LECTURE 3 – SU(2)

Contents

Messages

- **SU(2)** describes **spin angular momentum**.
- **SU(2)** is isomorphic to the description of **angular momentum – SO(3)**.
- **SU(2)** also describes **isospin** – for nucleons, light quarks and the weak interaction.
- We see how to describe **hadrons** in terms of several **quark wavefunctions**.

2D Representation of the Generators [3.1, 3.2, 3.3]

SU(2) corresponds to special unitary transformations on complex 2D vectors.

The natural representation is that of 2×2 matrices acting on 2D vectors – nevertheless there are other representations, in particular in higher dimensions.

There are 2^2-1 parameters, hence 3 generators: $\{J_1, J_2, J_3\}$.
The generators are traceless and Hermitian.
It is easy to show that the matrices have the form:

$$\begin{pmatrix} a & b^* \\ b & -a \end{pmatrix}$$ where a is real. There are three parameters: a, Re(b), Im(b).

A suitable (but not unique) representation is provide by the Paul spin matrices: $J_i = \sigma_i/2$ where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These have the properties that: $\sigma_i \sigma_j = \delta_{ij} + i\, \varepsilon_{ijk}\, \sigma_k$

The **Lie algebra** is:

$$[J_i, J_j] = i\, \varepsilon_{ijk}\, J_k$$

This is a sum over k, but in this case is trivial: the commutator just gives the "other" generator.
E.g. $[J_1, J_2] = i\, J_3$

Quantum Numbers [3.2, 3.3]

A **Casimir Operator** is one which commutes with all other generators.

In SU(2) there is just one Casimir: $J^2 = J_1^2 + J_2^2 + J_3^2$

Since $[J^2, J_3] = 0$, they can have simultaneous observables and can provide suitable QM eigenvalues by which to label states.

We can define **Raising & Lowering Operators**: $J_\pm = J_1 \pm iJ_2$
Can show $[J_3, J_\pm] = \pm J_\pm$

Define **eigenstates** $|j\ m\rangle$ with their corresponding **eigenvalues**:

$$J^2\ |j\ m\rangle = j(j+1)\ |j\ m\rangle$$
$$J_3\ |j\ m\rangle = m\ |j\ m\rangle$$

Recall from angular momentum in QM, $J_\pm$ generates different states within a multiplet:

$$J_3\ J_\pm|j\ m\rangle = (m\pm1)\ J_\pm|j\ m\rangle$$

and

$$J_\pm\ |j\ m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ |j\ m\pm1\rangle$$

Exercise

Prove $[J^2, J_i] = 0$ and $[J_3, J_\pm] = \pm J_\pm$
And $J_3\ J_\pm|j\ m\rangle = (m\pm1)\ J_\pm|j\ m\rangle$ and $J_\pm\ |j\ m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\ |j\ m\pm1\rangle$
And $J_\pm\ |j\ \pm j\rangle = 0$

2D Representation [3.2, 3.3]

While we started in 2D to motivate the group structure (which is defined originally in 2D), we have derived the **Lie algebra** and scheme for labelling eigenstates of the generators … which can be expressed in various vector spaces.
Further, we recalled results from angular momentum theory, yet we have not explicitly restricted ourselves to angular momentum.

SU(2) is motivated in 2D (although groups of transformations can operate in higher dimension vector spaces).
We can form a 2D representation by choosing two orthogonal states as base vectors:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\tfrac{1}{2} + \tfrac{1}{2}> \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv |\tfrac{1}{2} - \tfrac{1}{2}>$$

By inspection

$$J_3 = \begin{pmatrix} \tfrac{1}{2} & 0 \\ 0 & -\tfrac{1}{2} \end{pmatrix}$$

To find J_+, consider: $J_+ \,|\tfrac{1}{2} + \tfrac{1}{2}> = 0$ and $J_+ \,|\tfrac{1}{2} - \tfrac{1}{2}> = |\tfrac{1}{2} + \tfrac{1}{2}>$

$$\Rightarrow J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Recalling the definition of $J_\pm$ and the fact that the generators are Hermitian:

$$J_- = J_1 - iJ_2 = (J_1 + iJ_2)^H = J_+{}^H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Hence:

$$J_1 = \tfrac{1}{2}(J_+ + J_-) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \tfrac{1}{2i}(J_+ - J_-) = \begin{pmatrix} 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

and we recover $J_i = \sigma_i/2$

Alternative 2D Representation

The previous result arose from a particular choice (the most natural) of base states.

What if we made another choice such that

$$\begin{pmatrix}1\\0\end{pmatrix} \equiv \tfrac{1}{\sqrt{2}}(|\tfrac{1}{2}+\tfrac{1}{2}> + |\tfrac{1}{2}-\tfrac{1}{2}>) \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \equiv \tfrac{1}{\sqrt{2}}(|\tfrac{1}{2}+\tfrac{1}{2}> - |\tfrac{1}{2}-\tfrac{1}{2}>) \text{ ?}$$

Then

$$|\tfrac{1}{2}+\tfrac{1}{2}>= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}-\tfrac{1}{2}>= \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

We find the new representation of the generators by recalling that the matrix of eigenvectors E for a matrix M relates to the diagonal matrix of eigenvalues Λ by ME = EΛ.
For a Hermitian matrix M, Λ is real and E is Unitary.
So M = EΛE^H

Exercise

By considering (ME)^{H}E and E^H(ME), show that Λ is real and E is Unitary.
You will need to remember that Λ is diagonal by construction and look at components.

Hence

$$J_3 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}$$

To find J_+ and J_-, recall $J_+ \,|\, \tfrac{1}{2} + \tfrac{1}{2} >= 0$ and $J_+ \,|\, \tfrac{1}{2} - \tfrac{1}{2} >=|\, \tfrac{1}{2} + \tfrac{1}{2} >$ and $J_- = {J_+}^H$.

So by inspection:

$$J_+ = \tfrac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} \quad \text{and} \quad J_- = \tfrac{1}{2}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}$$

Hence:

$$J_1 = \begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix}0 & \frac{i}{2}\\ \frac{-i}{2} & 0\end{pmatrix}$$

So with this basis: $J_1 = \sigma_3/2$, $J_2 = -\sigma_2/2$, $J_3 = \sigma_1/2$

It looks like a "rotation" about the "1=3 axis".

With a different choice of basis, one would not necessarily obtain the Pauli matrices.

3D Representation [3.3]

This corresponds to j = 1 with m = –1,0,+1.

Choose

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \equiv |1 +1> \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \equiv |1 0> \qquad \begin{pmatrix}0\\0\\1\end{pmatrix} \equiv |1 -1>$$

Then by observation

$$J_3 = \begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}$$

By firstly constructing J_+ and J_-:

$$J_1 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix} \quad \text{and} \quad J_2 = \tfrac{i}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\1 & 0 & -1\\0 & 1 & 0\end{pmatrix}$$

So SU(n) can have representations in vector spaces of various dimensions – in each dimension, there will be an infinite number of representations. The lowest dimensional representation is in nD.
The matrices which correspond to the simple base vectors (1,0,0…), (0,1,0,…) etc. are called the **Fundamental Representation**, of which there are (n–1).
(For n>2, there are more quantum numbers and additional ways in which they can be associated to base vectors.)
For SU(2), there is just one fundamental representation, namely $\{\sigma_1/2, \sigma_2/2, \sigma_3/2\}$.

2D Transformations [3.3]

Transformations under SU(2) are performed by $\exp(i\alpha_i J_i)$ (implicit sum over i).

In 2D, we have identified the generators $\{J_i\}$ with the Pauli spin matrices $\{\sigma_i/2\}$ which correspond to the spin ½ angular momentum operators.

Furthermore, the operators have the form we would expect from our consideration of 3D transformations of spatial wavefunctions in QM (see Lecture 1) – i.e. the form of the operators **L** compared to **J,** and the corresponding eigen-states and eigen-values.
However, there are some subtle differences.

	Spatial Rotations	Spin Rotations
Vector Space	$\psi(x)$ – spatial w/f	ψ – spinor
Symmetry Group	SO(3)	SU(2)
Operation	$\exp(i\theta L)$ where $L = x\times(-i\nabla)$ operator is scaler	$\exp(i\alpha J)$ operator is matrix
Operates on	x – space	spin

There is an isomorphism between SO(3) and SU(2).

The transformation operator is $U = \exp(i\alpha_i\sigma_i/2)$.
Writing $\alpha_i = 2\omega n_i$, where n is a unit vector, gives $U = \exp(i\omega n\cdot\sigma)$.
Expanding the exponential:

$$U = 1 + (i\omega n\cdot\sigma) + (i\omega n\cdot\sigma)^2/2! + (i\omega n\cdot\sigma)^3/3! \ldots$$

$$(n\cdot\sigma)^2 = n_1^2\, \sigma_1^2 + \ldots + n_1\, n_2\, (\sigma_1\, \sigma_2 + \sigma_2\, \sigma_1) + \ldots$$

$$\sigma_1^2 = 1 \quad \text{and} \quad (\sigma_1\, \sigma_2 + \sigma_2\, \sigma_1) = 0 \quad \Rightarrow \quad (n\cdot\sigma)^2 = n_1^2 + n_2^2 + n_3^2 = 1$$

So

$$U = \{1 - \omega^2/2! + \omega^4/4! + \ldots\} + i\{\omega - \omega^3/3! + \omega^5/5! - \ldots\}\, n\cdot\sigma = \cos\omega + i\sin\omega\, n\cdot\sigma$$

$$\exp(i\tfrac{\theta}{2} n\cdot\sigma) = \cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2} n\cdot\sigma$$

We can now prove the Lie nature of SU(2) explicitly.
We should expect that the product of two operators is a third one of a similar form, where the parameters of the result are a function of the parameters of the first two operations.

$$\exp(i\alpha a\cdot\sigma)\exp(i\beta b\cdot\sigma) = (\cos\alpha + i\sin\alpha a\cdot\sigma)(\cos\beta + i\sin\beta b\cdot\sigma)$$

Using $a\cdot\sigma\; b\cdot\sigma = a\cdot b + i a\times b\cdot\sigma$, which is derived from $\sigma_i\, \sigma_j = \delta_{ij} + i\, \varepsilon_{ijk}\, \sigma_k$

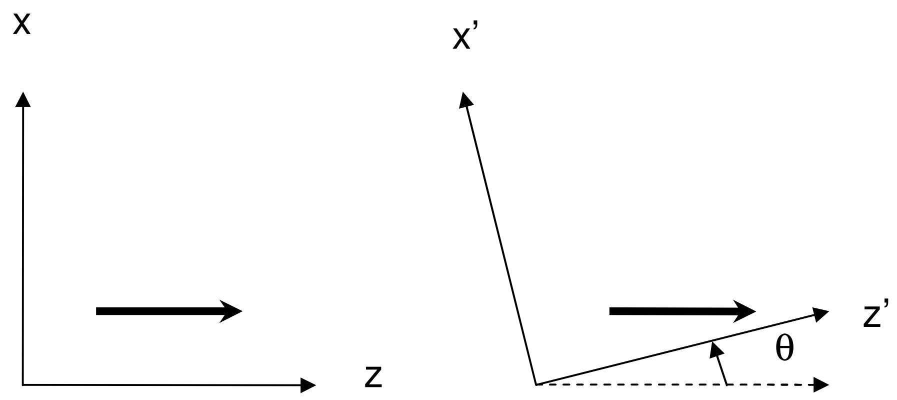
$$\exp(i\alpha a\cdot\sigma)\exp(i\beta b\cdot\sigma) = \cos\alpha\cos\beta - \sin\alpha\sin\beta(a\cdot b + i a\times b\cdot\sigma) + i(\sin\alpha\cos\beta\; a\cdot\sigma + \sin\beta\cos\alpha\; b\cdot\sigma)$$

$$= \{\cos\alpha\cos\beta - \sin\alpha\sin\beta a\cdot b\} + i\{a\times b + \sin\alpha\cos\beta\; a + \sin\beta\cos\alpha\; b\}\cdot\sigma \equiv C + iS\cdot\sigma$$

To show that this has the form $\cos\gamma + i\sin\gamma c\cdot\sigma$, it is sufficient to show that $C^2+S^2 = 1$ – this straightforward, albeit tedious !

Rotation Matrices

Rotate the coordinated frame about the y-axis:



A state |j m> in the first frame transforms to a different description in the second frame (same state):

$$| jm > \rightarrow R_y(\theta) \,|\, jm >$$

Note: because J^2 commutes with $R_y(\theta)$, the j quantum number is unchanged.

The **<u>Rotation Matrices</u>** d(θ) are defined by: $d^j_{mm'}(\theta) = < jm' | R_y(\theta) | jm >$ where $R_y(\theta) = \exp(i\theta J_y)$

Note the order is $m \rightarrow m'$.

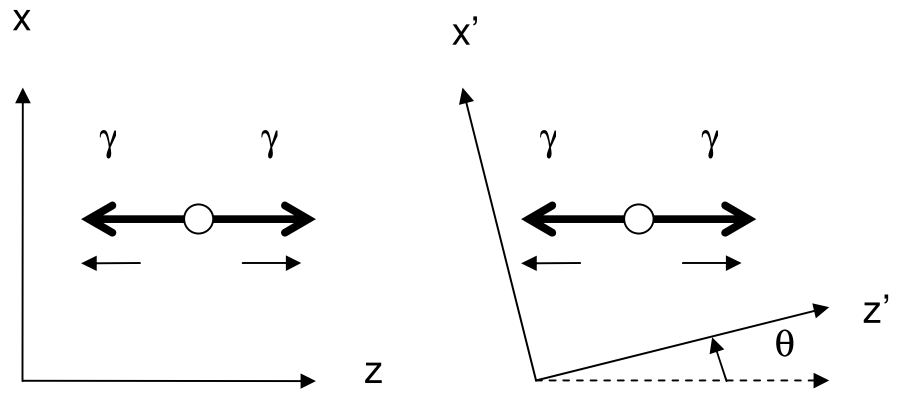
Why rotate about y-axis ? It leads to real matrices.
If use x-axis, get same physical probabilities when square amplitudes.

These matrices can be found alongside the Clebsch Gordon Coefficients (see later) in the **PDG**.

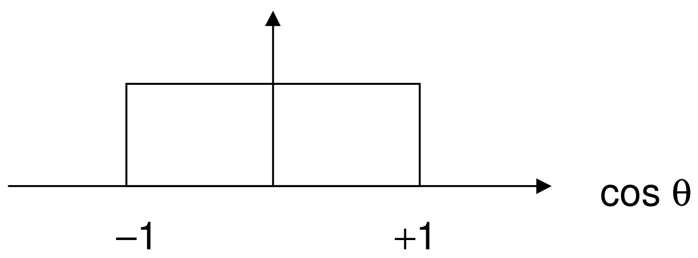
Spin-0

Spin-0 wavefunctions are scalars and have a trivial transformation, i.e. there is no change

E.g. $\pi^0 \to \gamma\gamma$
The π^0 has no spin, and therefore no preferred orientation.



The decay rate in the two frames will be the same, and hence flat in phase space:



$$d^0_{00}(\theta) = 1$$

Spin-½

From the above:

$$R_y(\theta) = \cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\,\sigma_2 = \cos\tfrac{\theta}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin\tfrac{\theta}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}\cos\tfrac{\theta}{2} & \sin\tfrac{\theta}{2}\\ -\sin\tfrac{\theta}{2} & \cos\tfrac{\theta}{2}\end{pmatrix}$$

So

$$|\tfrac{1}{2}+\tfrac{1}{2}> \equiv \begin{pmatrix}1\\0\end{pmatrix} \rightarrow \cos\tfrac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} - \sin\tfrac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad |\tfrac{1}{2}-\tfrac{1}{2}> \equiv \begin{pmatrix}0\\1\end{pmatrix} \rightarrow \sin\tfrac{\theta}{2}\begin{pmatrix}1\\0\end{pmatrix} + \cos\tfrac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

So

$$d^{\frac{1}{2}}_{+\frac{1}{2}+\frac{1}{2}}(\theta) = \cos\tfrac{\theta}{2} \quad d^{\frac{1}{2}}_{+\frac{1}{2}-\frac{1}{2}}(\theta) = -\sin\tfrac{\theta}{2}$$

$$d^{\frac{1}{2}}_{-\frac{1}{2}+\frac{1}{2}}(\theta) = +\sin\tfrac{\theta}{2} \quad d^{\frac{1}{2}}_{-\frac{1}{2}-\frac{1}{2}}(\theta) = \cos\tfrac{\theta}{2}$$

Spin-1

From the above:

$$J_y = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

<u>Homework</u>

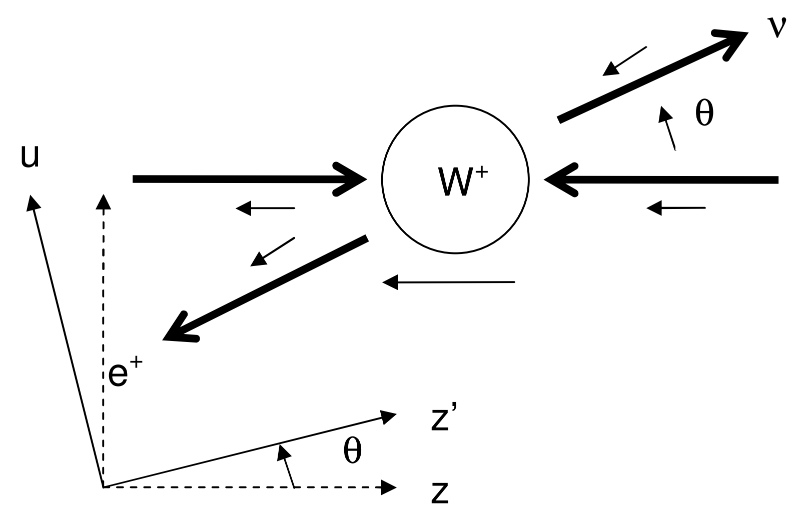
Show that $R_y(\theta) = \exp(i\theta J_y) = \tfrac{1}{2}\begin{pmatrix} 1+\cos\theta & 0 & 1-\cos\theta \\ 0 & 2\cos\theta & 0 \\ 1-\cos\theta & 0 & 1+\cos\theta \end{pmatrix} + \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sin\theta & 0 \\ -\sin\theta & 0 & \sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix}$

Hence

$$d^1_{+1\,+1}(\theta) = \tfrac{1}{2}(1+\cos\theta) \quad d^1_{+1\,0}(\theta) = \tfrac{-1}{\sqrt{2}}\sin\theta \quad d^1_{+1\,-1}(\theta) = \tfrac{1}{2}(1-\cos\theta)$$

$$d^1_{0\,0}(\theta) = \cos\theta \quad \text{etc}$$

W production and decay $u\bar{d} \to W^+ \to e^+\nu$



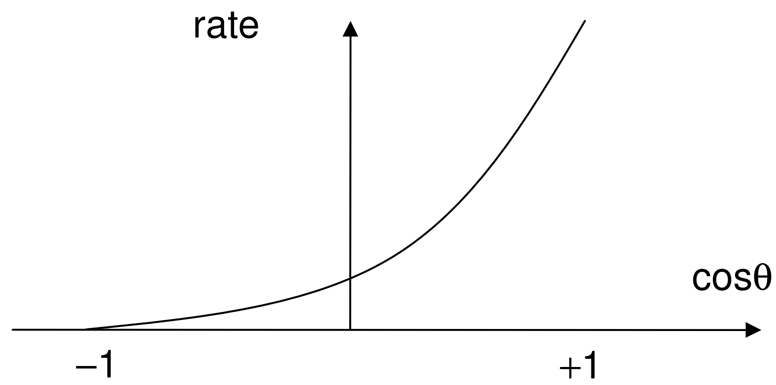
The initial state consists of two aligned spin-½ states in a (j=1,m=1) configuration; so does the final state.
What is the projection of the initial |11> state in the initial frame onto a |11> state in the final frame ?

$$d^1_{+1\ +1}(\theta) = \tfrac{1}{2}(1+\cos\theta)$$

So the decay rate is

$$\frac{dN}{d\cos\theta} \sim (d^1_{+1\ +1}(\theta))^2 \sim (1+\cos\theta)^2$$

The famous V-A distribution.



Gauge Transformations and the Adjoint

Here we will sketch out **Gauge Transformations** in **non-Abelian groups**.

We consider transformations $\exp(i\alpha_i(x)X_i)$ – the indices are summed over, and will be dropped.

Under an infinitisimal gauge transformation:

$\psi \to \exp(i\alpha X)\psi = (1+i\alpha X)\psi$ and the conjugate state transforms $\overline{\psi} \to \overline{\psi}\exp(-i\alpha X) = \overline{\psi}(1-i\alpha X)$

To maintain the invariance of the Lagrangian, the derivative is replaced by the covariant derivative

$$D^\mu = \partial^\mu - iXW^\mu$$

where a vector field W^μ has been introduced. This field transforms:

$$W^\mu \to W^\mu + \delta W^\mu$$

The Lagrangian $L \sim \overline{\psi}\not{D}\psi$ (drop slash corresponding to contraction with γ matrices) transforms:

$$L \to \overline{\psi}(1-i\alpha X)(\partial - iXW - iX\delta W)(1+i\alpha X)\psi$$

Expanding to first order in α and δW:

$$L \to \overline{\psi}(\partial - iXW - iX\delta W + i(\partial\alpha)X + i\alpha XXW - i\alpha XWX)\psi = \overline{\psi}(\partial - iXW - iX\delta W + i(\partial\alpha)X + i\alpha[X,X]W)\psi$$

If this is to be invariant:

$$-iX\delta W + i(\partial\alpha)X + i\alpha[X,X]W = 0$$

Replacing [X,X] with ifX – where the f's are the structure constants

$$iX\delta W = i(\partial\alpha)X - \alpha fXW$$

Equating all the coefficients of the X's (they form a basis):

$$\delta W = \partial\alpha + i\alpha fW$$

Since the structure constants are effectively the adjoint, the gauge field transforms "according to the adjoint representation" [Aitchison & Hey].

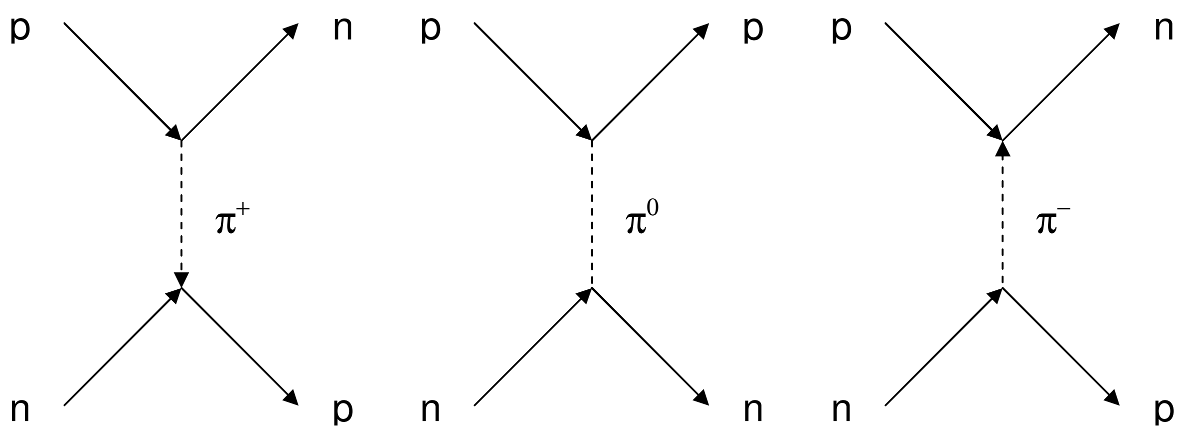
For particular quantum numbers, we consider a suitable vector space and associate the matter fields (fermion, spin ½) with the base vectors of the vector space – sometimes called confusingly the **fundamental representation**.
The associated intermediate gauge bosons are associated with the adjoint.

Local symmetry	**Base vectors**	**Adjoint**
Nuclear Isospin	(n,p)	π^+,π^0,π^-
Quark Isospin	(u,d)	π^+,π^0,π^- – made from quarks!
Weak Isospin	(u,d), (ν,e) etc	W^+, W^0, W^-
QCD	(r,b,g)	8 gluons

Hadronic Isospin [5]

Experimentally, the Hamiltonian describing nucleon interactions is (fairly) independent of whether a nucleon is a proton or a neutron – is has a **symmetry** which is called **Isospin**.

As far as strong interactions are concerned, the proton and neutron are indistinguishable and hence in QM, they are interchangeable.
This interchange is brought about by the gauge bosons of the symmetry,



It turns out that the adjoint of SU(2) is a triplet which transforms like a representation of SO(3).
They also form a basis in 3D for SU(2).

The symmetry between the three pions, means that in high energy collisions, they are produced in approximately equal numbers.

We can treat the nucleon as particle which has two possible states: p or n, and we denote the state by a vector (p,n).

In terms of quantum numbers (I,I_3) – along the same lines as the angular momentum labels:

Doublet	$p = (½,+½)$		$n = (½,–½)$
Triplet	$\pi^+ = (1,+1)$	$\pi^0 = (1,0)$	$\pi^- = (1,–1)$

In reality, the hadrons are distinguishable (else we would not have been able to label them) – they are distinguished by their electric charge and mass. These break the symmetry of the Hamiltonian, but since the EM effects are ~1/100th the size of the strong interaction, the symmetry is fairly good.

By observation, we can write the charge operator:

$$Q = ½ B + I_3$$

This corresponds to a symmetry $U(1)_B \otimes SU(2)_I$

Any complete Hamiltonian contains a Q. Since Q commutes with I_3 and I^2, so I_3 and I^2 will correspond to conserved quantum numbers; however Q does not commute with I (vector), and so will not be invariant under general isospin transformations, such as those corresponding to p-n interchange.

Historically, a lot was made of this symmetry to understand nucleon scattering and production.
We will examine how one can use the “machinery” when we look at the quark model.

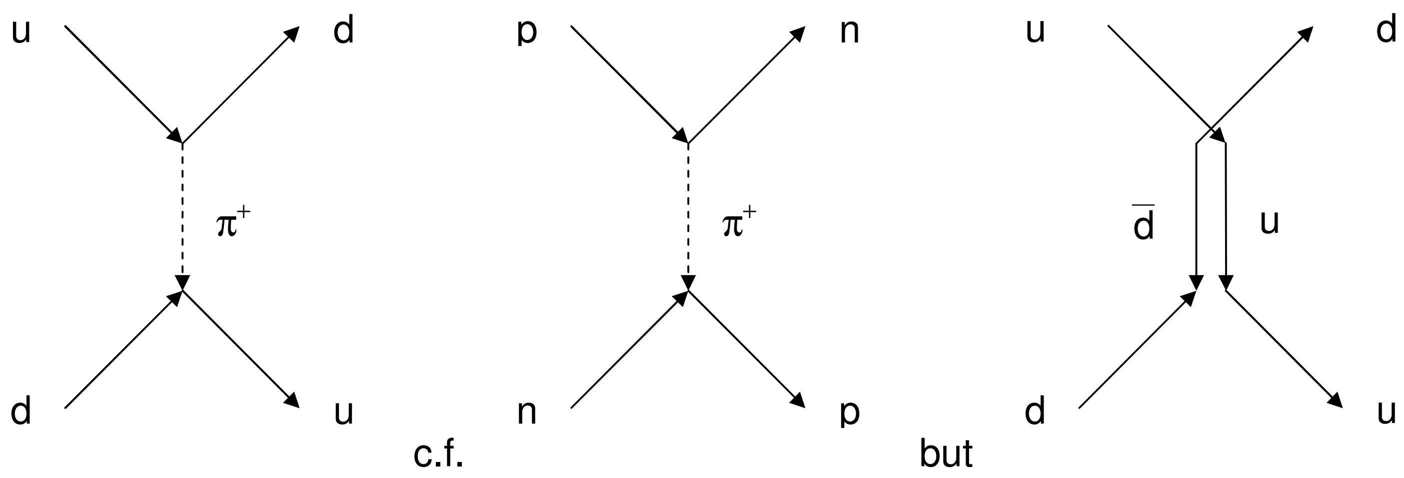
As an example [5.6], the **deuteron** is made from a proton and a neutron in a ground-state s-wave (symmetric spatial wave-function). It appears as a single state (no other comparable particles) and this is in an I=0 antisymmetric state.
To ensure the complete wave-function for two “identical” particles is antisymmetric, the spins must be in a symmetric state, namely S = 1.

Quark Isospin

The reason that Isospin is a good symmetry at the nucleon level is because of the underlying symmetry associated with the quarks, in particular {u, d}:

- The principle interactions are via QCD interactions, which are flavour-blind
- They have similar (negligible?) masses

Consequently there is an SU(2) isospin symmetry associated with the base states {u, d}.
The adjoint can be associated with exchange pions – although though this is actually a "trivial" exchange of quarks which is found at lower energies, while at higher energies, the principle exchange is via gluons – the gauge bosons of $SU(3)_{colour}$ – between colour charges.



SU(2) for strong interactions of quarks is less useful.
However, it will prove useful for hadron classification.

Weak Isospin

Weak Isospin is an exact symmetry of the SM (by construction).
The base vectors are formed from ($I_3 = +½$, $I_3 = -½$) – there are 6 particle states which have these quantum numbers:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L$$

Note, all these are left-handed fermion states: left-handed refers to the **chirality**, corresponding to the operator $½(1-\gamma_5)$.

The local SU(2) gauge symmetry leads to a vector of fields: $\{W_1, W_2, W_3\}$
These relate to the fields $\{W^+, W^0, W^-\}$, where $W^\pm = \frac{1}{\sqrt{2}}(W_1 \mp iW_2)$ and the W^0 field is transformed into the Z-boson field.

Conjugate States

In SU(2), define the **base state 2** $\equiv \begin{pmatrix} u \\ d \end{pmatrix}$ (sometimes called the "fundamental representation").

We would like to form a **conjugate state** which will correspond to the antiquarks and which will have *convenient* transformation properties.

The conjugate will contain the antiquark wave-functions, and to be consistent with normal properties, we expect the quantum numbers of the antiquarks to be negated with respect to those of the quarks. This is also consistent with the expression Q = ½ B + I_3 (in the case of quark isospin).

	Q	B	$I_3 = Q - \frac{1}{2} B$
u	$+\frac{2}{3}$	$+\frac{1}{3}$	$+\frac{1}{2}$
$\bar{u}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{1}{2}$
d	$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{2}$
$\bar{d}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$+\frac{1}{2}$

Under SU(2), the **2** transforms like:

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \exp(i\tfrac{\theta}{2} n \cdot \sigma) \begin{pmatrix} u \\ d \end{pmatrix} = (\cos\tfrac{\theta}{2} + i \sin\tfrac{\theta}{2} n \cdot \sigma) \begin{pmatrix} u \\ d \end{pmatrix}$$

Antiparticles behave like the complex conjugates [Halzen & Martin, section 5.4]:

$\bar{u} \sim (u)^*$ and $\bar{d} \sim (d)^*$

(Consider creation and annihilation operators in Quantum Field Theory.)

Using the expression for the transformation of **2** ≡ (u, d) and taking the complex conjugate (not Hermitian):

$$\begin{pmatrix} u'^* \\ d'^* \end{pmatrix} = (\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\,n\cdot\sigma^*)\begin{pmatrix} u^* \\ d^* \end{pmatrix}$$

So the transformation of the antiquark states is given by:

$$\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = (\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\,n\cdot\sigma^*)\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

It will prove to be convenient if the conjugate state will have the same transformation properties in SU(2) as the **2**.

To this end, we define the conjugate to be: $\bar{\mathbf{2}} = M\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$ where M is a 2×2 matrix and we would like:

$$M\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = (\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\,n\cdot\sigma)M\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

The challenge is to find a suitable M.

Premultiplying the earlier expression by M:

$$M\begin{pmatrix} \bar{u}' \\ \bar{d}' \end{pmatrix} = M(\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\,n\cdot\sigma^*)\begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}$$

So

$$M(\cos\tfrac{\theta}{2} - i\sin\tfrac{\theta}{2}\,n\cdot\sigma^*) = (\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2}\,n\cdot\sigma)M \Rightarrow -M\sigma^* = \sigma M \Rightarrow \sigma^* = -M^{-1}\sigma M$$

$$\sigma^* = \begin{pmatrix} \sigma_1 \\ -\sigma_2 \\ \sigma_3 \end{pmatrix} = \begin{pmatrix} -M^{-1}\sigma_1 M \\ -M^{-1}\sigma_2 M \\ -M^{-1}\sigma_3 M \end{pmatrix}$$

Bearing in mind $\sigma_i\sigma_j = \sigma_j\sigma_i$ for i=j and $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$, we can satisfy the above if M = *phase*$\times\sigma_2$ (consider the terms $\sigma_i M$ and consequences of commuting the two terms to remove M via $M^{-1}M = I$).

We take $M = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$ so $\overline{\mathbf{2}} = \begin{pmatrix} \overline{d} \\ -\overline{u} \end{pmatrix}$ will transform like $\mathbf{2} \equiv \begin{pmatrix} u \\ d \end{pmatrix}$.

This will turn out to be very useful for the formation of $q\overline{q}$ states.
It is the result of some good fortune in SU(2) and is not the case in other SU(n) groups.

Note that above, we took the complex conjugate, not the Hermitian conjugate, since we wanted to understand how the column vector $\begin{pmatrix} \overline{u} \\ \overline{d} \end{pmatrix}$, not the row vector $(\overline{u}, \overline{d})$, transforms.

Combining States [3.5]

We wish to construct multiparticle states which have well-defined properties under transformations, in this case, under SU(2).

First question: How do we undertake transformations on multiparticles states ?

The transformations we wish to consider are on internal degrees of freedom (associated with quantum numbers, rather than space-time).
It is helpful to think of the analogy with spatial transformations, for example rotations:

Consider a state corresponding to individual particles a and b: $\psi_{ab}(x) = \psi_a(x)\psi_b(x)$
The wavefunctions are functions of the common space-time.

If we consider spatial rotations about any particular axis by an angle θ, with corresponding angular momentum operator $L = \hat{\theta}\cdot r \times (-i\nabla)$, then the transformed state is:

$$\psi_{ab}'(x) = \exp(i\theta L)\psi_a(x)\psi_b(x) = \Sigma \tfrac{(i\theta L)^n}{n!}\psi_a(x)\psi_b(x)$$

From the product rule for differentiation:

$$L\psi_a(x)\psi_b(x) = \{L\psi_a(x)\}\psi_b(x) + \psi_a(x)\{L\psi_b(x)\} = (L_a + L_b)\psi_a(x)\psi_b(x)$$

$$L^2\psi_a(x)\psi_b(x) = \{L^2\psi_a(x)\}\psi_b(x) + 2\{L\psi_a(x)\}\{L\psi_b(x)\} + \psi_a(x)\{L^2\psi_b(x)\} = (L_a + L_b)^2\psi_a(x)\psi_b(x)$$

where L_a is the angular momentum operator which is restricted to operate only on $\psi_a(x)$.
Therefore

$$\psi_{ab}'(x) = \Sigma \tfrac{(i\theta(L_a + L_b))^n}{n!}\psi_a(x)\psi_b(x) = \exp(i\theta(L_a + L_b))\psi_{ab}(x)$$

So we find that the angular momentum operator for the combined state is equal to the sum of operators for individual states.
We will assume the same for the generators in SU(2) and assume $J^{ab} = J^a + J^b$ (vector relationship) and for the third component: $J_3^{ab} = J_3^a + J_3^b$.

Therefore:

$$J^2 = J^{a^2} + J^{b^2} + 2J^a \cdot J^b = J^{a^2} + J^{b^2} + 2(J_1^{\,a}J_1^{\,b} + J_2^{\,a}J_2^{\,b} + J_3^{\,a}J_3^{\,b}) = J^{a^2} + J^{b^2} + J_+^{\,a}J_-^{\,b} + J_-^{\,a}J_+^{\,b} + 2J_3^{\,a}J_3^{\,b}$$

and

$$J_\pm = J_\pm^{\,a} + J_\pm^{\,b}$$

We label the states $u \equiv |\tfrac{1}{2} + \tfrac{1}{2}>$ and $d \equiv |\tfrac{1}{2} - \tfrac{1}{2}>$ – think of these as spin or isospin (nucleon or quark flavour) states.

Start with the fully aligned state: $\psi_{ab} = \psi_a \psi_b$ with $\psi_a = u$ and $\psi_b = u$

$$J_3 uu = (J_3^{\,a} + J_3^{\,b})uu = \{J_3^{\,a}u\}u + u\{J_3^{\,b}u\} = \{\tfrac{1}{2}u\}u + u\{\tfrac{1}{2}u\} = uu$$

$$J^2 uu = (J^{a^2} + J^{b^2} + J_+^{\,a}J_-^{\,b} + J_-^{\,a}J_+^{\,b} + 2J_3^{\,a}J_3^{\,b})uu = (\tfrac{3}{4} + \tfrac{3}{4} + 0 + 0 + 2\tfrac{1}{2}\tfrac{1}{2})uu = 2uu = 1(1+1)uu$$

We see this state has (j, m) = (1, 1), and so $|1 + 1 >= uu$

Other states can be obtained using J_-. With $J_- u = d$:

$$J_- uu = (J_-^{\,a} + J_-^{\,b})uu = \{J_-^{\,a}u\}u + u\{J_-^{\,b}u\} = du + ud$$

This state can be shown to have (j, m) = (1, 0), and so $|10 >= \tfrac{1}{\sqrt{2}}(ud + du)$

Finally, we can obtain $|1 - 1 >= dd$

The missing combination which can be obtained by orthogonality is $|00 >= \tfrac{1}{\sqrt{2}}(ud - du)$

Multiplets

The benefit of constructing multiplets consisting of states of a given j and well-defined m is that under SU(2) transformations, states transform to states of the same multiplet – and therefore the members of the multiplet have related properties, such as masses or decay properties.

A state $|\psi\rangle = |j\ m\rangle$ transforms in SU(2) to $|\psi'\rangle = U|\psi\rangle \equiv \exp(i\theta\cdot J)\ |j\ m\rangle$.
Since J^2 commutes with all the J_is and their powers, it will commute with $U \equiv \exp(i\theta\cdot J)$, and thus the J^2 quantum number of the transformed state will be unchanged.

If the (mass) Hamiltonian H is invariant under the SU(2) transformations, then the mass of the transformed state (also in the multiplet) is

$$M' = \langle \psi' | H | \psi' \rangle = \langle \psi U^H | H | U\psi \rangle = \langle \psi | U^H H U | \psi \rangle = \langle \psi | H | \psi \rangle = M$$

(Do not add dash to H, else M′≡M by definition of H′. Instead, consider H for new states.)

Illustration for the singlet $|00\rangle \sim ud - du = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$, subject to a rotation $\exp(i\alpha\cdot\sigma)$.

$$\exp(i\alpha\cdot\sigma) = \exp(i\alpha\cdot\sigma^a)\exp(i\alpha\cdot\sigma^b) \quad \text{where } \sigma^a \text{ acts on the first particle etc.}$$

So

$$\exp(i\alpha\cdot\sigma)|00\rangle = \exp(i\alpha\cdot\sigma^a)\exp(i\alpha\cdot\sigma^b)\{\begin{pmatrix}1\\0\end{pmatrix}_a\begin{pmatrix}0\\1\end{pmatrix}_b - \begin{pmatrix}0\\1\end{pmatrix}_a\begin{pmatrix}1\\0\end{pmatrix}_b\}$$

Recalling that $\exp(i\alpha\cdot\sigma) = \cos\alpha + i\sin\alpha\ n\cdot\sigma = \begin{pmatrix}\cos\alpha + i\sin\alpha\, n_3 & i\sin\alpha\,(n_1 - in_2)\\ i\sin\alpha\,(n_1 + in_2) & \cos\alpha - i\sin\alpha\, n_3\end{pmatrix}$

$$\exp(i\alpha\cdot\sigma)\,|\,00> = \begin{pmatrix}\cos\alpha+i\sin\alpha\; n_3\\ i\sin\alpha\;(n_1+in_2)\end{pmatrix}\begin{pmatrix}i\sin\alpha\;(n_1-in_2)\\ \cos\alpha-i\sin\alpha\; n_3\end{pmatrix}-\begin{pmatrix}i\sin\alpha\;(n_1-in_2)\\ \cos\alpha-i\sin\alpha\; n_3\end{pmatrix}\begin{pmatrix}\cos\alpha+i\sin\alpha\; n_3\\ i\sin\alpha\;(n_1+in_2)\end{pmatrix}$$

It is helpful to think what these vectors look like in terms of the base vectors. If we look at the first vector for particle "a":

$$\begin{pmatrix}\cos\alpha+i\sin\alpha\; n_3\\ i\sin\alpha\;(n_1+in_2)\end{pmatrix}\equiv(\cos\alpha+i\sin\alpha n_3)\begin{pmatrix}1\\0\end{pmatrix}+i\sin\alpha(n_1+in_2)\begin{pmatrix}0\\1\end{pmatrix}$$

So the coefficients of $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ are

$$(\cos\alpha+i\sin\alpha n_3)\times i\sin\alpha(n_1-in_2)-i\sin\alpha(n_1-in_2)\times(\cos\alpha+i\sin\alpha n_3)=0$$

And the coefficients of $\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$ are

$$(\cos\alpha+i\sin\alpha\; n_3)(\cos\alpha-i\sin\alpha\; n_3)-i\sin\alpha\;(n_1-in_2)\times i\sin\alpha\;(n_1+in_2)=(\cos^2\alpha+\sin^2\alpha\; n_3{}^2)+\sin^2\alpha\;(n_1{}^2+n_2{}^2)$$

$$\Rightarrow \exp(i\alpha\cdot\sigma)\,|\,00>=0\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+\{(\cos^2\alpha+\sin^2\alpha\; n_3{}^2)+\sin^2\alpha\;(n_1{}^2+n_2{}^2)\}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$-\{\sin^2\alpha\;(n_1{}^2+n_2{}^2)+(\cos^2\alpha+\sin^2\alpha\; n_3{}^2)\}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+0\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=|\,00>$$

So we see the $|\,00>$ singlet state is unchanged by an SU(2) transformation.

Exercise

Using the above, change the sign to give $|10> \sim ud + du = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ and show that this transforms to an admixture of j=1 states.

Do the same for $|1{+}1> \sim uu = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$ and $|1{-}1> \sim dd = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$

In general, a transformation $\exp(i\theta\cdot J)$ can be expanded into a power series in J^n and any term can be expressed as a product of J^2 and zero or one J. While J, expressed as (J_1,J_2,J_3) or (J_+,J_-,J_3), can change the m quantum number, it will not change the j quantum number and hence the effect of $\exp(i\theta\cdot J)$ on a state |j m> will be to produce an admixture of states |j m'> – all having the same j.

Meson States

If we now interpret the states in terms of the u & d quark wavefunctions, we can build the states:

$$|1\,{+}1> = uu$$
$$|1\,0> = \tfrac{1}{\sqrt{2}}(ud + du)$$
$$|1\,{-}1> = dd$$

$$|0\,0> = \tfrac{1}{\sqrt{2}}(ud - du)$$

However, there are no observed qq states. Instead, if we wish to create the mesons, which are $q\bar{q}$ states, we recall that $\bar{\mathbf{2}} = \begin{pmatrix} \bar{d} \\ -\bar{u} \end{pmatrix}$ transforms like $\mathbf{2} \equiv \begin{pmatrix} u \\ d \end{pmatrix}$.

So to obtain the $q\bar{q}$ states, we replace $u \rightarrow \bar{d}$ and $d \rightarrow -\bar{u}$:

$$|1\,{+}1> = u\bar{d}$$
$$|1\,0> = \tfrac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$
$$|1\,{-}1> = (-)d\bar{u}$$
$$|0\,0> = \tfrac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

while some of the signs are important (those which are “internal”), there are global phases which are not significant and can be dropped.

With have a triplet with I = 1 and a singlet with I = 0.
We associate these states with the triplet of pions: $\{\pi^+, \pi^0, \pi^-\}$, but what about the singlet ?
Is it the η or the η' ? We will see when we look at $SU(3)_{Isospin}$.

Weights [6.1, 6.3, 6.4]

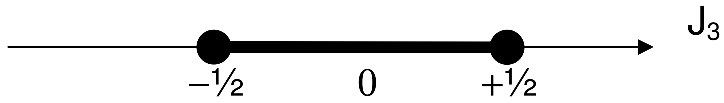
We want to find the set of commuting Hermitian operators for a given system, since these correspond to the independent, simultaneous observables.
The set of commuting generators is called the **Cartan Subalgebra**.
The associated eigenvalues are the **Weights**.
Weight Vectors are constructed from the eigenvalues from all of the commuting generators for a given label of the base vector.
(The weights of the adjoint representation are called the **Roots** – they are related to the raising and lowering operators.)

In SU(2), the simplest representation is in terms of the $(2^2-1)=3$ 2×2 σ matrices.
There is 1 commuting operator, J_3, giving rise to 2 weight vectors, ($m=-½$) and ($m=+½$) – these are just points on a line.
In SU(3), the simplest representation is in terms of the $(3^2-1)=8$ 3×3 λ matrices.
There are 2 commuting operators, λ_3 and λ_8, giving rise to 3 weight vectors, of the form (m,Y) – these are points in a plane.

The weights are useful for classifying states and combinations of states.

Weights in SU(2)

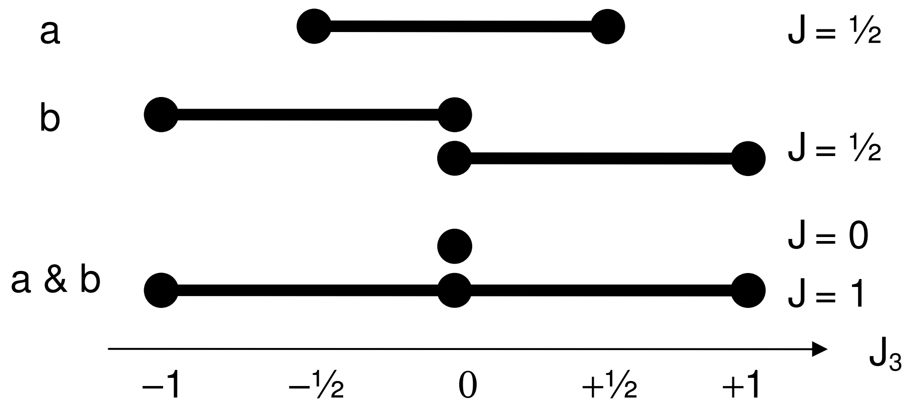
There are no commuting generators in SU(2) – so the Cartan subalgebra has only one generator which trivially commutes with itself – we typically choose J_3. The weights are –½ and +½.
We can represent this as a pair of points on a line:



When we combine **two particles** (meson state or two spin-½ states), in the language of spin:

Spin	$\frac{1}{2} \otimes \frac{1}{2}$	$= 1 \oplus 0$
Multiplicity	$2 \otimes 2$	$= 3 \oplus 1$

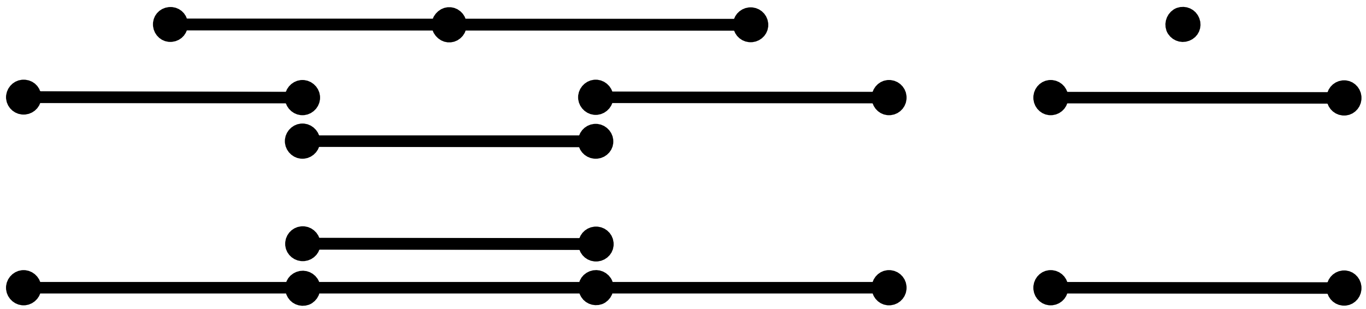
We add the weight diagrams for the second particle to the nodes of the first:



Baryon States [3.5]

When we combine **three particles** (baryon state or three spin-½ states), in the language of spin:

Spin	$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$	$= (1 \oplus 0) \otimes \frac{1}{2}$	$= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$
Multiplicity	$2 \otimes 2 \otimes 2$		$= 4 \oplus 2 \oplus 2$



J = 1 & J = 0

J = ½

J = 3/2 & J = ½
& J = ½

The weight diagrams help us to identify the quantum numbers of the resultant states.

To identify the quark description of the states, we proceed as with the mesons:

1. Construct the maximally aligned state: uuu
2. Use the lowering operator: $I_- = I_-^a + I_-^b + I_-^c$
3. Construct other states using orthogonality and the lowering operators (this is not unambiguous)

We can create states:

I = 3/2 (= 1 + ½) Symmetric in $1 \leftrightarrow 2 \leftrightarrow 3$

$I_3 = +3/2$ uuu

$I_3 = +½$ $\frac{1}{\sqrt{3}}(uud + udu + duu)$

$I_3 = -½$ $\frac{1}{\sqrt{3}}(udd + dud + ddu)$

$I_3 = -3/2$ ddd

I = ½ (= 1 − ½) Symmetric in $1 \leftrightarrow 2$

$I_3 = +½$ $\sqrt{\frac{2}{3}}(uud - \frac{udu + duu}{2})$

$I_3 = -½$ $\sqrt{\frac{2}{3}}(ddu - \frac{dud + udd}{2})$

I = ½ (= 0 + ½) Asymmetric in $1 \leftrightarrow 2$

$I_3 = +½$ $\frac{1}{\sqrt{2}}(ud - du)u$

$I_3 = -½$ $\frac{1}{\sqrt{2}}(du - ud)d$

We can identify the I = 3/2 multiplet as the particles: $(\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-})$
Which states correspond to (p, n), since there are two I = ½ multiplets ?

<u>Exercise</u>

Show that the effect of the lowering operator on the state (I=½, I_3=+½) is to produce the state (I=½, I_3=−½).

Clebsch Gordon Coefficients

The **Clebsch Gordon Coefficients** are used to describe states in terms of the combination of other pairs of states.

They are defined by:

$$| jm >= \sum_{m_1,m_2} C^{j\,j_1\,j_2}_{m\,m_1\,m_2}\, | j_1 m_1 >| j_2 m_2 > \quad \text{with} \quad m = m_1 + m_2$$

These numbers are tabulated in the **Particle Data Book.**

So for example:

$$\begin{pmatrix} \frac{3}{2} \\ \frac{+1}{2} \end{pmatrix} = \sqrt{\tfrac{1}{3}} \begin{pmatrix} 1 \\ +1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix} + \sqrt{\tfrac{2}{3}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{+1}{2} \end{pmatrix} = \sqrt{\tfrac{1}{3}}(uu)d + \sqrt{\tfrac{2}{3}}\, \frac{ud+du}{\sqrt{2}} u = \tfrac{1}{\sqrt{3}}(uud + udu + duu)$$

and

$$\begin{pmatrix} \frac{1}{2} \\ \frac{+1}{2} \end{pmatrix} = \sqrt{\tfrac{2}{3}} \begin{pmatrix} 1 \\ +1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix} - \sqrt{\tfrac{1}{3}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{+1}{2} \end{pmatrix} = \sqrt{\tfrac{2}{3}}(uu)d - \sqrt{\tfrac{1}{3}}\, \frac{ud+du}{\sqrt{2}} u = \sqrt{\tfrac{2}{3}}(uud - \tfrac{1}{2}(ud+du)u) \quad - \text{sym in } 1 \leftrightarrow 2$$

What about the antisymmetric form in $1 \leftrightarrow 2$?

$$\begin{pmatrix} \frac{1}{2} \\ \frac{+1}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{+1}{2} \end{pmatrix} = \frac{ud-du}{\sqrt{2}} u$$