# LECTURE 2 – GROUPS

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- Introduction to Groups
- Lie Groups
- Unitary Groups
- U(1)

## Messages

- QM is characterised by **vector spaces** of **quantum numbers**
- We are interested in the properties of QM systems under **transformations** operating on the vector spaces
- Transformations form **groups**
- Group theory gives us the language and tools for understanding the properties

# Introduction [1.1]

A **Group** consists of:

- Set of objects: G = {a, b, c …}
- Binary operation for combining them: • often the operation is "follows". Often one drops the operation sign.

It must satisfy:

- Closure: $\forall$ a, b $\in$ G, a•b $\in$ G
- Associativity: (a•b)•c = a•(b•c)
- Identity: $\exists$ e $\in$ G s.t. $\forall$ a $\in$ G, a•e = a
- Inverse: $\forall$ a $\in$ G, $\exists$ $a^{-1} \in$ G, s.t. $a \bullet a^{-1} = e$

Note: one can prove that the identity is also a left-identity, and the inverse is a left-inverse:

Introduce b, the inverse of $a^{-1}$: $a^{-1} \bullet b = e$

Then $e \bullet a = e \bullet (a \bullet e) = e \bullet (a \bullet (a^{-1} \bullet b)) = e \bullet ((a \bullet a^{-1}) \bullet b) = e \bullet (e \bullet b) = (e \bullet e) \bullet b = e \bullet b = (a \bullet a^{-1}) \bullet b = a \bullet (a^{-1} \bullet b) = a \bullet e = a$

And $a^{-1} \bullet a = a^{-1} \bullet (a \bullet e) = a^{-1} \bullet (a \bullet (a^{-1} \bullet b)) = a^{-1} \bullet ((a \bullet a^{-1}) \bullet b) = a^{-1} \bullet (e \bullet b) = (a^{-1} \bullet e) \bullet b = a^{-1} \bullet b = e$

The operator need not be commutative, however, if

- Commutivity: $\forall$ a, b $\in$ G, a•b = b•a

G is said to be **Abelian**.

# Example – Finite Group [1.2]

Consider a set of rotations:

- R0 = no rotation
- R+ = rotation of $+120^o$
- R– = rotation of $-120^o$

Operation is "follows".

| "Follows" | | $2^{nd}$ rotation | | |
|---|---|---|---|---|
| | | R0 | R+ | R– |
| $1^{st}$ rotation | R0 | R0 | R+ | R– |
| | R+ | R+ | R– | R0 |
| | R– | R– | R0 | R+ |

Closed ✓
Associative ✓
Identity: R0
Inverses: $(R0)^{-1} = R0$
$(R+)^{-1} = R-$
$(R-)^{-1} = R+$

This an example of the **Cyclic Group** of order 3: $\mathbf{Z_3}$.
It is also **finite** since it has a finite number of elements.

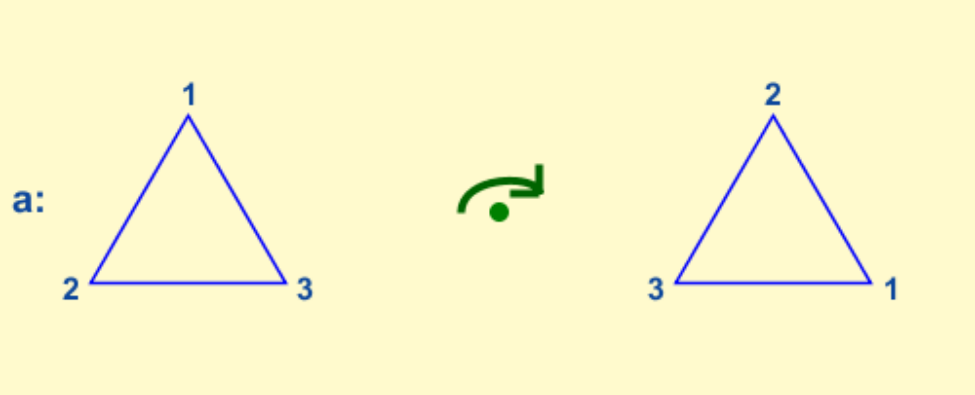
# Subgroups & Isomorphism [1.7, 1.10]

Consider permutations of a set of 3 objects {O1, O2, O3} – 3! permutations
Permutations:

e ≡ (1,2,3) corresponds to O1→O1, O2→O2, O3→O3
a ≡ (2,3,1) corresponds to O1→O3, O2→O1, O3→O2
b ≡ (3,1,2)
x ≡ (1,3,2)
y ≡ (3,2,1)
z ≡ (2,1,3)

Helps to think of rotations (a,b) and reflections (x,y,z) of an equilateral triangle:

Eg: a ≡ (2,3,1)



| | e | a | b | x | y | z |
|---|---|---|---|---|---|---|
| e | e | a | b | x | y | z |
| a | a | b | e | z | x | y |
| b | b | e | a | y | z | x |
| x | x | y | z | e | a | b |
| y | y | z | x | b | e | a |
| z | z | x | y | a | b | e |

This corresponds to the **Symmetry Group** of order 3: $\mathbf{S_3}$.

We see the set {e, a, b} forms a **Subgroup**.
There are 3 other subgroups: {e, x}, {e, y}, {e, z} – all examples $\mathbf{S_2}$.

Furthermore, the subgroup {e, a, b} has the same form as $Z_3$.
Groups which have different physical origins, and yet the same group structure are **Isomorphic**.
Two groups

G = {a, b, c} with •
G' = {a', b', c'} with ▪

are Isomorphic if there is a one-to-one correspondence between their members, such that the "products" also correspond to each other:

If c = a • b and c' = a' ▪ b'
Then for $\forall$ a, b, a', b'
$a \leftrightarrow a'$, $b \leftrightarrow b'$ and $c \leftrightarrow c'$

Note: $S_2$ and $Z_2$ are isomorphic.

# Example – Continuous Group

Consider continuous rotations in 2D.
A rotation by an angle $\alpha$ about the origin can be represented by a rotation matrix:

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

These rotations can readily be shown to form a group.
In 2D it is Abelian … but not in 3D.
This is an example of a **Special Orthogonal** transformation in 2D: **SO(2)**.

Consider multiplication of complex numbers, in particular by phasors: $e^{i\alpha}$
This is an example of a **Unitary** transformation in 1D: **U(1)**.

It is obvious by consideration of the Argand plane that SO(2) and U(1) are isomorphic.

# Representations [1.1]

The **Representation** of a group G = {e, a, b} is a mapping D onto a set of **linear operators** acting on a vector space, such that:

D(e) = I – the identity in the vector space

D(a)D(b) = D(a•b)

In other words, the group "product" is mapped onto the multiplication operation in the vector space.

This mapping is usually isomorphic.

Often the operators are chosen to be matrices.
Their form is not unique and depends explicitly on the vector space, in particular its dimensionality.

We are more interested in the representations than the abstract groups.

Note: Some authors (e.g. Close) use "representation" to refer to the vector space itself.

# Example – Representations

U(1) has a natural representation in 1D, corresponding to phasors.
This will also form a representation for SO(2).

SO(2) – rotations in 2D – have a natural representation as 2×2 matrices (the fundamental representation), but there are others:

| Vector space | Representation |
|---|---|
| 2D space (x,y) | $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ |
| 3D space (x,y,z) | $\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| Hilbert space for wave-functions | $\exp(i\alpha L_z) = \exp(\alpha(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}))$ |
| Complex numbers | $\exp(-i\alpha)$ |

# Regular Representations [1.3]

How do we form representations ?
Sometimes the physical concept behind the group will suggest a representation – as was obvious when we considered $S_3$ in terms of rotations and reflections.

Another way is to form the Regular Representation:

Take the group elements to form a basis: {| e >, |a>, |b>, …}
In vector notation:

$$|e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \dots \end{pmatrix}, |a\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \dots \end{pmatrix}, \text{etc}$$

Define **Regular Representation** {D(a)} by  D(a) |b> ≡ |a•b>
This is indeed a representation:

D(e) |a> = |ea> = |a>  ⇒  D(e) = I

D(a•b) |c> = |(a•b)c> = |a(bc)> = D(a) |bc> = D(a) D(b) |c>  ∀ c  ⇒  D(a•b) = D(a) D(b)

The matrices can be constructed:  $D(a)_{ij} \equiv \langle i|\ (D(a)\ |j\rangle = \langle i|a•j\rangle$ – the projection of the vector corresponding to the product |a•j> onto |i>

# Example – Regular Representation [1.3]

Consider $Z_3$:

| | R0 | R+ | R– |
|---|---|---|---|
| R0 | R0 | R+ | R– |
| R+ | R+ | R– | R0 |
| R– | R– | R0 | R+ |

To find D(R+), look at |R+•j> - corresponds to shaded row; the column headings provide the "j" label. Obtain matrix element $R+_{ij}$ by

D(R+)

| "i" label | "j" label: R0 | R+ | R– |
|---|---|---|---|
| R0 | 0 | 0 | 1 |
| R+ | 1 | 0 | 0 |
| R– | 0 | 1 | 0 |

So $D(R0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(R+) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, D(R-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

These are matrices operating in 3D corresponding to rotations:

D(R0): identity; D(R+): x→y, y→z, z→x and D(R–): x→z, y→x, z→y

# Irreducible Representations [1.4]

The representations are active on a **vector space**.

If the action of all the operators on vectors v in some **subspace** S is to produce vectors in the same subspace, then the representation is said to be **reducible**.

For $\forall$ v $\in$ S and $\forall$ g $\in$ G, D(g) v $\in$ S

For example in $Z_3$, there is an invariant subspace of all the points with x = y = z, i.e. all the points on this axis are left unchanged by the three rotations D(R0), D(R+) and D(R–).
Thinking of what $Z_3$ represents, this corresponds to (cyclic) permutations of 3 identical objects (think pieces of fruit !).
(Care should be taken in always assuming the space is a real x-y-z space – it may not be !)

It turns out that the subspaces will correspond to **multiplets** of particles.

An **irreducible** representation is one which is not reducible.
A **completely reducible representation** is one which can be broken down in to a sum of independent irreducible representations (see [1.4]).

# Example – Parity [1.6]

Parity is an example of $Z_2$:

| | e | P |
|---|---|---|
| e | e | P |
| P | P | e |

The regular representation is:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can interpret D(P) as a reflection in x=y. (One can envisage an infinite number of reflections in alternative lines, corresponding to alternative representations.)

To reduce the representation, we apply a similarity transformation S: $D(a) \to D'(a) = S^{-1}D(a)S$

This will lead to an alternative representation:

$$D'(a)D'(b) = S^{-1}D(a)S \cdot S^{-1}D(b)S = S^{-1}D(a)D(b)S = S^{-1}D(a \bullet b)S = D'(a \bullet b)$$

Here we make a 45° rotation with $S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow D'(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D'(P) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So we can identify two independent subspaces:

line y=0, corresponding to $\Rightarrow D'(e) = 1, D'(P) = +1$

line x=0, corresponding to $\Rightarrow D'(e) = 1, D'(P) = -1$

Note: every group has a trivial representation: D(a) = 1 for all group members a.

If we have Hamiltonian which is symmetric in the spatial coordinate x, then if we consider parity transformations, P: $x \rightarrow x' = -x$, then [H,P] = 0, implying that parity and energy can be simultaneous observables, and we can construct energy eigenfunctions which have well defined parity.
In particular:

the symmetric functions correspond to the representation with $D(P) = +1$ and

the antisymmetric functions correspond to the representation with $D(P) = -1$

# Lie Groups [2]

(Pronounced “lee”.)

Consider a **continuous group** with a finite number N of parameters – these could be labelled with a vector.
For example:

- Multiplication by a phasor $e^{i\alpha}$
- Rotation in 3D with parameters $\{\theta_x, \theta_y, \theta_z\}$

We can write the group members as $g(\underline{a})$, where $\underline{a}$ is a vector of N parameters.
If the product is $g(\underline{a}'') = g(\underline{a}) \bullet g(\underline{a}')$, then the group is a **Lie Group** if $\underline{a}'' = \Gamma(\underline{a},\underline{a}')$, where $\Gamma$ is an analytic function.

In practice groups which are characterised by parameters in an analytic way will inevitably be Lie groups.
The above examples are Lie groups:

- For phasors, the function is simply the addition of the two phases
- For the rotations in 3D, the function is complicated

# Generators [2.1]

What is most interesting to us is to consider the effects of transformations on particle wavefunctions $\psi(x)$.

Along the lines of what we did in Lect 1, but with a bit more generality:
Let us consider a group of transformations which act on the spatial coordinates x (implicitly a vector):

$$x \to x' = f(x,a)$$

where a is also implicitly a **real vector**. For example, this might correspond to a rotation or a Lorentz transformation.

If we choose the identity transformation to be characterised by a = 0, and we consider an infinitesimal transformation $\delta a$, then

$$x' = x + \delta x = f(x,\delta a) = f(x,0) + \frac{\partial f}{\partial a}\delta a \quad \Rightarrow \delta x = \frac{\partial f}{\partial a}\delta a$$

So

$$\psi(x) \to \psi'(x') \equiv \psi(x) = \psi(x'-\delta x) = \psi(x') - \frac{\partial \psi}{\partial x}\delta x$$

Replacing the dummy index x' by x:

$$\Rightarrow \psi'(x) = \psi(x) - \frac{\partial \psi}{\partial x}\frac{\partial f}{\partial a}\delta a = (1 - \frac{\partial f}{\partial a}\frac{\partial}{\partial x}\delta a)\psi \quad \text{cf } \psi' = U\psi = \exp(i\delta a X)\psi \approx (1 + iX\delta a)\psi$$

Where we identify

$$X = i\frac{\partial f}{\partial a}\frac{\partial}{\partial x}$$ as the **Generator** for the transformation associated with a coordinate transformation.

The above is implicitly summed over the vector indices.

For each parameter (in the vector) there is one generator, so there are a total of N generators for the group of transformations.

Alternatively, one can consider a representation of the transformation acting in the Hilbert space:

$$\psi \rightarrow \psi' = D(\delta a)\psi = (1 + \frac{\partial D}{\partial a}\,\delta a)\psi \quad \text{with} \quad X = -i\frac{\partial D}{\partial a}$$

This is valid for transformations which do not operate on spatial coordinates, but on internal quantum numbers, for example spin or flavour indices.

As we saw in the last lecture, finite transformations can be built up by exponentiating the generators:

$$D(a_i) = \exp(i a_i X_i) \quad \text{no summation.}$$

Care needs to be taken with non-commuting generators.

With a suitable choice of parameters, **Hermitian** generators can be chosen, leading to **Unitary** representations.

To summarise:
Often the easiest way to identify the generator is to look at the infinitesimal transformation:

$$D(\delta a) \approx 1 + iX\delta a$$

# Lie Algebra [2.2]

Lie demonstrated that properties of Lie group can be derived from consideration of elements which differ infinitesimally from the identity.

We have seen when one considers one of the parameters from the vector $\underline{a}$ = ($a_1$, $a_2$, $a_3$, …), the representation corresponding to the transformation is $D(a_i) = \exp(ia_i X_i)$ – no summation.

It is therefore plausible (although not obvious) that the general representation corresponding to $\underline{a}$ is $D(a) = \exp(i\sum_i \Lambda_i(a) X_i)$. The sum is now shown explicitly and the functions $\Lambda_i(a)$ are not trivial as they were for a single parameter due to the potentially non-Abelian nature of the generators.

Expressed differently, any group member can be expressed by a representation which is the exponential of a linear combination of the generators.

So if we consider the combination of two representations, this should result in a third:

$$D(a_1,0,0,...)D(0,a_2,0,...) = \exp(ia_1X_1)\exp(ia_2X_2) = \exp(i\sum_i b_i X_i)$$

Expressing this a bit more generally, and using vector notation for $\{b_i\}$ and $\{X_i\}$:

$$\exp(ipX_p)\exp(iqX_q) = \exp(i\underline{b}\cdot\underline{X})$$

Expanding this to second order:

$$(1+ipX_p - \tfrac{1}{2}p^2X_p^{\,2})(1+iqX_q - \tfrac{1}{2}q^2X_q^{\,2}) = 1+i\underline{b}\cdot\underline{X} - \tfrac{1}{2}(\underline{b}\cdot\underline{X})^2$$

Multiplying out:

$$1+ipX_p - \tfrac{1}{2}p^2X_p^{\,2} + iqX_q - \tfrac{1}{2}q^2X_q^{\,2} - pqX_pX_q = 1+i\underline{b}\cdot\underline{X} - \tfrac{1}{2}(\underline{b}\cdot\underline{X})^2$$

We want to solve implicitly for $\underline{b}\cdot\underline{X}$. Approximating to first order:

$$\underline{b}\cdot\underline{X} = pX_p + qX_q$$

Substituting this into the above:

$$1+ipX_p-\tfrac{1}{2}p^2X_p^{\,2}+iqX_q-\tfrac{1}{2}q^2X_q^{\,2}-pqX_pX_q=1+i\underline{b}\cdot\underline{X}-\tfrac{1}{2}(pX_p+qX_q)^2$$

Expanding the last term and removing some of the obvious terms:

$$ipX_p+iqX_q-pqX_pX_q=i\underline{b}\cdot\underline{X}-\tfrac{1}{2}pX_pqX_q-\tfrac{1}{2}qX_qpX_p$$

Moving quadratic terms to RHS:

$$ipX_p+iqX_q-i\underline{b}\cdot\underline{X}=\tfrac{1}{2}pqX_pX_q-\tfrac{1}{2}pqX_qX_p=\tfrac{1}{2}pq[X_p,X_q]$$

So we see the commutator of the generators is a linear combination of the generators.

$$[X_a,X_b]=i\sum_c f_{abc}X_c$$

The exact definition varies slightly, sometimes “i” is missing.

(I have been a bit hand-waving in the above derivation, but apparently the logic is valid to all orders.)

This is the **Lie Algebra**, which characterises the group.
$f_{abc}$ are the **Structure Constants** which characterise the Lie algebra, and hence the group.

By observation $f_{bac}=-f_{abc}$

# Adjoint Representation [2.2]

The **Adjoint** is derived from the **structure constants**:

$$T^a{}_{bc} = -i\, f_{abc}$$

$T^a$ is a matrix of dimension N×N. The above defines the element (b,c).
There are different definitions for the coefficient (–i).
There are N adjoint matrices.

Using the **Jacobi Identity**: $[X_a,[X_b,X_c]] + [X_b,[X_c,X_a]] + [X_c,[X_a,X_b]] = 0$

$$[X_a, f_{bcp}X_p] + [X_b, f_{cap}X_p] + [X_c, f_{abp}X_p] = 0 \quad \text{with summation over p}$$

$$\Rightarrow f_{apq}f_{bcp}X_q + f_{bpq}f_{cap}X_q + f_{cpq}f_{abp}X_q = 0 \quad \text{with summation over p and q.}$$

Assuming the generators form a basis:

$$\Rightarrow f_{apq}f_{bcp} + f_{bpq}f_{cap} + f_{cpq}f_{abp} = 0$$

Swapping two indices, using $f_{bac} = -f_{abc}$:

$$\Rightarrow f_{apq}f_{bcp} - f_{bpq}f_{acp} - f_{pcq}f_{abp} = 0$$

Replacing structure constants with Adjoint components:

$$-T^a{}_{pq}T^b{}_{cp} + T^b{}_{pq}T^a{}_{cp} - iT^p{}_{cq}f_{abp} = 0 \quad \Rightarrow T^a{}_{cp}T^b{}_{pq} - T^b{}_{cp}T^a{}_{pq} - if_{abp}T^p{}_{cq} = 0$$

Bearing in mind the summation over p and observing that we are seeing the (c,q) elements within a matrix equation:

$$\Rightarrow T^a T^b - T^b T^a - i f_{abp} T^p = 0 \quad \Rightarrow [T^a, T^b] = i f_{abp} T^p$$

So we see the Adjoint also satisfies the Lie algebra and therefore provides an alternative set of generators of the representation.

It is usual to chose bases in which the Adjoint satisfies:

$$\mathrm{Tr}(T^a T^b) = \lambda \, \delta_{ab}$$

With this, it is easy to show that

$$f_{abc} = -i \, \lambda^{-1} \, \mathrm{Tr}([T^a, T^b] T^c)$$

Since traces are unchanged by cyclically changing order of matrices:

$$\mathrm{Tr}([T^a, T^b] T^c) = \mathrm{Tr}(T^a T^b T^c - T^b T^a T^c) = \mathrm{Tr}(T^b T^c T^a - T^c T^b T^a) = \mathrm{Tr}([T^b, T^c] T^a) \text{ etc}$$

and

$$\mathrm{Tr}([T^a, T^b] T^c) = -\mathrm{Tr}([T^b, T^a] T^c) \text{ etc}$$

$f_{abc}$ is unchanged by cyclic permutations and negated by anticyclic permutations – it is a completely antisymmetric tensor like Levi-Cevita tensor $\varepsilon_{abc}$

$$f_{abc} = f_{bca} = f_{cab} = -f_{acb} = -f_{cba} = -f_{bac}$$

# Unitary Transformations

The **Unitary Group** of order n **U(n)** is the group associated with n×n unitary matrices under matrix multiplication.
The matrices operate on nD complex vectors.

Definition:

$$U^H U = U U^H = I$$

where superscript H indicates Hermitian conjugate.

A complex n×n matrix has 2×n×n parameters.
$U^H U = I$ imposes n×n constraints.

This is not totally trivial ! Consider two column vectors $a \neq b$.
The diagonal terms are like $a^H a = 1$. Since $a^H a$ is already real, these provides one constraint each.
The off-diagonal terms are like $a^H b = 0$ and $b^H a = 0$. If the first is satisfied, the second is also true. However $a^H b$ is not necessarily real, and corresponds to two constraints: that both real and imaginary parts are zero. So these terms provide n(n−1)/2 pairs of constraints.
So there are $n \times 1 + n(n-1)/2 \times 2$ constraints.

So there are $n^2$ free parameters and hence $n^2$ generators of U(n).

# Special Unitary Transformations

$\det(U^H U) = \det(I) = 1 \quad \Rightarrow \quad \det(U^H)\det(U) = 1$

since

$U^H \equiv U^{T*} \quad \Rightarrow \quad \det(U^H) = \det(U^T)^* = \det(U)^*$

$\Rightarrow \det(U)^* \det(U) = 1 \quad \Rightarrow \quad |\det(U)|^2 = 1 \quad \Rightarrow \quad |\det(U)| = 1$

The **Special Unitary Group** is defined by $\det(U) = +1$

This additional constraint means **SU(n)** has $n^2-1$ generators.

Homework

Show that U(n) and SU(n) satisfy group axioms.

Consider the generators {X} of the Unitary groups for an infinitesimal transformation, corresponding to a parameter $\delta\alpha$:

$$U(\delta\alpha) = \exp(i\delta\alpha X) = 1 + i\delta\alpha X$$

Remembering that $\delta\alpha$ is chosen to be real:

$$U^H U = 1 \;\Rightarrow\; (1-i\delta\alpha\, X^H)\,(1+i\delta\alpha\, X) = 1 \;\Rightarrow\; X^H = X$$

So the generators are Hermitian.

For SU(n), and only keeping terms of order $\delta\alpha$:

$$\det(U) = 1 \;\Rightarrow\; \det(1+i\delta\alpha\, X) = 1 \;\Rightarrow\; 1+i\delta\alpha\, \mathrm{Tr}(X) = 1 \;\Rightarrow\; \mathrm{Tr}(X) = 0$$

Proof

$\mathrm{Det}(1+\varepsilon) = 1 + \mathrm{Tr}(\varepsilon)$ to first order in $\varepsilon$:

# Orthogonal Transformations

Important subgroups of U(n) or SU(n) are the **Orthogonal groups** **O(n)** and **SO(n)** respectively.
These are the real number versions of the unitary transformations and correspond to **isometries** in nD – and isometry does not change the shape of an object, e.g. rotations and reflections.

Definition:

$$O^T O = O O^T = I$$

Corresponding to the discussion for the unitary matrices, there are $n^2$ parameters
The diagonal terms are like $a^T a = 1$. These provide one constraint each.
The off-diagonal terms are like $a^T b = b^T a = 0$. These provide one constraint each.
So there are $n \times 1 + n(n-1)/2 \times 1 = n(n+1)/2$ constraints.
This gives $n(n-1)/2$ free parameters.

As before, we can show $|\det(O)| = 1 \Rightarrow \det(O) = \pm 1$.

SO(n) is derived from O(n) with the constraint: $\det(O) = +1$
These correspond to rotations, while those with $\det(O) = -1$ correspond to reflections (combined with rotations).

This does not change the number of free parameters, since it relates to sign changes of the parameters.

For example: $\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$ has one parameter. If u=+1, it is the identity; If u=–1, it is a reflection in y=0.

# Generators of U(n)

There are different “representations” (infinitely many) of the generators corresponding to the different “representations” of the unitary matrices.
(“representations” here means in the common sense of “sets”, as opposed to the isomorphic set of linear operators corresponding to a group.)
(By analogy, the $\gamma^{\mu}$ matrices can be in the Dirac-Pauli or Weyl representations.)

For U(n), one can choose a representation of the generators which includes the unit matrix.
Recall we are talking about the set of generators – not the group members which must include the identity.

However Tr(I) = n, so I cannot be a generator for SU(n) … although of course I is a member of SU(n): I = exp(i0X), where X is any generator.

The generator I corresponds to a phase change: U = exp(i$\alpha$I) = exp(i$\alpha$) – implicitly multiplied by the n$\times$n unit matrix.

This corresponds to the group U(1).

So U(n) is made up of SU(n) and a U(1) – U(n) is isomorphic to the combination of the two and is written as the product: U(n) = SU(n) $\otimes$ U(1).

# U(1)

**U(1)** is the group of **Unitary Transformations** of 1D vectors, i.e. complex numbers.
This corresponds to a phase transformation and a suitable generator is the unit operator 1.

However, rather than a vector space consisting of complex numbers, we can choose a different vector space, namely one in which we choose to operate on a single measurable quantity characterised by some quantum number.
In this case, the generator is not the unit operator, but the quantum operator corresponding to the vector space of choice.

Let us consider the quantity of electric charge:
We will consider groups of transformations of the form $\exp(i\alpha Q)$ acting on Hilbert space, namely the set of particle wavefunctions.
The generator is Q, the charge operator, and this acts on the 1D vector space labelled with the electric charge.

We choose to consider transformations which are

**Global**: $\alpha$ does not depend on x
**Local**: $\alpha$ does depend on x

# Global U(1) Symmetry

Consider the example of electric charge, with an operator Q.

If the (interaction) Hamiltonian H contains a symmetry with respect to electric charge, then it commutes with Q, and we know the charge of a state will be conserved. Previously we considered the time variation of an observable. Alternatively, we can examine the evolution of a state $\psi$ of some well defined charge, q.

$$\psi \rightarrow \psi' = e^{-itH}\psi$$

Consider the charge of the final state, determined by the charge operator:

$$Q\psi' = Qe^{itH}\psi = e^{itH}Q\psi = e^{itH}q\psi = qe^{itH}\psi = q\psi'$$

We see the final state is also a state of well-defined charge, being the same as the original one.

By construction, the SM Lagrangian is invariant under the Global transformation. It consists of terms like $L = \psi_1\psi_2\psi_3...$ where the sum of the charges of the corresponding particles is $q_1 + q_2 + q_3 + ... = 0$.

Under a U(1) transformation $\psi \rightarrow \psi' = e^{i\alpha Q}\psi$ which for a single particle gives $\psi' = e^{i\alpha q}\psi$

For multiple particles, $Q = Q_1 + Q_2 + Q_3 + \ldots$

Since the operators are all independent they commute, hence:

$$e^{i\alpha Q} = e^{i\alpha Q_1}e^{i\alpha Q_2}e^{i\alpha Q_3}...$$

and

$$L' = e^{i\alpha Q}\psi_1\psi_2\psi_3... = e^{i\alpha Q_1}\psi_1 e^{i\alpha Q_2}\psi_2 e^{i\alpha Q_3}\psi_3... = e^{i\alpha q_1}\psi_1 e^{i\alpha q_2}\psi_2 e^{i\alpha q_3}\psi_3... = e^{i\alpha(q_1+q_2+q_3+...)}\psi_1\psi_2\psi_3... = L$$

Note: this seems pretty trivial, but wait for the local transformations.

Also, we have not got something for nothing … this is all by construction.

# Baryon and Lepton Number Conservation

By observation, **Baryon and Lepton numbers** are conserved in interactions.
Therefore the SM is constructed to ensure this.

Baryon and Lepton numbers are additive, just like charge.

| **Particle** | **Baryon Number (B)** | **Lepton Number (L)** |
|---|---|---|
| quark | +1/3 | 0 |
| anti-quark | −1/3 | 0 |
| lepton, neutrino | 0 | +1 |
| anti-lepton, anti-neutrino | 0 | −1 |
| Others | 0 | 0 |

By construction the baryons have baryon number +1.

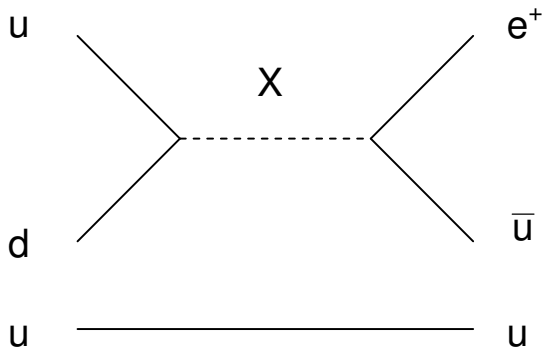
The conservation law means that within the SM, a quark can not be destroyed, except by an anti-quark.

Until the recent observation of neutrino oscillations, it was suggested that the different flavours of lepton had their own separate conservation laws.
The existence of right-handed neutrinos allows the possibility that neutrinos have mass and that the corresponding mass eigenstates are no longer eigenstates of the Weak Interaction, allowing mixing, as found in the quark sector.

However, in models derived with alternative symmetries, constructed in attempt to provide a GUT, baryon number and lepton number may not be conserved.

In many models (such as SU(5)), B and L are not conserved but B–L is:

Proton decay: $p \rightarrow e^+\pi^0$ mediated by a new gauge boson, X:




Proton: B = +1, L = 0, B–L = +1

Final state: B = 0, L = –1, B–L = +1

Is this true ?
Only experiment will tell.
So far (Soudan) no sign of proton decay.

# Local U(1) Symmetry

Now we consider **local U(1) transformations** $\exp(i\alpha(x)Q)$ – these are **Gauge Transformations**.
("Gauge" ↔ size, as in railway track gauge … misnomer for phase transformation.)

Then what happens in the SM Lagrangian when considering a charged particle (will be considered more in SM course) ?

$$L \sim \bar{\psi}\partial\!\!\!/\psi \quad \text{where} \quad \partial\!\!\!/ \equiv \gamma_\mu \partial^\mu = \gamma_\mu \frac{\partial}{\partial x_\mu}$$

Under a local gauge transformation:

$$L \rightarrow L' \sim L + \bar{\psi}(iQ\partial\!\!\!/\alpha)\psi \neq L$$

If we wish to ensure invariance, must replace $\partial^\mu$ by the covariant derivate $\partial^\mu - iQA^\mu$, such that $A^\mu$ transforms like $A^\mu \rightarrow A^\mu + \frac{1}{Q}\partial^\mu\alpha$.

Q is associated with the electric charge of the charged particle, while $A^\mu$ is identified as the field describing the photon – the quantum of EM interactions.

Assume: local U(1) gauge symmetry → vector field

Historically, it was the other way round:

Observe: E and B fields and construct Maxwell's Equations → gauge invariance

How does this help ?

Assume: local SU(n) gauge symmetry → new vector fields

In particular, SU(2) led to the predictions of $W^\pm$ and Z bosons … but not in absence of experiments.