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Wormholes in the Kaluza-Klein theory

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We discuss wormholes in Kaluza-Klein theory, and deduce the corresponding wormhole equation. By solving this equation, we give an analytic solution of the equation.

A wormhole is an Euclidean field configuration in some field theory containing gravity, consisting of two asymptotically flat regions connected by a tube, or throat. Wormholes had been introduced in the 1950s by Wheeler [1]. Originally, wormholes were three-dimensional spaces. A good example is the Schwarzschild bridge which is a slice through a black hole joining two asymptotic regions. More recently, people have introduced four-dimensional wormholes. Their effects on spacetime coupling constants have provoked great interest among theoretical physicists [2–8]. Since this kind of wormhole can join spaces with different topologies, they represent tiny quantum fluctuations of space. From the point of view of mathematics, the condition of a wormhole existing on a four-dimensional asymptotic flat manifold M_4 is that the Ricci tensor of M_4 has negative eigenvalues somewhere on M_4 [9]. In the pure gravitation case, Hawking discussed wormholes in which two baby universes are connected [2,3]. However, in general the Euclidean action of the wormhole in the pure gravitation case is unstable. The wormhole with a stable Euclidean action was first discovered by Giddings and Strominger [10] in the theory with a spontaneously broken Abelian internal symmetry—the theory of a Goldstone boson, or axion, minimally coupled to Einstein gravity. There is a Giddings-Strominger wormhole for every value of L, the wormhole size. (Here, the so-called wormhole size is defined as the geodesic length of $2\pi L$ around the throat.) Recently, Coleman and Lee studied a wormhole [11] in a theory with an unbroken Abelian internal symmetry—the theory of a complex scalar field, ψ, of mass m, minimally coupled to Einstein gravity, but with no other interactions. They have obtained the corresponding wormhole equation and the wormhole solution. In Refs. [12] and [13], wormholes in a scalar field (axion field) with an axion charge are discussed. In Refs. [14–16], wormholes are discussed in gauge field theories. In Ref. [17], wormholes in the Skyrme model are discussed. In Ref. [18], wormholes in scalar-tensor gravitation theory are discussed. In Ref. [19], wormholes in higher-dimensional theory are discussed. In Ref. [20], wormholes with a spinor field are discussed.

In this paper, we discuss the wormhole in Kaluza-Klein theory, and deduce the corresponding wormhole equation. Solving this equation, we obtain an analytic solution of this equation and the action of the wormhole solution.

We consider the spacetime with the topology

$$R^1 \otimes S^3 \otimes \mathcal{M}^M \ . \tag{1}$$

The spacetime dimensions D are $D=1+3+M$. The corresponding metric takes the form

$$dS^2 = d\tau^2 + a^2(\tau)d\Omega_3^2 + b^2(\tau)\tilde{g}_{ab}(y)dy^a dy^b \ , \tag{2}$$

where $\tau = t/i$, $a(\tau)$ and $b(\tau)$ are scale factors, $d\Omega_3^2$ is a three-dimensional sphere metric, $\tilde{g}_{ab}(y)$ is the metric of M-dimensional compact constant-curvature space and satisfies $\tilde{R}_{ab}(\tilde{g}) = k\tilde{g}_{ab}$, and K is a curvature constant. In this paper, we only consider $K>0$.

We take the Euclidean action as

$$S_E = \frac{1}{16\pi G}\int_M d^D x \sqrt{g}\,(R - 2\Lambda) + \text{surface terms} - \int_M d^D x \sqrt{g}\,\tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \ , \tag{3}$$

in which R is the curvature scalar, Λ is the cosmological constant, ϕ is a D-dimensional scalar field, and $\phi = \phi(\tau)$.

From the scalar field equation, it is given that

$$\dot{\phi}=\frac{im}{a^3b^M} . \tag{4}$$

The Einstein field equations are

$$-\frac{3\ddot{a}}{a}-\frac{M\ddot{b}}{b}=\left[-\frac{m^2}{a^6b^{2M}}+\frac{2}{D-2}\Lambda\right]8\pi G ,$$

$$-\frac{\ddot{a}}{a}+2\left[\frac{1}{a^2}-\frac{\dot{a}^2}{a^2}\right]-M\frac{\dot{a}\dot{b}}{ab}=\frac{2\Lambda}{D-2}8\pi G , \tag{5}$$

$$-\frac{\ddot{b}}{b}+(M-1)\left[\frac{k}{(M-1)b^2}-\frac{\dot{b}^2}{b^2}\right]-3\frac{\dot{a}\dot{b}}{ab}=\frac{2\Lambda}{D-2}8\pi G ,$$

where an overdot expresses the derivative with respect to τ.

Let

$$b^2=\frac{k(D-2)}{2\Lambda}\frac{1}{8\pi G} . \tag{6}$$

Then we have

$$\dot{a}^2=1-\frac{L^4}{a^4}-H^2a^2 , \tag{7}$$

in which

$$L^4=\frac{(8\pi G)^{M+1}m^2}{6}\left[\frac{2\Lambda}{k(D-2)}\right]^M ,$$

$$H^2=\frac{8\pi G}{3}\frac{2\Lambda}{D-2} .$$

The parameters L and H have the following physical meaning: L is a wormhole radius in D-dimensional background spacetime, and H is a Hubble constant in D-dimensional de Sitter spacetime.

Only when

$$\Lambda^{M+2}<\frac{k^M(D-2)^{M+2}m^2}{2^{4M+8}(\pi G)^{M+3}}$$

does Eq. (7) have the following solution:

$$H\tau=\frac{\beta}{[\alpha(\beta-\gamma)]^{1/2}}\times\Pi\left[h\left[\frac{a^2}{L^2}\right],\frac{\alpha-\beta}{\alpha},\left[\frac{\gamma(\beta-\alpha)}{\alpha(\beta-\gamma)}\right]^{1/2}\right] , \tag{8}$$

where,

$$h(x)=\arcsin\left[\frac{\alpha(x-\beta)}{x(\alpha-\beta)}\right]^{1/2} , \tag{9}$$

Π is the elliptic integral of the third kind, α, β, and γ are three real roots of the equation of third order,

$$x^3-\frac{1}{(HL)^2}x^2+\frac{1}{(HL)^2}=0 ,$$

and $\alpha>\beta>0>\gamma$. Equation (9) is a wormhole which connects the corresponding points in two Euclidean higher-dimensional de Sitter spacetimes.

When $m\neq 0$, the axion charge in $S^3\otimes\mathcal{M}^M$ space can be given as

$$q=2\pi^2V_0m , \tag{10}$$

where V_0 is the volume of $\mathcal{M}^M$.

The action of this wormhole is

$$S_b=-\frac{c_0LH}{[\alpha(\alpha-\gamma)]^{1/2}}\left[\alpha E\left[\frac{\pi}{2},p\right]+\alpha\gamma F\left[\frac{\pi}{2},p\right]-3\Pi\left[\frac{\pi}{2},\frac{\alpha-\beta}{\alpha},p\right]\right] , \tag{11}$$

where

$$p=[(\alpha-\beta)(\alpha-\gamma)]^{1/2} ,$$

and F and E are the elliptic integrals of the first and second kinds, respectively, and C_0 is a constant which relates to dimensions.

The above calculated results show that there may exist wormhole solutions in a spontaneous compactification spacetime with $R^1\otimes S^3\otimes\mathcal{M}^M$.

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