spin wavefunctions to create states of new symmetry:

$\phi_{flavour}$ \ χ_{spin}	S	M
S	S	M
M	M	S, M, A
A	A	M

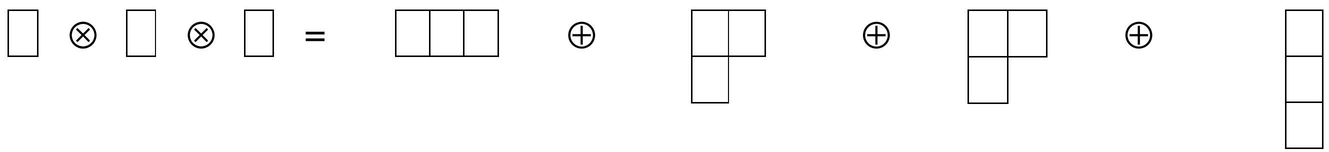
Note: there is no χ_A to connect to ϕ_A.

We construct symmetric states:

Decuplet (J=3/2): $\phi_S \, \chi_S$ — 10 flavour states $\otimes$ 4 spin states = 40 states

Octet (J=1/2): $\frac{1}{\sqrt{2}}(\phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A})$ — 8 flavour states $\otimes$ 2 spin states = 16 states

A total of 56 states.



Totally Symmetric | Mixed Symmetry | Mixed Symmetry | Totally Antisymmetric

$$SU(6) \quad 6 \otimes 6 \otimes 6 = \boxed{56} \oplus 70 \oplus 70 \oplus 20$$

Examples

$\phi_S \, \chi_S$

- Δ^{++}, $J_z = 3/2$
  $uuu \uparrow\uparrow\uparrow$

- Δ^{+}, $J_z = 1/2$
  $\frac{1}{\sqrt{3}}(uud + udu + duu)\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$

$\frac{1}{\sqrt{2}}(\phi_{M,S}\chi_{M,S} + \phi_{M,A}\chi_{M,A})$

- p, $J_z = 1/2$
  $\frac{1}{\sqrt{2}}\frac{1}{\sqrt{6}}\{(ud + du)u - 2uud\}\frac{1}{\sqrt{6}}\{(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow\} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(ud - du)u\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow$
  $= \frac{1}{\sqrt{2}}\{+\frac{2}{3}udu - \frac{1}{3}duu - \frac{1}{3}uud\}\uparrow\downarrow\uparrow + \frac{1}{\sqrt{2}}\{-\frac{1}{3}udu + \frac{2}{3}duu - \frac{1}{3}uud\}\downarrow\uparrow\uparrow + \frac{1}{\sqrt{2}}\{-\frac{1}{3}udu - \frac{1}{3}duu + \frac{2}{3}uud\}\uparrow\uparrow\downarrow$
  $= \frac{1}{\sqrt{2}}\{u\uparrow u\uparrow d\downarrow + u\uparrow d\downarrow u\uparrow + d\downarrow u\uparrow u\uparrow - \frac{1}{3}(uud + udu + duu)(\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)\}$

It can be seen that this term is indeed symmetric.

Final Comments [16.3,16.4]

One can add **heavier quarks** to {u,d,s}:

+ c $\quad SU(4)_{flavour}$
+ b $\quad SU(5)_{flavour}$
+ t $\quad SU(6)_{flavour}$

However, since these quarks are much heavier than their lighter brothers and exceed the QCD scale Λ_{QCD}, the symmetry is badly broken.

It is best to treat the heavy quarks Q separately from the lighter ones {q}:

Mesons: $Q\overline{q}$ where q transforms according to SU(3)
Baryons: Qq_1q_2 where q_1 and q_2 transform according to SU(3)

So far, all the hadron states considered have L=0 (s-wave).
Excited states can be obtained by considering higher orbital angular momentum.

Consequences of Group Theory

$SU(3)_{colour}$

- Z Decay

$$
\begin{array}{llll}
Z & \rightarrow & 3\ l^+l^- & e, \mu, \tau \\
 & + & 3\ \nu\bar{\nu} & e, \mu, \tau \\
 & + & 5\times 3\ q\bar{q} & u, d, c, s, \cancel{t}, b \ \otimes\ 3\ \underline{\text{colours}}
\end{array}
$$

So the branching ratio for neutrinos is 3/21 = 1/7 not 3/11 (plus ElectroWeak factors).
Significant when measuring **visible cross-section** at LEP – led to conclusion $N_\nu = 3$.

- **Anomalies** – cancellations required for renormalisable theories

$$\sum_{\text{fermions}} Q = 0$$

$$
\begin{array}{lll r}
e, \mu, \tau & 3 \times (-1) & = & -3 \\
\nu_e, \nu_\mu, \nu_\tau & 3 \times (0) & = & 0 \\
u, c, t \ \otimes\ 3\ \underline{\text{colours}} & 3 \times 3 \times (+2/3) & = & +6 \\
d, s, b \ \otimes\ 3\ \underline{\text{colours}} & 3 \times 3 \times (-1/3) & = & -3 \\
\hline
 & & \Sigma & 0
\end{array}
$$

SU(3)$_{\text{flavour}}$ or SU(6)$_{\text{flavour}\otimes\text{spin}}$

- **Multiplets** of comparable mass … better still, there are patterns in the masses which are well described – see Gell-Mann Okubu formula [11.2]

- **Magnetic Moments** – see Lecture #6 [15.4]

- **Decay Rates** [Close, Lichtenberg]

- **Scattering Amplitudes** [Close, Lichtenberg]

The Standard Model

For a Dirac fermion: $L \sim \bar{\psi}\partial\psi = \bar{\psi}\gamma_\mu\partial^\mu\psi$

We impose local gauge symmetries $\partial^\mu \rightarrow \partial^\mu - igX\cdot F^\mu$

"Charge"	**Symmetry Group**	**Generator**	**Coupling**	**Bosons**
Weak Hypercharge	U(1)	Y	g_Y	B
Weak Isospin	SU(2)	T	g_T	W^+, W^0, W^-
Quark Colour	SU(3)	λ	g_c	G – 8 gluons

Left-handed Isospin Doublets:

$$L_L = \left\{ \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \right\}$$

$$Q_L = \left\{ \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \right\}$$

Right-handed Isospin Singlets:

$$\nu_R = \nu_{eR}, \nu_{\mu R}, \nu_{\tau R} \qquad l_R = e_R, \mu_R, \tau_R$$

$$u_R = u_R, c_R, t_R \qquad d_R = d_R, s_R, b_R$$

- Left-handed refers to the chirality, resulting from the projection operator $\frac{1}{2}(1-\gamma_5)$
- Right-handed refers to the chirality, resulting from the projection operator $\frac{1}{2}(1+\gamma_5)$

Furthermore

- Any quark is a colour triplet: $q = (q_r, q_b, q_g)$
- All particle states are Dirac spinors (4 components)

So all of the particles are complicated tensors, and implicitly carry several indices.

Following the *observations* of neutrino oscillations, the **neutrinos** must be massive and there must exist right-handed states.
This is claimed to be evidence for **New Physics Beyond the Standard Model**.
However, the Standard Model (which originally was constructed to include a description of massless neutrinos) can trivially be extended to included right-handed states and masses.

Having said this, because of a hierarchy problem associated with $m(\nu) << m(l,q)$, many theorists like the idea of neutrinos with a **Majorana** mass term in the Lagrangian (for which the neutrino is its own antiparticle). The masses of the neutrinos are then manifestations of new physics at the GUT or Planck scale.

Quantum numbers:

	Hypercharge Y	Isospin T	λ
L_L	-1	1/2	0
Q_L	1/3	1/2	1
ν_R	0	0	0
l_R	-2	0	0
u_R	4/3	0	1
d_R	$-2/3$	0	1

$Q = T_3 + \frac{1}{2}Y$

The **Standard Model Lagrangian**:

$$
\begin{array}{llllllll}
L \sim & \bar{L}_L & \gamma_\mu(\partial^\mu & - g_Y B^\mu & + g_T T\cdot W^\mu) & & L_L \\
+ & \bar{Q}_L & \gamma_\mu(\partial^\mu & + \tfrac{1}{3} g_Y B^\mu & + g_T T\cdot W^\mu & + g_c \lambda\cdot G^\mu) & Q_L \\
+ & \bar{\nu}_R & \gamma_\mu(\partial^\mu) & & & & \nu_R \\
+ & \bar{l}_R & \gamma_\mu(\partial^\mu & - 2g_Y B^\mu) & & & l_R \\
+ & \bar{u}_R & \gamma_\mu(\partial^\mu & + \tfrac{4}{3} g_Y B^\mu & & + g_c \lambda\cdot G^\mu) & u_R \\
+ & \bar{d}_R & \gamma_\mu(\partial^\mu & - \tfrac{2}{3} g_Y B^\mu & & + g_c \lambda\cdot G^\mu) & d_R
\end{array}
$$

$+$ Boson terms
$+$ Higgs terms, rendered Gauge Invariant, giving Boson mass terms
$+$ Higgs-Fermion terms, giving Fermion mass terms

(Drop $i = \sqrt{-1}$; explicitly evaluate Y and ignore ½ with λ to avoid clutter.)

Baryon and lepton number conservation is explicitly built into the model.

The ν_R has no gauge couplings – it is "sterile" – the only interactions it has are with the Higgs.

Could $U(1)_Y = U(1)_{EM}$?
No – because terms like $\bar{\nu}_L B^\mu \nu_L$ indicate a coupling of the ν to the $U(1)_Y$ field B^μ, and since the ν has no electrical charge it does not couple to the exchange boson on $U(1)_{EM}$, namely the photon.

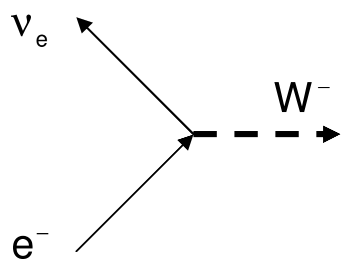
Quantum Numbers

In order to be gauge invariant, that is: unaffected by the symmetry transformations corresponding to $U(1)_Y \otimes SU(2)_T \otimes SU(3)_c$, terms in the SM Lagrangian must carry no net quantum numbers.

Therefore

- B has Y=0, T=0
- W has Y=0, T=1 and Q = +1,0,−1

For example in a term like $\overline{e}^- W^- \nu_e$ corresponding to a vertex



$T_3(e^-)$	$\rightarrow$	$T_3(\nu_e)$	+	$T_3(W^-)$
$-\frac{1}{2}$	$\rightarrow$	$\frac{1}{2}$	+	-1

The Higgs Mechanism

The Higgs doublet $\begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ has Y=1, T=½ (choice ensures $Q = T_3 + \frac{1}{2}Y$).

We recall for SU(2), that the $\overline{\mathbf{2}}$ transforms like the $\mathbf{2}$.
So the conjugate state is $\begin{pmatrix} \overline{\phi}^0 \\ -\phi^- \end{pmatrix}$ and has Y=–1, T=½.

In the Higgs Mechanism, the real part of ϕ^0 is written as v+h, where v is the Higgs vev and h is the real Higgs field.
ϕ^+ and ϕ^- and the imaginary part of ϕ^0 are zero. (Actually, they have zero vev's and the fields correspond to the Goldstone bosons, which are absorbed by the W bosons to provide the longitudinal polarisation).
This gives $\mathbf{2} \sim \begin{pmatrix} 0 \\ v+h \end{pmatrix}$ and $\overline{\mathbf{2}} \sim \begin{pmatrix} v+h \\ 0 \end{pmatrix}$

The Higgs Mechanism results from imposing Gauge Invariance on the Higgs Lagrangian terms:

$$\phi^H \partial_\mu \partial^\mu \phi + V(\phi)$$

giving terms like v^2WW which look like W mass terms $m(W)^2WW$.

By design, the Higgs Mechanism provides masses for the Vector Bosons $\{W^+, W^0, W^-\}$ while causing the mixing of the W^0 and B gauge fields resulting in the Z and γ fields.

Fermion Masses

The part of the Lagrangian describing the fermion masses (ignoring the different couplings which lead to different fermion masses) is

$$\overline{L}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} l_R + \overline{L}_L \begin{pmatrix} \overline{\phi}^0 \\ -\phi^- \end{pmatrix} \nu_R + \overline{Q}_L \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} d_R + \overline{Q}_L \begin{pmatrix} \overline{\phi}^0 \\ -\phi^- \end{pmatrix} u_R + hc + \text{non-diagonal terms}$$

All the terms above give rise to zero net quantum numbers:

For example, consider the first term ~ $\overline{\nu}_L \phi^+ l_R + \bar{l}_L \phi^0 l_R$

The above expression needs to be repeated for each of the generations (ν_e,e), (ν_μ,μ), (ν_τ,τ), (u,d), (c,s), (t,b).

"Non-diagonal" terms which combine different generations give rise to mixing, as found in the CKM matrix.

The second term is new for the Standard Model, corresponding to Dirac masses for the neutrinos (if this is the correct mechanism for neutrino masses).

The fermion mass terms appear when the ϕ fields are replaced by their vev's, giving terms like $v\bar{l}_L l_R$ which looks like the lepton mass term $m(l)\,\bar{l}_L l_R$.

The different assignments of weak quantum numbers to the left- and right-handed fermions allows the possibility of C, P and CP violation.

As CPT is conserved in Quantum Field Theories, this implies T can also be violated.

Beyond the Standard Model

Undesirable features of the Standard Model include:

- Large number of unconstrained **constants** (couplings, charges)
- No explanation of **generations**
- No explanation as to **charge quantisation**: $Q(u) = -2/3\ Q(e^-)$ and $Q(d) = 1/3\ Q(e^-)$

It would seem *elegant* to contain all of the SM in some single theory (group) – rather than as the product of three seemingly disconnected groups.
While the Electroweak force is supposedly unified, it nevertheless represents the product of two groups $SU(2)_T$ and $U(1)_Y$, albeit that they are mixed through the Higgs Mechanism.

If the **coupling constants** of the three groups $U(1)_Y$, $SU(2)_T$ and $SU(3)_c$ are evolved via the Renormalisation Group Equations (RGE), they meet at a scale of ~10^{15} GeV.
It turns out that this convergence is even more precise if **SuperSymmetry** is included.

SU(5) [18]

The fundamental representation is taken as $\mathbf{5} = \begin{pmatrix} d_r \\ d_b \\ d_g \\ e^+ \\ \overline{\nu}_e \end{pmatrix}_R$

The particles of the Standard Model (in the absence of neutrino mass and hence ν_R) can be contained in multiplets of SU(5):

$$\overline{\mathbf{5}} = \begin{pmatrix} \overline{d}_r \\ \overline{d}_b \\ \overline{d}_g \\ e^- \\ -\nu_e \end{pmatrix}_L \qquad \mathbf{10} = \begin{pmatrix} 0 & \overline{u}_g & -\overline{u}_b & -u_r & -d_r \\ -\overline{u}_g & 0 & \overline{u}_r & -u_b & -d_b \\ \overline{u}_b & -\overline{u}_r & 0 & -u_g & -d_g \\ u_r & u_b & u_g & 0 & -e^+ \\ d_r & d_b & d_g & e^+ & 0 \end{pmatrix}_L$$

Reproduced for each of the three generations.

Where the **10** an antisymmetric multiplet derived from **5**⊗**5**:

$$\square \otimes \square = \square\square \oplus \begin{array}{c}\square\\ \square\end{array}$$

$$\mathbf{5} \otimes \mathbf{5} = \mathbf{15} \oplus \mathbf{10}$$

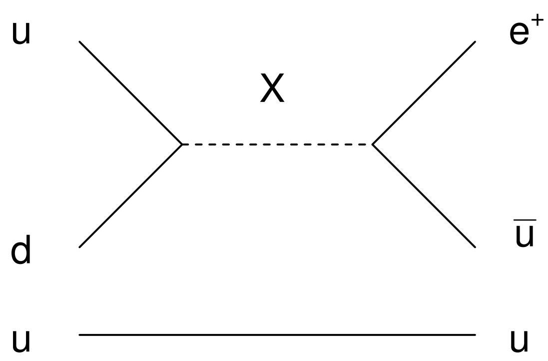
Totally Symmetric	Totally Antisymmetric

There are $5^2-1 = 24$ gauge bosons. In terms of the fundamental representation they have the form:

	d_r	d_b	d_g	e^+	$\bar{\nu}_e$
d_r	g, γ, Z	g	g	$\bar{X}_r$	$\bar{Y}_r$
d_b	g	g, γ, Z	g	$\bar{X}_b$	$\bar{Y}_b$
d_g	g	g	g, γ, Z	$\bar{X}_g$	$\bar{Y}_g$
e^+	X_r	X_b	X_g	γ, Z	W^+
$\bar{\nu}_e$	Y_r	Y_b	Y_g	W^-	Z

So there are 8 gluons, 3 of W^+, W^-, Z and 1 γ, along with 12 X, Y bosons.
The X, Y bosons are coloured and transform leptons to quarks and v.v. – they are leptoquarks.

As we saw in Lecture #2, the X, Y bosons can mediate proton decay:



B and L are not conserved, but B–L is.

The good points about SU(5) are:

- It is the smallest group which contains $U(1)_Y \otimes SU(2)_T \otimes SU(3)_c$, and therefore has the greatest predictivity.
- $Q = T_3 + \frac{1}{2}Y$ and T_3 and Y are now both generators of SU(5) and hence traceless.
  (In the Standard Model, Y was a generator of $U(1)_Y$, and was not traceless).
  Hence Q is traceless and in the fundamental representation this leads:
  $$Q(d_r)+Q(d_b)+Q(d_g)+Q(e^+)+Q(\bar{\nu}_e) = 0 \Rightarrow 3Q(d)+Q(e^+) = 0 \Rightarrow Q(d) = -1/3\ Q(e^+) \text{ etc}$$
- It predicts $\sin^2\theta_W$ very accurately.
- The couplings of the low-energy gauge group representations converge at a scale ~ M(X).
- Since there is only room for one neutrino type in the $\bar{\mathbf{5}}$ and **10**, i.e. there is no ν_R, hence the only way to generate a mass term is via a Majorana term which requires $\nu = \bar{\nu}$ – which violates B–L conservation. Hence the neutrino must be massless. But …

The problems are:

- The neutrino appears to have mass.
- The predicted decay rate for the proton is much larger than the current limits.
- There is no explanation of the three generations.

So despite its elegance, SU(5) is now excluded.

Other Symmetry Groups [24, 27]

Another possibility is SO(10) which contains SU(5) – this is not yet excluded.

Going further, other possibilities include the **Exceptional Groups**: E_6 and E_8.
The latter is of relevence to the heterotic superstring (combining bosonic and fermionic modes).
These groups are associated with the algebra of **Octonians**, which are generalisations of $i = \sqrt{-1}$.
E_6 contains SO(10).

SuperSymmetry

SUSY provides an extension of the Poincaré groups (associated with Translations, Rotations and Lorentz Boosts).
It is the one remaining symmetry in QFT, not exploited in the Standard Model Lagrangian, whereby transformations take place which treat bosons and fermions as indistinguishable states.

Crudely speaking, the generator can be thought of as the square root of the momentum operator!

The corresponding super-fields multiplets are:

gauge: $\begin{pmatrix} J = 1 \\ J = \frac{1}{2} \end{pmatrix}$ chiral (matter): $\begin{pmatrix} J = \frac{1}{2} \\ J = 0 \end{pmatrix}$

In many SUSY models, there is a conserved quantity, **R-parity**, which distinguishes between the standard particles and their super-partners.
This ensures that SUSY particles can only be created in pairs, and that once created, a SUSY particle will always leave a SUSY particle in the final state, leading to a stable **Lightest Supersymmetric Particle** (LSP), commonly considered to be the neutralino – a mixture of the photino, Zino and neutral Higgsinos.
These may be one of the constituents of **Dark Matter**.