LECTURE 1 – SYMMETRIES & CONSERVATION

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Messages

- **Symmetries** give rise to **conserved quantities**.

Introduction

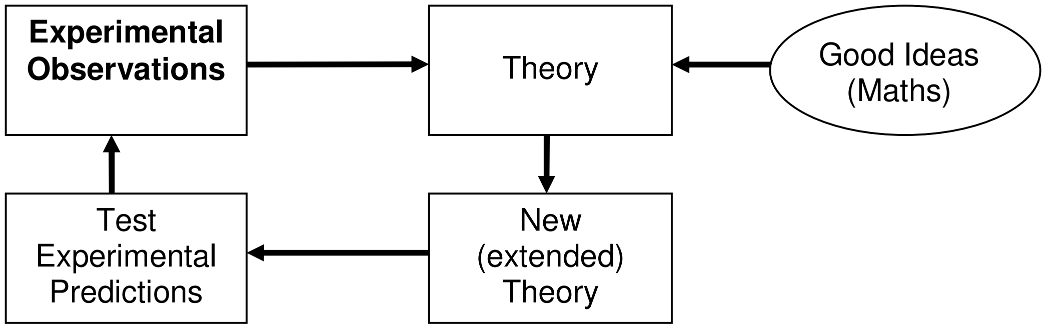
The benefits of Group Theory are in understanding:

- The structure and properties of **hadrons** and their interaction … this is mainly historical, although useful background.
- The structure of the **Standard Model Lagrangian** and in particular the tensor nature of the particle fields (i.e. whether they are singlets, doublets, etc.) and their couplings.
  This will feed into the Standard Model course.
- Possible extensions **beyond the SM** and the particle content of GUTs.

We will focus on **ideas** rather than developing formalism.
The course will draw on concepts and results from **Group Theory**.
There is a lot of mathematics behind what we need to know and it is not worthwhile us learning all of this – instead, I will seek to make things seem plausible and will try to make as much connection as possible with concrete examples.

In what follows, we will use a classical approach to QM, rather than QFT.

At times we will appear to get “something for nothing”. In reality:



Symmetry & Transformations

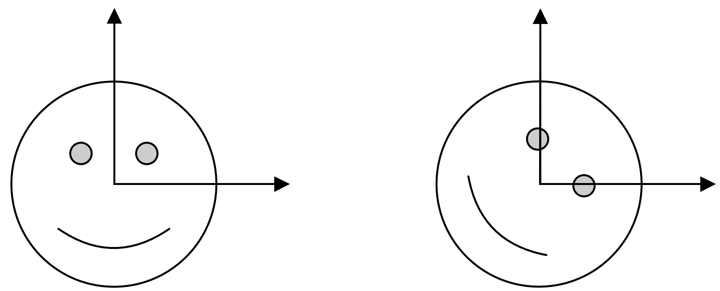
Systems contain **<u>Symmetry</u>** if they are unchanged by a **Transformation**.

This symmetry is often due to an absence of an **absolute reference** and corresponds to the concept of **indistinguishability**.

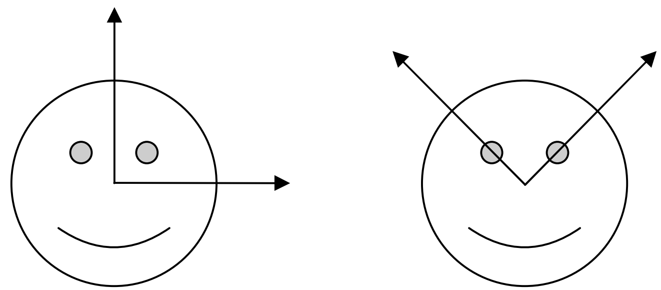
It will turn out that symmetries are often associated with **conserved quantities**.

Transformations may be:

Active:
- Move object
- More physical



Passive:
- Change “description”
  Eg. Change Coordinate Frame
- More mathematical



We will consider two classes of Transformation:

Space-time:
- Translations in (x,t) } Poincaré Transformations
- Rotations and Lorentz Boosts }
- Parity in (x,t) (Reflections)

Internal: associated with quantum numbers

Translations:

$$x \rightarrow x' = x - \Delta_x$$
$$t \rightarrow t' = t - \Delta_t$$

Rotations (e.g. about z-axis):

$$x \rightarrow x' = x\cos\theta_z + y\sin\theta_z \;\; \& \;\; y \rightarrow y' = -x\sin\theta_z + y\cos\theta_z$$

Lorentz (e.g. along x-axis):

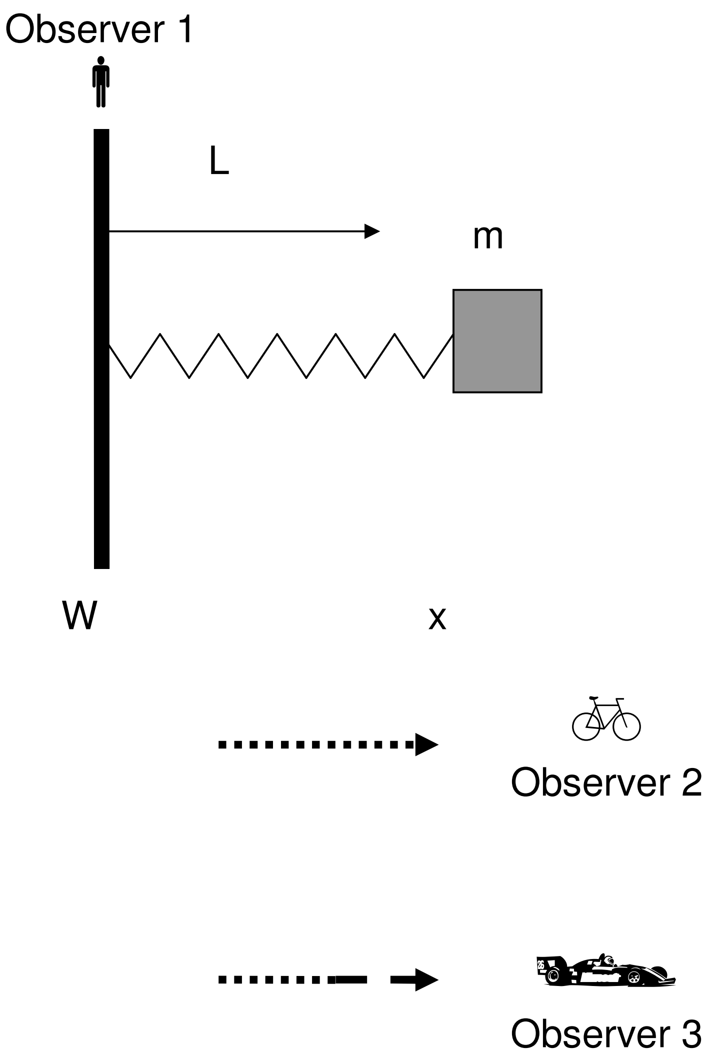
$$x \rightarrow x' = \gamma(x - \beta t) \;\; \& \;\; t \rightarrow t' = \gamma(t - \beta x)$$

Parity:

$$x \rightarrow x' = -x$$
$$t \rightarrow t' = -t$$

For **physical laws** to be useful, they should exhibit a certain generality, especially under symmetry transformations.
In particular, we should expect invariance of the laws to change of the status of the observer – all observers should have the same laws, even if the evaluation of measurables is different. Put differently, the laws of physics applied by different observers should lead to the same observations.
It is this principle which led to the formulation of Special Relativity.

Illustration from Non-Relativistic Mechanics



For Observer 1:

Mass m at position x, attached to Spring.
Spring fixed to Wall at x = W.
Natural length of Spring is L, and spring constant k.

Boundary conditions: $x = W+L+A$ & $\dot{x} = 0$ at $t = 0$.

Observers 1 & 2 are in inertial frames.
Observer 3 is in an accelerating frame wrt 1 & 2.

Observer 1 measures things in a coordinate frame {x}, in which the Wall is stationary.

Newton: $F = m\ddot{x}$

$$-k(x-(W+L)) = m\ddot{x}$$

Implicitly replace $x-(W+L)$ by ξ,the extension of the Spring, and solve in terms of $\sin(\omega t)$ and $\cos(\omega t)$.

Solution: $x = W + L + A\cos(\omega t)$, where $\omega^2 = k/m$

Observer 2 measures in a coordinate frame {x'}, in which Observer 1 is moving with a uniform velocity: $x' = x + \Delta + vt$.

As far as Observer 2 is concerned,

$$-k(x'-(W'+L)) = m\ddot{x}'$$

The expression of for the Force (and mass and spring constant) is unchanged.

Again, implicitly replace $x'-(W'+L)$ by ξ – this works because the derivative of $W' = W + \Delta + vt$ vanishes, allowing us to replace $\ddot{x}'$ with $\ddot{\xi}$.

Solution: $x' = W' + L + A\cos(\omega t)$, where $\omega^2 = k/m$

The frames of the two observers are equally good, there is no way of absolutely distinguishing the frames – this indistinguishability corresponds to a symmetry (or equality) between the two frames – associated with the transformation between the two.

The physical predictions of the two observers are identical: the amplitude and frequency are the same, although the description of the position of the mass is different.

However, Observer 3 is accelerating: $x'' = x + \frac{1}{2}at^2$, the equation of motion *assumed* by Observer 3, is

$$-k(x''-(W''+L)) = m\ddot{x}''$$

This time, one cannot implicitly replace $x''-(W''+L)$ by ξ – because the derivative of $W'' = W + \frac{1}{2}at^2$ does not vanish.

The solution is a little more complicated:

Solution: $x'' = W''+L - ma/k + (A + ma/k)\cos(\omega t)$, where $\omega^2 = k/m$

This is a different solution – the amplitude is different.

The reason is that the force was incorrectly attributed as $-k(x''-(W''+L))$ – as far as Observer 3 is now concerned, there is an additional force ma required to accelerate the Mass (and the Wall) – like a gravitational force.

The best way to solve for the motion of the Mass, is to solve in frame {x} and *then* transform.

Why the lack of symmetry between the Observers ?

We know that Observer 3 (not 1 & 2) is accelerating because he is forced into the back of his car seat.

Newton's Law is only applicable for observers in inertial frames.

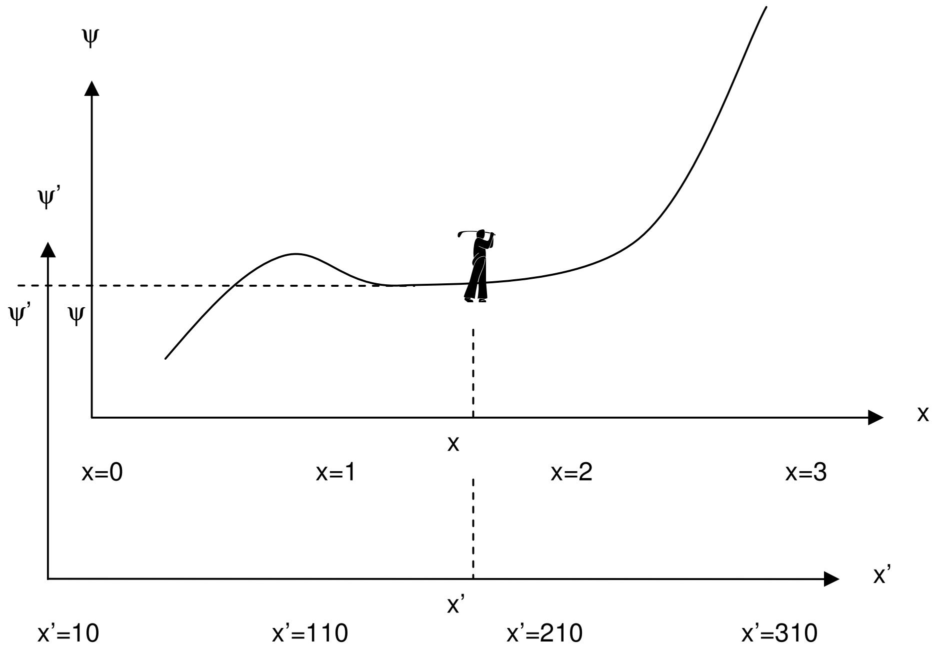
One must be careful to understand the validity of given equation under a transformation.

Transformations in Quantum Mechanics

Consider a scalar wavefunction: $\psi(x)$

Make a transformation from one coordinate system to another: $x \rightarrow x'$

Define the transformed wavefunction in the new frame by: $\psi'(x') \equiv \psi(x)$
The intention is that x and x' correspond to the *same point* in space-time and the wavefunctions ψ' and ψ describe the *same event*.



At the point where the event happens, the wavefunction has some (relatively) well-defined value.

Note ψ' will be a different function from ψ:

If $x' = f(x)$, and the inverse transformation is f^{-1}, then $x = f^{-1}(x')$. Hence

$$\psi'(x') \equiv \psi(x) = \psi(f^{-1}(x')) \quad \text{so} \quad \psi'() = \psi(f^{-1}())$$

For example, if $\psi(x) = x$ and $x' = \exp(x)$, then $\psi'(x') \equiv \psi(x) = x = \log(x')$.

In general, assume that the new wavefunction can be derived from the old one by a transformation of the wavefunction itself: $\psi' = U\psi$ where U is an operator

In general, ψ can be expressed as a linear superposition of base states $\{\phi_i\}$: $\psi = \sum c_i \phi_i$

In the new description, $\psi \to \psi'$, $\phi \to \phi'$, so $\psi' = \sum c'_i \phi'_i$.
But since ψ and ψ' correspond to the same states, as do ϕ and ϕ', then we would expect $c'_i = c_i$

$$\Rightarrow \psi' = \sum c_i \phi'_i \Rightarrow U(\sum c_i \phi_i) = \sum c_i U(\phi_i)$$

This is the definition of a **linear operator.**

Note there are two distinct transformations:

a. the transformation describing the change in “description” (coordinate frame): $x \to x' \equiv f(x)$
b. the associated transformation of the wavefunction: $\psi \to \psi' = U\psi$

Furthermore, the overlap between any states ψ_a and ψ_b is an observable and should be independent of the description.

Using the bra-ket notation for compactness: $\psi_a \leftrightarrow |a>$, then

$$\psi'_a = U\psi_a \leftrightarrow |a'>= U|a> \text{ and } \psi'^{H}_a = \psi_a^{H}U^{H} \leftrightarrow <a'|=<a|U^{H}$$

where ^{H} denotes the Hermitian conjugate (often shown by a dagger).

The overlap is $<b'|a'>=<b|U^{H}U|a>$ and if this is equal to $<b|a>$ for all ψ_a and ψ_b, then $U^{H}U = I$.
The transformation of the wavefunctions is **Unitary**.

How do operators transform ?
Consider the observable $<b|A|a> \equiv \int \psi_b^{H} A \psi_a$.
Want $<b'|A'|a'>=<b|A|a>$ for all ψ_a and ψ_b.

$$\Rightarrow <b|U^{H}A'U|a>=<b|A|a> \Rightarrow U^{H}A'U = A \Rightarrow A' = UAU^{H}$$

If A is invariant, then

$$A' = A \Rightarrow UAU^{H} = A \Rightarrow UA = AU \Rightarrow [A,U] = 0$$

Lastly, if states $\{|i>\}$ form an orthonormal basis:

- $<j|i> = \delta_{ij}$
- All states $|a>$ in the vector space can be written as a linear superposition $|a> = \sum \alpha_i |i>$

Then the transformed states $|i'> = U|i>$ also form an orthonormal basis:

$$<j'|i'> = <j|U^H U|i> = <j|I|i> = \delta_{ij}$$

and since $|i'>$ is derived from $|i>$ etc, then if i=j, the states $|i'>$ and $|j'>$ must be equal.

$$|i'> = U|i> = \sum <j|U|i>|j>$$

Since $<j|U|i>$ is just (the set of coefficients of) a unitary matrix, the matrix can be inverted:

$$|i> = \sum <j|U^H|i>|j'>$$

Hence any state $|a>$ can be written as:

$$|a> = \sum \alpha_i |i> = \sum \alpha_i \sum <j|U^H|i>|j'> = \sum \beta_i |j'>$$

which is the requirement for a basis.

Generators

Under a transformation, $\psi \to \psi' = U\psi$.
Assume that the unitary operator can be expressed as: $U = \exp(iaX)$, where $a \in \Re$.

What is X ? Naively it is the “log” of U, but this is non-trivial, since we are dealing with operators and functions need to be defined by their power series.

X is defined as the **Generator** of the transformation.

Exponentiation of Operators

Define:

$$\exp(A) \equiv 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{p=0}^{\infty} \frac{1}{p!} A^p \quad \text{where } A^0 = I$$

Then

$$\exp(A)^H = \left(\sum_{p=0}^{\infty} \frac{1}{p!} A^p\right)^H = \sum_{p=0}^{\infty} \frac{1}{p!} (A^H)^p = \exp(A^H)$$

If $U = \exp(iaX)$, then it is easy to show $U^{-1} = \exp(-iaX)$ – it follows from the normal rules for multiplying exponentials of scalars, which in turn can be proved by expanding the exponentials. The result follows easily because all the terms commute.

If the U is unitary, then $U^{-1} = U^H$, hence

$$\exp(-iaX) = (\exp(iaX))^H = \exp((iaX)^H) = \exp(-iaX^H)$$

Equating terms order by order in a implies $X^H = X$, i.e. X is **Hermitian**.

These results can be derived by

- Looking at the series expansions of exponentials and considering Binomial expressions
- Considering infinitisimal terms: $\exp(\varepsilon) \approx 1 + \varepsilon$ for small ε
- Building finite transformations from a product of infinitisimal ones: $\exp(A) = \lim_{N \to \infty} (1 + \frac{A}{N})^N$

So the generator of a **Unitary transformation is a Hermitian operator**.
In QM, Hermitian operators are postulated to correspond to observables.

Generator for Translations

Consider a translation in 1D in the x-direction $x \to x' = x - \Delta_x \Rightarrow x = x' + \Delta_x$
So $\psi'(x') = \psi(x) = \psi(x' + \Delta_x)$

By Taylor expansion (and changing the dummy variable x' back to x for neatness):

$$\psi'(x) = \psi(x + \Delta_x) = \psi(x) + \Delta_x \frac{\partial}{\partial x}\psi + \tfrac{1}{2!}\Delta_x^{\ 2}\frac{\partial^2}{\partial x^2}\psi + \tfrac{1}{3!}\Delta_x^{\ 3}\frac{\partial^3}{\partial x^3}\psi + \ldots = \exp(\Delta_x \frac{\partial}{\partial x})\psi$$

So we identify $U = \exp(\Delta_x \frac{\partial}{\partial x})$

Since in QM, $p_x = -i\hbar \frac{\partial}{\partial x}$

$\Rightarrow U = \exp(i\Delta_x p_x / \hbar)$ … we often choose units so that $\hbar = 1$, and so it can be dropped.

So we see that the **generator of a translation is the momentum operator**.

This can be generalised to 3D: $x \to x' = x - \Delta \Rightarrow x = x' + \Delta$ – where the quantities are 3-vectors.

Then

$$\psi'(x) = \psi(x + \Delta) = \psi(x) + \Delta \cdot \nabla\psi + \tfrac{1}{2!}(\Delta \cdot \nabla)^2\psi + \tfrac{1}{3!}(\Delta \cdot \nabla)^3\psi + \ldots = \exp(\Delta \cdot \nabla)\psi$$

where ∇ is the grad vector derivative.

Since in QM, $p = -i\hbar\nabla \quad \Rightarrow U = \exp(i\Delta \cdot p / \hbar)$

Generator for Rotations

Consider a rotation about the z-axis $x \to x' = x\cos\theta_z + y\sin\theta_z$ & $y \to y' = -x\sin\theta_z + y\cos\theta_z$
It proves to be much easier to consider infinitisimal rotations: $x \to x' = x + y\theta_z$ & $y \to y' = -x\theta_z + y$

So $\psi'(x', y') = \psi(x, y) = \psi(x' - y'\theta_z, y' + x'\theta_z)$ and by Taylor expansion to first order:

$$\psi'(x, y) = \psi(x - y\theta_z, y + x\theta_z) = \psi(x, y) - y\theta_z \frac{\partial}{\partial x}\psi + x\theta_z \frac{\partial}{\partial y}\psi = \exp(\theta_z (x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}))\psi$$

In QM, $L_z = (x \times p)_z = (xp_y - yp_x) = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$

So we see that the **generator of a rotation is the angular momentum operator**.

Note: one should be careful about generalising this to 3D, since a rotation cannot be built up trivially of three rotations about the three axes. The combination of three such rotations depends on the order.
This will be manifested in QM if three operators are combined:

$$\exp(i\theta_x L_x / \hbar)\exp(i\theta_y L_y / \hbar)\exp(i\theta_z L_z / \hbar) \neq \exp(i(\theta_x L_x + \theta_y L_y + \theta_z L_z) / \hbar)$$

The reason is that $\exp(A)\exp(B) = \exp(A + B)$ only if A and B commute – which us not the case for the angular momentum operators.
By contrast, the above *is* true for the momentum operators when generating 3D translations.

Finally, extending what was done for spatial translations, we find **the generator of a time translation is the Hamiltonian operator** $-i\frac{\partial}{\partial t} = -H/\hbar$

Symmetry in Quantum Mechanics

Equation of motion:

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi \quad \text{where} \quad \psi = \psi(x,t) \text{ and } H = H(x)$$

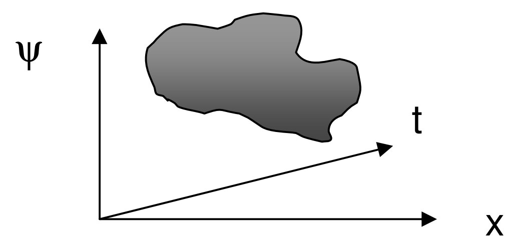
Under a transformation:

$$\psi \to \psi' = U\psi \quad \text{and} \quad H \to H' = UHU^{H}$$

So in the new description:

$$i\hbar\frac{\partial}{\partial t}\psi' = H'\psi'$$

By definition, the system is said to have a **Symmetry** if $H' = H$
Note: this is a symmetry of the Hamiltonian, not of the vector space (Hilbert Space) of solutions $\{\psi\}$.
Of course, the symmetry contained within H will be reflected in the individual solutions.
H defines the dynamics of the system, i.e. the interactions, the ψ's provide the way of describing the position of the particles.



Symmetry: $H' = H \quad \Rightarrow UHU^{H} = H \Rightarrow UH = HU \Rightarrow [H,U] = 0$

Symmetries & Conservation Laws

If a Unitary Transformation U is generated by X: $U = \exp(iaX)$

Then if H has a symmetry associated with the transformation U:

$$[H,U] = 0 \Rightarrow [H, \exp(iaX)] = 0 \Rightarrow [H, \sum \tfrac{1}{p!}(iaX)^p] = 0$$

For this to be true for all orders of a: $\Rightarrow [H, X] = 0$.

Now consider the time variation of observables formed from X: $< b \,|\, X \,|\, a >$

$$\tfrac{\partial}{\partial t} \,|\, a > = -\tfrac{i}{\hbar} H \,|\, a > \quad \text{and} \quad \tfrac{\partial}{\partial t} < b \,|\, = \tfrac{\partial}{\partial t} \{|\, b >\}^H = +\tfrac{i}{\hbar} < b \,|\, H$$

$$\tfrac{d}{dt} < b \,|\, X \,|\, a > = < b \,|\, \tfrac{\partial}{\partial t} X \,|\, a > - \tfrac{i}{\hbar} < b \,|\, X\, H \,|\, a > + \tfrac{i}{\hbar} < b \,|\, H\, X \,|\, a > = < b \,|\, \tfrac{\partial}{\partial t} X \,|\, a > + \tfrac{i}{\hbar} < b \,|\, [H, X] \,|\, a >$$

So if X has no explicit time-dependence and $[H, X] = 0$, then $< b \,|\, X \,|\, a >$ is constant in time.

Summary: If the **Hamiltonian** of a system is **invariant** under a **Unitary transformation** U generated by an (Hermitian) operator X, then there will **<u>conserved observables</u>** associated with X.

Symmetry	Conserved Observable
Translation in Space-time (x,t)	Momentum-energy (p,E)
Rotation in Space	Orbital Angular-momentum $L = x \times p$
Reflections in Space	Parity
Gauge Transformation	Charge

Note: there are symmetries associated with

- Reflections in time – Time reversal
- "Rotations" in Space-time – Lorentz boosts

which do not correspond to unitary transformations and hence do not have conserved observables.

Time reversal is anti-unitary: $UU^H = -I$ (by construction … that's QM for you!)

For Lorentz boosts, it is not sufficient to say $\psi'(x',t') = \psi(x,t)$, since the boost is not an isommetry – it does not preserve the volume d^3x and hence the normalisation needs to be modified to preserve probability. (Corresponds to a rotation in imaginary Minkowski space (x,it), but not a rotation in (x,t).)

If $\psi \to \psi' = L\psi$ and $d^3x \to d^3x' = \gamma d^3x$, then the normalisation condition is:

$$\int \psi'^H \psi' d^3x' = \int \psi^H \psi d^3x = 1 \Rightarrow \int \psi^H L^H L \psi \gamma d^3x = 1 \Rightarrow L^H L \gamma = 1$$

So L is not unitary.

(The generator ~ $-tp_x + xH$ c.f. ang mom, but adding a scale factor for the normalisation introduces an imaginary and therefore non-Hermitian term to the generator.)

Conservation Laws in Classical Mechanics [Goldstein]

We have seen in Quantum Mechanics that **Symmetries** lead to **Conservation Laws**.

Some of the mathematical motivation for QM lies in the formulation of **Classical Mechanics**, which we will touch on briefly here.
However, these formulations seem slightly perverse and at best are a manifestation of the "Real World" which is of course the Quantum World. In the limit of large numbers of particles, the Real World approximates to the description of CM.

Therefore, it is not worth pursuing this too far.

Construct **Lagrangian**: $L(t,q,\dot{q}) = T - V$, where T is the **Kinetic Energy** and V is the **Potential Energy** and q is a **Generalised Coordinate**.

By minimising the Action: $A = \int L(t,q,\dot{q})dt$, one can derive the **Euler-Lagrange** equation of motion:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

The Canonical Momentum is defined $p = \frac{\partial L}{\partial \dot{q}}$.

If the Kinetic Energy can be written as $T = \frac{1}{2}\mu\dot{q}^2$ and V does not depend on $\dot{q}$, then $p = \mu\dot{q}$ – this looks like mass×velocity … although there is no reason for q to be a spatial coordinate, and hence $\dot{q}$ does not need to be velocity.

If V and hence L does not depend on q (KE does not usually depend on q), then $\frac{\partial L}{\partial q} = 0$, and hence the E-L equation becomes $\frac{d}{dt} p = 0$, which implies p is constant.
This situation corresponds to a uniform potential, i.e. not having derivatives with respect to q. Since we would tend to identify forces with the (spatial) derivatives of the potential, this corresponds to systems where in the absence of external forces, the momentum is conserved.

The **Hamiltonian** is constructed: $H(t, q, p) = p\dot{q} - L$

The Hamiltonian equations of Motion are:

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p} \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

For many systems, this leads to a Hamiltonian which is equal to $T + V$, which we identify with the total energy of the system.

If V and hence L does not depend on time (KE does not usually depend on t), $\frac{\partial H}{\partial t} = 0$, and hence H is constant. We would tend to think of this as a situation where forces/potentials were not time dependent and hence the total energy of the system is conserved.

Connections with Quantum Mechanics [Goldstein]

Taking this further, it is possible to formulate the time variation of a quantity $Q(t,q,p)$:

$$\frac{dQ}{dt} = [Q,H] + \frac{\partial Q}{\partial t}$$ where the **Poisson Bracket** is defined by $$[Q,H] \equiv \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}$$

So if $H = \frac{1}{2}p^2/\mu$

$$\dot{q} = \frac{\partial H}{\partial p} = p/\mu \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q} = 0$$

Then if Q does not depend on q and t, but only p, $\frac{dQ}{dt} = 0$ and Q is a constant in time.

As was done with QM, it is possible to identify **generators** of transformations and from the invariance of a Hamiltonian, deduce the presence of **conserved quantities**, such as momentum and angular momentum.
Further, it is possible to identify analogies between the Poisson Bracket formulation and the Commutators of QM, as well as their corresponding Lie Algebras.
(Lie Algebra will be discussed in the following lecture.)

Parity

The **Parity** transformation is spatial reflection (in all 3 dimensions): $x \to x' = -x$

$$\psi(x) \to \psi'(x') = \psi(x) = P\psi(x') \quad \text{so} \quad \psi' = P\psi$$

For any isometry $d^3x \to d^3x' = d^3x$ (there is an implicit modulus) – since by definition an isometry does not alter the shape of an object.

From the normalisation:

$$\int \psi'^H \psi' d^3x' = \int \psi^H P^H P\psi d^3x = 1 \Rightarrow P^H P = 1$$

So we see the transformation is associated with a unitary transformation.

Further, since $\psi'(x') = \psi(x) = P\psi(x')$ and $x = -x'$, $P\psi(x') = \psi(-x')$ or replacing the dummy variable x' with x: $P\psi(x) = \psi(-x)$.

$$\Rightarrow P^2\psi(x) = PP\psi(x) = P\psi(-x) = \psi(x) \Rightarrow P^2 = I \quad \text{so} \quad P^{-1} = P = P^H$$

hence P is not only unitary but also Hermitian and hence corresponds to an observable.

If P has eigenstates $\{|\lambda>\}$ with eigenvalues $\{\lambda\}$, then

$$P|\lambda>=\lambda|\lambda> \quad \text{and} \quad P^2|\lambda>=\lambda^2|\lambda>=|\lambda>$$

Hence $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$

So if Parity is a symmetry of the Hamiltonian, there exist states of well-defined Parity (± 1) which will be conserved.

<u>Type of object</u>	<u>Transformation under Parity – examples</u>	
Vector (or Polar vector)	Spatial coordinate Momentum	$x \rightarrow -x$ $p = -i\hbar\nabla \rightarrow -p$
Axial vector (or Pseudovector)	Ang momentum also spin and tot ang mom	$L \equiv x \times p \rightarrow +L$
Scalar	Scalar product	$x \cdot x \rightarrow +x \cdot x$
Pseudoscalar	Helicity	$H \equiv \dfrac{s \cdot p}{\lvert p \rvert} \rightarrow -H$

Note: Symmetries of the Hamiltonian must be verified experimentally.
They may be postulated because they seem “sensible” and elegant, but this does not guarantee that they exist.
For example, Parity is *not* a symmetry of the Weak Interaction.

An Example

Consider the Hamiltonian corresponding to

a) Particle in a vacuum $H = \frac{1}{2m} p^2$

b) Particle subjected to a central force $H = \frac{1}{2m} p^2 + V(r)$

What happens under

i) Translations ?

ii) Rotations (about the origin, on which the central force is centred) ?

The invariance of the Hamiltonian under transformations
$\leftrightarrow$ commutation with the corresponding generator
$\leftrightarrow$ conservation of the corresponding physical observable.

Firstly <u>Classical Mechanics</u>:

Translation: $x \to x + \Delta$, $p = m\dot{x} \to p$

so p^2 is unchanged, but $r = \sqrt{x \cdot x}$ is changed

Rotation: $x \to Rx$, $p \to Rp$

so $p^2 = p^T p \to p^T R^T R p = p^T I p = p^T p$ is unchanged, as is r

So under Translations:

a) p is conserved

b) p is not conserved – particles do not travel in straight lines

and under Rotations:

a) L is conserved

b) L is conserved

Next Quantum Mechanics:

Does the Hamiltonian commute with the generators of the transformations, namely

i) Momentum

ii) Ang Momentum?

Momentum:

a) p clearly commutes with p^2

b) What is [V(r),p] ?

$$[V(r),p] \sim [V(r),\nabla] = V(r)\nabla - \nabla V(r) = -\nabla(V(r))$$

In spherical coordinates, $\nabla \sim \hat{r}\,\frac{\partial}{\partial r}$, so $[V(r),p] \sim \hat{r}\,\frac{\partial V}{\partial r} \neq 0$

Ang momentum:

a) $[p^2,L] \sim [p^2,x\times p] = p^2(x)\times p + 2p(x)\times p\cdot p$ – being a bit cavalier with the vectors and their indices, but it can all be followed through logically.

$$\nabla^2(x) = 0 \quad \text{and} \quad p(x)\times p\cdot p \sim \varepsilon_{abc}\partial_i(x_a)p_b p_i = \varepsilon_{abc}\delta_{ia}p_b p_i = \varepsilon_{abc}p_b p_a = 0 \quad \text{so } [p^2,L] = 0$$

b) $[V(r),L] \sim [V(r),x\times p] = x\times\nabla(V(r)) = x\times\hat{r}\,\frac{\partial V}{\partial r} = x\times\frac{x}{r}\frac{\partial V}{\partial r} = 0$

So the conclusions are the same as in the Classical case for which quantities are conserved.