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SEARCH FOR A REALISTIC KALUZA–KLEIN THEORY*

Edward WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

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An attempt is made to construct a realistic model of particle physics based on eleven-dimensional supergravity with seven dimensions compactified. It is possible to obtain an $SU(3) \times SU(2) \times U(1)$ gauge group, but the proper fermion quantum numbers are difficult to achieve.

In 1921 Kaluza suggested [1] that gravitation and electromagnetism could be unified in a theory of five-dimensional riemannian geometry. The idea was further developed by Klein [2] and was the subject of considerable interest during the classical period of work on unified field theories [3]. Readable expositions of some of the classical work have been given in text books by Bergmann and by Lichnerowicz [4]; more recent discussions have been given by Rayski and by Thirring [5].

While the Kaluza–Klein approach has always been one of the most intriguing ideas concerning unification of gauge fields with general relativity, it has languished because of the absence of a realistic model with distinctive and testable predictions. Yet the urgency of the unification of gauge fields with general relativity has surely greatly increased with the growing importance of gauge fields in physics. Moreover, the Kaluza–Klein theory has generalizations to non-abelian gauge fields which actually were first proposed [6] well before real applications were known for Yang–Mills fields in physics.

In the last few years this approach has been revived by Scherk and Schwarz and by Cremmer and Scherk, originally in connection with dual models [7]. These authors introduced many new ideas as well as new focus. In contrast to much of the classical literature, they advocated that the extra dimensions should be regarded as true, physical dimensions, on a par with the four observed dimensions. Cremmer and Scherk suggested that the obvious differences between the four observed dimensions and the extra microscopic ones could arise from a spontaneous breakdown of the vacuum symmetry, or, as they called it, from a process of "spontaneous compactification" of the extra dimensions.

These ideas have motivated much recent work. The idea of spontaneous compactification has been developed in more detail by Luciani [8]. An interesting idea by Palla [9] about massless fermions in theories with extra compact dimensions

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will figure in some of the discussion below. Manton [10] has discussed some questions that arise in trying to generate Higgs fields as components of the gauge field in extra dimensions. The idea of extra hidden dimensions has stimulated much work in supersymmetry theory, including the successful construction of the $N=8$ supergravity theory by Cremmer, Julia and Scherk and by Cremmer and Julia [11]. This work has been generalized to give models with broken sypersymmetry [12].

In many respects, of course, the modern approaches to this subject tend to differ from the classical point of view. In view of the proliferation of new particles in the last thirty years, one may be more willing today than in the past to postulate the infinite number of new degrees of freedom that must exist if extra dimensions really exist. Much of the classical literature focussed on the need to eliminate a massless spin-zero particle that naturally exists in the original Kaluza–Klein theory; the question seems less urgent today because the obvious answer is that quantum mechanical mass renormalization could easily account for the failure to observe this particle (a mass of 10^{-4} eV would make it undetectable). Some of the early work was motivated by the hope that the fifth dimension could provide the hidden variables that would eliminate indeterminacy from quantum mechanics. Despite the many generalizations and changes in emphasis that have occurred, I will refer generically to theories in which gauge fields are unified with gravitation by means of extra, compact dimensions as Kaluza–Klein theories.

It has often been suggested that spontaneous compactification and supergravity could be usefully combined together. The $N=8$ supergravity theory was constructed by "dimensional reduction" starting from an eleven-dimensional theory. In this context, "dimensional reduction" just means that the fields are taken to be independent of seven of the original eleven coordinates, to which physical reality need not be attributed. However, Cremmer and Julia [11] suggested that one might wish to consider seriously the eleven dimensions and interpret seven of them as compact dimensions in the spirit of Kaluza and Klein. This idea has been raised, on occasion, by various other theorists. In this paper, I will describe an attempt – not completely successful, but not completely unsuccessful either – to construct a realistic theory of Kaluza–Klein type, based on eleven-dimensional supergravity.

As discussed by some of the authors mentioned above, from a modern point of view the Kaluza–Klein unified theory of gravitation and electromagnetism is probably best understood as a theory of spontaneous symmetry breaking in which the group of general coordinate transformations in five dimensions is spontaneously broken to the product of the four-dimensional general coordinate transformation group and a local U(1) gauge group.

Let us review how this arises. One considers standard general relativity in five dimensions with the standard Einstein–Hilbert action

$$A=\int \mathrm{d}^5x\sqrt{g}R. \tag{1}$$

Instead of assuming that the ground state of this system is five-dimensional Minkowski space, which we will denote as M^5, one takes the ground state to be the product $M^4 \times S^1$ of four-dimensional Minkowski space M^4 with the circle S^1. The space $M^4 \times S^1$ is, like M^5, a solution of the five-dimensional Einstein equations. Classically it is difficult to decide which of the spaces M^5 and $M^4 \times S^1$ is a more appropriate choice as the ground state, since they both have zero energy, insofar as energy can be defined in general relativity*. Conventionally, one might assume that the ground state is M^5. In the Kaluza–Klein approach one assumes, instead, that the ground state is $M^4 \times S^1$, and the physical spectrum is determined by studying small oscillations around this ground state. One assumes that the radius of the circle S^1 is microscopically small, perhaps of order of the Planck length, and this accounts for why the existence of this fifth dimension is not noted in everyday experience.

The symmetries of the Kaluza–Klein ground state $M^4 \times S^1$ are the four-dimensional Poincaré symmetries, acting on M^4, and a U(1) group of rotations of the circle S^1. These symmetries would be observed as local or gauge symmetries in the apparent four-dimensional world because the whole theory started with the Einstein action (1) which is generally covariant. In fact, if one considers small oscillations around the "ground state" $M^4 \times S^1$, one finds an infinite number of massive excitations, the masses being of order the inverse of the circumference of S^1. One finds also a finite number of massless modes, which presumably would constitute the low-energy physics. The massless modes turn out to be a spin-two graviton and a spin-one photon, which are gauge particles of the symmetries of $M^4 \times S^1$, and a Brans–Dicke scalar.

The ansatz which exhibits the massless modes is the following. The metric tensor of this theory is a five by five matrix $g_{AB}(x^\mu, \phi)$ which in general may depend on the four coordinates x^μ, $\mu = 1 \cdots 4$, of M^4, and on the angular coordinate ϕ of S^1. The massless modes are those for which g_{AB} is a function of x^μ only. One can then write g_{AB} in block form

$$g_{AB}(x^\mu, \phi) = \left(\begin{array}{c|c} g_{\mu\nu}(x) & A_\mu(x) \\ \hline A_\mu(x) & \sigma(x) \end{array}\right), \tag{2}$$

where $g_{\mu\nu}$ is a four by four matrix (the first four rows and columns of g_{AB}), $A_\mu = g_{\mu 5}$, and $\sigma = g_{55}$. Then $g_{\mu\nu}$ is the ordinary metric tensor of the apparent four-dimensional world, and describes a massless spin-two particle; A_μ is the gauge field of the U(1) symmetry, and σ is the Brans-Dicke scalar.

In the classical work on the Kaluza–Klein theory, it is shown that the five-dimensional Einstein action (1), when expanded in terms of $g_{\mu\nu}$, A_μ, and σ (and the other modes, which decouple from these at low energies) contains a four-dimensional Einstein action $\sqrt{g}R^{(4)}$ for $g_{\mu\nu}$, a Maxwell action $F_{\mu\nu}^2$ for A_μ, and the usual

* The definition of energy in general relativity depends on the boundary conditions, so while both M^5 and $M^4 \times S^1$ have zero energy, a comparison between them is meaningless, like comparing zero apples to zero oranges.

kinetic energy for σ. Also, one can readily check that A_μ transforms as a gauge field $A_\mu \to A_\mu + \partial_\mu \varepsilon$ under coordinate transformations of the special type $(x^i, \phi) \to (x^i, \phi + \varepsilon(x^i))$ if the metric g_{AB} is transformed by the standard rule

$$g_{AB} \to g_{A'B'} \frac{\partial x'^{A'}}{\partial x^A} \frac{\partial x'^{B'}}{\partial x^B}\,.$$

The Kaluza–Klein theory thus unifies the metric tensor $g_{\mu\nu}$ and a gauge field A_μ into the unified structure of five-dimensional general relativity. This theory is surely one of the most remarkable ideas ever advanced for unification of electromagnetism and gravitation.

The Kaluza–Klein theory, as noted above, also has a non-abelian generalization, which has been extensively discussed over the years. In this generalization, one starts with general relativity in $4+n$ dimensions, possibly with additional matter fields or with a cosmological constant. Instead of assuming the ground state to be M^{4+n}, Minkowski space of $4+n$ dimensions, one assumes the ground state to be a product space $M^4 \times B$, where B is a compact space of dimension n. $M^4 \times B$ should be a solution of the classical equations of motion, or possibly, as will be discussed later, a minimum of some effective potential.

As in the previous discussion, symmetries of B will be observed as gauge symmetries in the effective four dimensional world. With a suitable choice of B, one may unify an arbitrary gauge group, abelian or non-abelian, with ordinary general relativity, in a $4+n$ dimensional theory.

The ansatz which generalizes (2) is the following. Let ϕ_i, $i = 1 \cdots n$, be coordinates for the internal space B. Let T^a, $a = 1 \cdots N$, be the generators of the symmetry group G of B. Let the action of the symmetry generator T^a on the ϕ_i be $\phi_i \to \phi_i + K_i^a(\phi)$, where $K_i^a(\phi)$ is the "Killing vector" associated with the symmetry T^a. Then the massless excitations of the candidate "ground state" $M^4 \times B$ correspond to an ansatz of the following form:

$$g_{AB}(x^\alpha, \phi^k) = \left(\begin{array}{c|c} g_{\mu\nu}(x^\alpha) & \sum_a A_\mu^a(x^\alpha) K_i^a(\phi^k) \\ \hline \sum_a A_\mu^a(x^\alpha) K_i^a(\phi^k) & \gamma_{ij}(\phi^k) \end{array} \right), \tag{3}$$

where γ_{ij} is the metric tensor of the internal space B. The fields $A_\mu^a(x^\alpha)$ are massless gauge fields of the group G. In this way one may obtain the gauge fields of an arbitrary abelian or non-abelian gauge group as components of the gravitational field in $4+n$ dimensions.

One may verify that the $4+n$ dimensional gravitational action really contains the proper kinetic energy term $\sum_a (F_{\mu\nu}^a)^2$. It is also straightforward to check that under infinitesimal coordinate transformations of the special form $(x^\alpha, \phi_i) \to (x^\alpha, \phi_i + \sum_a \varepsilon^a(x^\alpha) K_i^a(\phi))$, which is an x-dependent symmetry transformation of the internal space B, the field $A_\mu^a(x)$ transforms in the expected fashion, $A_\mu^a(x) \to A_\mu^a(x) + D_\mu \varepsilon^a(x)$. Thus, A_μ^a really has the properties expected of an ordinary four-dimensional gauge field. This gauge field is a remnant of the original coordinate invariance group in $4+n$ dimensions, which has been spontaneously broken down to the symmetries of $M \times B$.

As has been noted before, there is a fairly extensive literature on this construction. The case which has been discussed most widely is the case in which B is itself the manifold of some group H. It should be noted that, if H is a non-abelian group, the symmetry group G of the group manifold is not H but $H \times H$, since the group manifold can be transformed by either left or right multiplication. If one starts with general relativity in $4+n$ dimensions, the ansatz (3) will automatically give massless gauge mesons of the full symmetry group $H \times H$.

What problems arise if we try to construct a realistic theory along these lines? Known particle interactions can be described by the gauge group $SU(3) \times SU(2) \times U(1)$. So the symmetry group G of the compact space B must at least contain this as a subgroup,

$$SU(3) \times SU(2) \times U(1) \subset G\,. \tag{4}$$

So B must at least have $SU(3) \times SU(2) \times U(1)$ as a symmetry group.

To be as economical as possible, we may wish to choose B to be a manifold of minimum dimension with an $SU(3) \times SU(2) \times U(1)$ symmetry. What is the minimum dimension of a manifold which can have $SU(3) \times SU(2) \times U(1)$ symmetry?

$U(1)$ is the symmetry group of the circle S^1, which has dimension one. The lowest dimension space with symmetry $SU(2)$ is the ordinary two-dimensional sphere S^2. The space of lowest dimension with symmetry group $SU(3)$ is the complex projective space CP^2, which has real dimension four. (CP^2 is the space of three complex variables (Z^1, Z^2, Z^3), not all zero, with the identification $(Z^1, Z^2, Z^3) \simeq (\lambda Z^1, \lambda Z^2, \lambda Z^3)$ for any non-zero complex number λ. CP^2 can also be defined as the homogeneous space $SU(3)/U(2)$.) Therefore, the space $CP^2 \times S^2 \times S^1$ has $SU(3) \times SU(2) \times U(1)$ symmetry, and it has $4+2+1=7$ dimensions.

As we will see below, seven dimensions is in fact the minimum dimensionality of a manifold with $SU(3) \times SU(2) \times U(1)$ symmetry, although $CP^2 \times S^2 \times S^1$ is not the only seven-dimensional manifold with this symmetry. If, therefore, we wish to construct a theory in which $SU(3) \times SU(2) \times U(1)$ gauge fields arise as components of the gravitational field in more than four dimensions, we must have at least seven extra dimensions. With also four non-compact "space-time" dimensions, the total dimensionality of our world must be at least $4+7=11$.

This last number is most remarkable, because eleven dimensions is probably the maximum for supergravity. Eleven-dimensional supergravity has been explicitly constructed, and it is strongly believed that supergravity theories do not exist in dimensions greater than eleven. (The reason for this belief is that, on purely algebraic grounds [13], a supergravity theory in $d > 11$ would have to contain massless particles of spin greater than two. But there are excellent reasons, both S-matrix theoretic [14] and field theoretic [15], to believe that consistent field theories with gravity coupled to massless particles of spin greater than two do not exist.) It is consequently just barely possible to obtain $SU(3) \times SU(2) \times U(1)$ gauge fields as part of the gravitational field in a supergravity theory, if we use the unique, maximal, eleven-dimensional supergravity theory.

It is certainly a very intriguing numerical coincidence that eleven dimensions, which is the maximum number for supergravity, is the minimum number in which one can obtain $SU(3)\times SU(2)\times U(1)$ gauge fields by the Kaluza–Klein procedure. This coincidence suggests that the approach is worth serious consideration.

Let us now discuss in more detail the question of why seven dimensions is the minimum number of dimensions for a space with $SU(3)\times SU(2)\times U(1)$ symmetry – and the related matter of determining all seven-dimensional manifolds with this symmetry.

The space of lowest dimension with any symmetry group G is always a homogeneous space G/H, where H is a maximal subgroup of G. (The space G/H is defined as the set of all elements g of G, with two elements g and g' regarded as equivalent, $g \simeq g'$, if they differ by right multiplication by an element of H, that is, if $g = g'$ with $h \in$ H.) The dimension of G/H is always equal to the dimension of G minus the dimension of H.

In the case $G = SU(3)\times SU(2)\times U(1)$, the largest dimension subgroup that is suitable is $SU(2)\times U(1)\times U(1)$. Any larger subgroup of G would contain as a subgroup one of the three factors SU(3), SU(2), or U(1) of G, and this factor would then not have any non-trivial action on G/H – it would not really be a symmetry group of G/H. Since the dimension of $SU(3)\times SU(2)\times U(1)$ is $8+3+1=12$ and the dimension of $SU(2)\times U(1)\times U(1)$ is $3+1+1=5$, the dimension of $(SU(3)\times SU(2)\times U(1))/(SU(2)\times U(1)\times U(1))$ is $12-5=7$. It is for this reason that a space with $SU(3)\times SU(2)\times U(1)$ symmetry must have at least seven dimensions. However, there are many ways to embed $SU(2)\times U(1)\times U(1)$ in $SU(3)\times SU(2)\times U(1)$, and as a result there are many seven-dimensional manifolds with $SU(3)\times SU(2)\times U(1)$ symmetry.

To embed $SU(2)\times U(1)\times U(1)$ in $SU(3)\times SU(2)\times U(1)$ we first embed SU(2). SU(2) can be embedded in $SU(3)\times SU(2)\times U(1)$ in a variety of ways. The only embedding that turns out to be relevant is for SU(2) to be embedded in SU(3) as an "isospin" subgroup, so that the fundamental triplet of SU(3) transforms as $2+1$ under SU(2). [Other embeddings of SU(2) lead to spaces G/H on which some of the $SU(3)\times SU(2)\times U(1)$ symmetries act trivially, as discussed in the previous paragraph.] We still must embed $U(1)\times U(1)$ in $SU(3)\times SU(2)\times U(1)$.

$SU(3)\times SU(2)\times U(1)$ has three commuting U(1) generators which commute with the SU(2) subgroup of SU(3) that we have just chosen. There is a "hypercharge" generator of SU(3), which we may call λ_8, which commutes with the "isospin" subgroup. Also, we have the U(1) factor of $SU(3)\times SU(2)\times U(1)$, which will be called Y, and we may choose an arbitrary U(1) generator of the SU(2) factor, which will be called T_3.

So $SU(3)\times SU(2)\times U(1)$ contains an essentially unique subgroup $SU(2)\times U(1)\times U(1)\times U(1)$, where the three U(1) factors are λ_8, T_3, and Y. We do not want to divide

$SU(3)\times SU(2)\times U(1)$ by the full $SU(2)\times U(1)\times U(1)\times U(1)$ subgroup because this would yield a space ($CP^2\times S^2$, to be precise) on which the U(1) of $SU(3)\times SU(2)\times U(1)$ would act trivially and would not really be a symmetry. So we delete one of the three U(1) factors, and divide only by $SU(2)\times U(1)\times U(1)$.

The U(1) factor that is deleted may be an arbitrary linear combination $p\lambda_8+qT_3+rY$ of λ_8, T_3, and Y where p, q, and r are any three integers which have no common divisor*. So we define H as $SU(2)\times U(1)\times U(1)$, where the SU(2) is our "isospin" subgroup of SU(3), and the two U(1)'s are the two linear combinations of λ_8, T_3, and Y which are orthogonal to $p\lambda_8+qT_3+rY$. The space G/H is then a seven-dimensional space with $SU(3)\times SU(2)\times U(1)$ symmetry, which we may call M^{pqr}.

In a few cases the M^{pqr} are familiar spaces. M^{001} is our previous example $CP^2\times S^2\times S^1$. But in most cases the M^{pqr} are not familiar spaces, and are not products.

In a few cases the M^{pqr} have greater symmetry than $SU(3)\times SU(2)\times U(1)$. M^{101} is $S^5\times S^2$, which has the symmetry $O(6)\times SU(2)$. M^{011} is $CP^2\times S^3$, whose full symmetry is $SU(3)\times SU(2)\times SU(2)$. Except for these two cases, one cannot obtain from seven extra dimensions a symmetry "larger" than $SU(3)\times SU(2)\times U(1)$. Therefore, the observed gauge group in nature is practically the "largest" group one could obtain from a Kaluza–Klein theory with seven extra dimensions.

Although the M^{pqr} for general values of p, q, and r are not familiar spaces, it is possible to give a rather explicit description of them. Consider first the eight dimensional space $S^5\times S^3$ [S^n is the n-dimensional sphere, with symmetry group $O(n+1)$]. The symmetry group of $S^5\times S^3$ is $O(6)\times O(4)$. Let us introduce a particular generator of O(6),

$$K=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{5}$$

and a particular generator of O(4),

$$L=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{6}$$

Then the subgroup of O(6) that commutes with K is $SU(3)\times U(1)$ [the U(1) being

* And r should be non-zero to avoid obtaining a space on which U(1) is realized as the identity.

generated by K itself] and the subgroup of O(4) that commutes with L is $SU(2)\times U(1)$ [the U(1) being generated by L].

For any non-zero p and q, we now define $N=-qK+pL$. Then N generates a U(1) subgroup of $O(6)\times O(4)$, consisting of elements of the form $\exp tN$, $0\leq t\leq 2\pi$. We may now form from $S^5\times S^3$ a seven-dimensional space $M^{pq}=(S^5\times S^3)/U(1)$, where two points in $S^5\times S^3$ are considered to be identical if they are mapped into each other by the action of the U(1) subgroup generated by N.

This space M^{pq} is equal to the $r=1$ case of what we have previously called M^{pqr}. The M^{pq} are actually the most general simply connected seven-dimensional manifolds with $SU(3)\times SU(2)\times U(1)$ symmetry. To obtain M^{pqr} for $r\neq 1$ one must factor out from $S^5\times S^3$ an additional discrete subgroup consisting of elements of the form $\exp(2\pi qK/r)$ $(q=0,1,2,\dots,r-1)$. We define $M^{pqr}=M^{pq}/Z^r=(S^5\times S^3)/(U(1)\times Z^r)$.

To verify that the construction of the M^{pqr} just presented is equivalent to the previous definition as $(SU(3)\times SU(2)\times U(1))/(SU(2)\times U(1)\times U(1))$, one uses the fact that $SU(3)/SU(2)$ is S^5, while SU(2) is S^3, so $(SU(3)\times SU(2)\times U(1))/(SU(2)\times U(1)\times U(1))$ is $(S^5\times S^3\times U(1))/(U(1)\times U(1))$. Dividing out the two U(1) factors, one arrives at the above definition of M^{pqr} as $(S^5\times S^3)/(U(1)\times Z^r)$.

The M^{pqr} are not quite the most general seven-dimensional manifold with $SU(3)\times SU(2)\times U(1)$ symmetry, because for special values of p, q, and r it is possible to supplement $SU(2)\times U(1)\times U(1)$ with an additional twofold discrete symmetry. One obtains in this way some non-orientable manifolds with one of the M^{pqr} as a double covering space. These spaces are the following. Dividing M^{001} by a discrete symmetry one can get $CP^2\times P^2\times S^1$ (P^k is real projective space of dimension k), or $CP^2\times(S^2\times S^1)/Z_2$, where Z_2 is a simultaneous inversion of S^2 and S^1. From M^{101} one gets $S^5\times P^2$ and $(S^5\times S^2)/Z_2$, where the Z_2 is a simultaneous inversion of S^5 and S^2. Likewise, by dividing M^{10r} by an additional two-fold symmetry one can make $S^5/Z^r\times P^2$ and $(S^5/Z^r\times S^2)/Z_2$. These spaces are non-orientable. This completes the list of seven-dimensional manifolds with $SU(3)\times SU(2)\times U(1)$ symmetry.

If one is willing to suppose that the ground state of eleven-dimensional supergravity is a product of four-dimensional Minkowski space with one of the M^{pqr}, one can obtain an $SU(3)\times SU(2)\times U(1)$ gauge group, the gauge fields being components of the gravitational field, according to the ansatz of eq. (3). Of course, to describe nature, it is not sufficient to have the gauge group. It is also necessary to have quarks and leptons of essentially zero mass [very light compared to the energy scale of gravitation; massless in any approximation in which $SU(3)\times SU(2)\times U(1)$ is not spontaneously broken] which should be in the appropriate representation of the gauge group. And it is necessary to find Higgs bosons whose vacuum expectation value could ultimately trigger $SU(2)\times U(1)$ breaking.

How can one obtain massless quarks and leptons in the Kaluza–Klein framework? To understand the basic idea*, suppose that in a $4+n$ dimensional theory we have a

* See also a discussion by Palla [9].

massless spin one half fermion. It satisfies the $4+n$ dimensional Dirac equation,

$$\not{D}\psi = 0\,, \tag{7}$$

or explicitly

$$\sum_{i=1}^{4+n} \gamma^i D_i \psi = 0\,. \tag{8}$$

This Dirac operator can be written in the form

$$\not{D}^{(4)}\psi + \not{D}^{(\mathrm{int})}\psi = 0, \tag{9}$$

where $\not{D}^{(4)} = \sum_{i=1}^{4} \gamma^i D_i$ is the ordinary four-dimensional Dirac operator, and $\not{D}^{(\mathrm{int})} = \sum_{j=5}^{4+n} \gamma^j D_j$ is the Dirac operator in the internal space of n compact dimensions.

The expression (9) immediately shows that the eigenvalue of $\not{D}^{(\mathrm{int})}$ will be observed in practice as the four-dimensional mass. If $\not{D}^{(\mathrm{int})}\psi = \lambda\psi$, then ψ will be observed by four-dimensional observers who are unaware of the existence of the extra microscopic dimensions as a fermion of mass $|\lambda|$.

The operator $\not{D}^{(\mathrm{int})}$ acts on a compact space, so its spectrum is discrete. Its eigenvalues either are zero or are of order $1/R$, R being the radius of the extra dimensions. Since $1/R$ is, in the Kaluza–Klein approach, presumably of order the Planck mass, the non-zero eigenvalues of $\not{D}^{(\mathrm{int})}$ correspond to extremely massive fermions which would not have been observed. The observed quarks and leptons must correspond to the zero modes of $\not{D}^{(\mathrm{int})}$.

If, in eleven-dimensional supergravity, the ground state is a product of four-dimensional Minkowski space with one of the M^{pqr}, then the zero modes of the Dirac operator in the internal space will, if there are any zero modes at all, automatically form multiplets of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, since this is the symmetry of the internal space. It therefore is reasonable to wonder whether for an appropriate choice of p, q, and r, zero modes could exist and form the appropriate representation of the symmetry group, so as to reproduce the observed spectrum of quarks and leptons.

Of course, to reproduce what is observed in nature, we would need quite a few zero modes of the internal space Dirac operator. If the top quark exists, there are in nature at least 45 fermion degrees of freedom of given helicity, counting all colors and flavors of quarks and leptons. We would therefore need at least 45 Dirac zero modes. However, when a Dirac operator has zero modes, the number usually depends on topological invariants. Perhaps by choosing suitable values of p, q, and r we could suitably "twist" the topology and obtain the required 45 zero modes lying in the appropriate representation of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$.

Actually, if one has in mind eleven-dimensional supergravity, one must modify this program slightly. In eleven-dimensional supergravity, there is no fundamental spin one half field. The only fundamental Fermi field in that theory is the Rarita–Schwinger field $\psi_{\mu\alpha}$, of spin $\frac{3}{2}$ (μ is a vector index, α a spinor index).

Although this field has spin $\frac{3}{2}$ from the point of view of eleven dimensions, the components of ψ_μ with $5 \leqslant \mu \leqslant 11$ are spin one half fields from the point of view of ordinary four-dimensional physics. For $\mu \geqslant 5$, μ would be observed as an internal symmetry index, not a space-time index; it carries spin zero. Although the components ψ_μ with $\mu = 1 \cdots 4$ are spin-$\frac{3}{2}$ fields in the four-dimensional sense, the components with $\mu = 5 \cdots 11$ are spin one half fields. So zero-mode solutions of the spin-$\frac{3}{2}$ wave equation in the extra dimensions would be observed as massless spin-$\frac{1}{2}$ fermions in four dimensions. These would be the ordinary light fermions of the spontaneously compactified eleven-dimensional theory.

In one sense, it is an advantage to have to consider the Rarita–Schwinger operator rather than the Dirac operator. The Rarita–Schwinger operator can have zero modes more easily and in more abundance than the Dirac operator, because the Dirac operator has positivity properties which tend to suppress the number of zero modes. For instance, with four extra dimensions, it is known [16] that there is only a single non-flat compact solution of Einstein's equations on which the Dirac operator has zero modes. This is the Kahler manifold K3 (which has no Killing vectors). On this space there are two zero modes of the Dirac operator – but 42 zero modes of the Rarita–Schwinger operator. The large discrepancy is caused, in this case, by a much larger coefficient of the axial anomaly for Rarita–Schwinger fields. This example shows, incidentally, that the rather large number of zero modes that would be required to describe what is observed in physics is not necessarily out of reach.

In the approach considered here, the solution of the problem of flavor – the problem of the existence of several "generations" of fermions with the same quantum numbers – would be that the extra dimensions have a sufficiently complex topology that there are several zero modes with the same $SU(3) \times SU(2) \times U(1)$ quantum numbers. When an operator has several zero modes, they are not necessarily related by any symmetry. For instance, the isospinor Dirac operator in a Yang–Mills instanton of topological number K has K modes; these modes are not related by any symmetry. This is fortunate, because the various generations of fermions have very different masses and are not obviously related by any symmetry.

Unfortunately, there is a basic reason that this idea does not work, at least not in the form described above. The reason for this is related to one of the most basic facts about the observed quarks and leptons: the fermions of given helicity transform in a complex representation of the gauge group, or, to put it differently, right-handed fermions do not transform the same way that the left-handed fermions transform. For instance, left-handed color triplets (quarks) are $SU(2)$ doublets, but right-handed color triplets are $SU(2)$ singlets. This is the reason that quarks and leptons do not have bare masses but receive their mass from the Higgs mechanism – from $SU(2) \times U(1)$ symmetry breaking. This is a very important fact theoretically, because it is the basis for our theoretical understanding of why the quarks and leptons are very light compared to the mass scale of grand unification or the Planck mass. If left- and right-handed fermions transformed the same way under the the gauge group, bare masses would have been possible and could have been arbitrarily large.

In the framework that has been described above, right- and left-handed fermions would inevitably transform the same way under $SU(3)\times SU(2)\times U(1)$. The reason for this is that low mass fermions are supposed to arise as zero modes of the Rarita–Schwinger operator in the extra dimensions. But the Rarita–Schwinger operator in the seven extra dimensions does not "know" whether a spinor field is left- or right-handed with respect to four-dimensional Lorentz transformations. It treats four-dimensional left- and right-handed fermions in the same way. One therefore could not get the observed $SU(3)\times SU(2)\times U(1)$ representation. One would inevitably get vector-like rather than V–A weak interactions, with bare masses being possible for all fermions. (Indeed, precisely because bare masses would be possible for all fermions, it is not natural to get any massless fermions at all.)

There is an intriguing mechanism by which, at first sight, it seems that the internal space Rarita–Schwinger equation could treat left and right fermions differently. Eleven-dimensional spinors are constructed with eleven gamma matrices γ_i, $i = 1\cdots 11$. Let us define an operator $\Gamma_{11} = i\gamma_1\cdots\gamma_{11}$ which is a sort of eleven-dimensional helicity operator. Let us also define an operator $\Gamma_4 = i\gamma_1\gamma_2\gamma_3\gamma_4$ which measures the ordinary four-dimensional helicity, and an operator $\Gamma_7 = \gamma_5\cdots\gamma_{11}$ which one might think of as "helicity" in the internal eleven-dimensional space. Then $\Gamma_{11}^2 = \Gamma_4^2 = \Gamma_7^2 = 1$ and $\Gamma_{11} = \Gamma_4\Gamma_7$.

The Rarita–Schwinger field ψ of eleven-dimensional supergravity satisfies a Weyl condition $\psi = \Gamma_{11}\psi$. (This condition must be imposed; otherwise there would be more Fermi than Bose degrees of freedom and supersymmetry would not be possible.) This identity may equivalently be written $\Gamma_4\psi = \Gamma_7\psi$.

The latter equation shows that in eleven-dimensional supergravity the four-dimensional helicity of fermions is correlated with the seven-dimensional "helicity". Components with $\Gamma_4 = +1$ (or -1) have $\Gamma_7 = +1$ (or -1). If the quantum numbers of zero modes of the seven-dimensional Rarita–Schwinger equation depended on Γ_7, as one might intuitively expect, they would also depend on Γ_4.

Unfortunately, the spectrum of the seven-dimensional Rarita–Schwinger operator does not depend on Γ_7. The reason for this is very simple (and depends only on the fact that the number of extra dimensions is odd). In defining how spinors transform under coordinate transformations in riemannian geometry one needs the matrices $\sigma_{ij} = [\gamma_i, \gamma_j]$. One does not (on an orientable manifold) need the γ_i themselves. The transformation $\gamma_i \leftrightarrow -\gamma_i$ does not change the σ_{ij}, so it does not affect the definition of spinors. It does, however, change the sign of $\Gamma_7 = \gamma_1\gamma_2\cdots\gamma_7$. Consequently, spinors with opposite values of Γ_7 transform the same way under coordinate transformations. Since, in the approach discussed here, $SU(3)\times SU(2)\times U(1)$ transformations are coordinate transformations, spinors with opposite values of Γ_7 have the same $SU(3)\times SU(2)\times U(1)$ quantum numbers.

One could try to avoid this conclusion by taking the extra seven dimensions to be a non-orientable manifold. In a non-orientable manifold, the definition of spinors is subtle and involves the γ_i as well as σ_{ij}. However, seven-dimensional non-orientable manifolds with $SU(3) \times SU(2) \times U(1)$ symmetry are not abundant (they have all been listed above), and it is not difficult to show that none of them are suitable.

One might also try to avoid the above stated conclusion by going beyond riemannian geometry to include some variant of torsion. What possibilities this would offer is not very clear; the matter will be discussed at the end of this paper.

Obtaining the right quantum numbers for quarks and leptons is, of course, not the only problem that must be faced in order to obtain a realistic theory, although it may be the most difficult problem. We must also worry about spontaneous breaking of supersymmetry, spontaneous breaking of *CP*, spontaneous breaking of $SU(2) \times U(1)$ gauge symmetry, and obtaining the proper values of the low-energy parameters (coupling constants, masses, and mixing angles); and we must worry about what the true ground state of the theory really is. These questions will now be briefly discussed in turn.

For spontaneous breaking of supersymmetry the prospects are very bright; in fact, supersymmetry almost inevitably is spontaneously broken as part of any scheme in which there are compact dimensions with a non-abelian symmetry.

The reason for this is the following. Unbroken supersymmetry means that under a supersymmetry transformation the vacuum expectation values of the fields do not change. The vacuum expectation values of the Bose fields automatically are invariant under supersymmetry, since their supersymmetric variation would be proportional to the (vanishing) vacuum expectation values of the Fermi fields. The delicate question is whether the vacuum expectation values of the fermi fields change under supersymmetry.

To illustrate the point, let us ignore the possible presence in the theory of Bose fields other than the gravitational field. Then the transformation law for the Rarita–Schwinger field is $\delta\psi_\mu = D_\mu \varepsilon$, ε being the gauge parameter. An unbroken supersymmetry – a symmetry of the vacuum – must have $\delta\psi_\mu = 0$, so unbroken supersymmetry transformations correspond to solutions of $D_\mu \varepsilon = 0$.

On a curved manifold, this equation will almost certainly not have solutions, since $D_\mu \varepsilon = 0$ implies the integrability condition $[D_\mu, D_\nu]\varepsilon = 0$ or $R_{\mu\nu\alpha\beta}[\gamma^\alpha, \gamma^\beta]\varepsilon = 0$, which on most curved manifolds is not satisfied by any non-zero ε. For instance, on none of the M^{pqr} does a solution exist. (The properties of seven-dimensional manifolds admitting solutions of $D_\mu \varepsilon = 0$ have been discussed in the mathematical literature [17], but non-trivial examples do not seem to be known.) So in theories with curved extra dimensions, there will generally not be any unbroken supersymmetries.

The picture does not change greatly when one includes Bose fields other than the gravitational field. We now have $\delta\psi_\mu = \bar{D}_\mu \varepsilon$, where $\bar{D}_\mu = D_\mu$ plus non-minimal terms involving the vacuum expectation values of other Bose fields (and possibly involving the expectation values of fermion bilinears, as discussed below). Unbroken supersymmetries are now solutions of $\bar{D}_\mu \varepsilon = 0$, but solutions will still typically not exist because the integrability condition $[\bar{D}_\mu, \bar{D}_\nu]\varepsilon = 0$ will still not have solutions.

Although solutions will generally not exist, the extra dimensions and the vacuum expectation values of the fields may be just such that one or more solutions of $\bar{D}_\mu \varepsilon = 0$ would exist. Each solution of $\bar{D}_\mu \varepsilon = 0$ in the internal space would correspond to an unbroken supersymmetry charge in four dimensions. If there is precisely one such solution, and so only one unbroken supersymmetry generator, this corresponds to a theory in which $N = 8$ supersymmetry has been spontaneously broken down to $N = 1$ supersymmetry. If there are K solutions, there is an unbroken $N = K$ supersymmetry.

A particularly attractive possibility would be a theory in which the equation $\bar{D}_\mu \varepsilon = 0$ has precisely one solution in the extra dimensions, corresponding to unbroken $N = 1$ supersymmetry. With $N = 1$ supersymmetry it is possible to construct more or less realistic models of observed particle physics. With $N \geqslant 2$ it is not possible to make a realistic model, because the supersymmetry algebra for $N \geqslant 2$ forces left- and right-handed fermions to transform in the same way under the gauge group, in contrast with what is observed. It is attractive to believe that $N = 1$ supersymmetry might survive after compactification of seven dimensions because this would severely constrain the theory, would make many predictions that might be testable in accelerators, and [19] might shed light on $\mathrm{SU}(2) \times \mathrm{U}(1)$ breaking and the gauge hierarchy problem. Of course, we would then have to explain how $N = 1$ supersymmetry is eventually spontaneously broken at low energies.

In addition to supersymmetry breaking, we must also explain P and CP breaking in order to construct a realistic theory. The eleven-dimensional supergravity langrangian is invariant under inversions of space (or time) combined with a change of sign of the antisymmetric tensor gauge field that exists in this theory. After compactification of seven dimensions, the eleven-dimensional symmetry could be manifested as both P (inversion of space) and C (inversion of the compact dimensions). These potential invariances must be spontaneously broken.

A natural mechanism for spontaneous breaking of P, C, and CP involves the antisymmetry tensor gauge field of the eleven-dimensional supergravity theory. The curl $F_{\alpha\beta\gamma\delta}$ of this field may have a vacuum expectation value without breaking Lorentz invariance or $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. In fact, as discussed recently by several authors [20], a vacuum expectation value of F_{1234} is Lorentz invariant. It would violate P and CP but conserve C. The components F_{ijkm} for $i \cdots m \geqslant 5$ may also have expectation values, which would spontaneously break C and CP but conserve P. It is not difficult to see (by considering the little group of a point on M^{pqr}) that on any of the M^{pqr}, the most general $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ invariant vacuum expectation value of F_{ijkl} depends on two real parameters.

Although the eleven-dimensional theory can have spontaneous breaking of C, P, and CP, the strong interaction angle θ will inevitably vanish at the tree level. The reason for this is that in the eleven-dimensional theory, there is no operator which might be added to the lagrangian which reduces in four dimensions to $\theta \int \mathrm{d}^4 x F_{\mu\nu} \tilde{F}_{\mu\nu}$. There simply does not exist in eleven dimensions any topological invariant that can be written as the integral of a lagrangian density. Of course, the question of how large a vacuum angle might be generated by quantum corrections must wait until we understand how to do calculations in this (presumably) non-renormalizable theory.

It is also necessary, of course, to obtain $SU(2)\times U(1)$ symmetry breaking; this presumably means that we must find, at the tree level, a massless Higgs doublet which could later obtain a very tiny negative mass squared.

There are various ways that, in a Kaluza–Klein theory, one might obtain massless charged scalars. In the original Kaluza–Klein theory, with a single compact dimension (a circle) there is a massless scalar (at least at the tree level) because the classical field equations do not determine the radius of the circle. Space-time dependent fluctuations of this radius would be observed as a massless scalar degree of freedom.

If the equations that determine our hypothetical ground state $M^4\times M^{pqr}$ admit not a unique solution for the metric of M^{pqr} but a whole family of solutions, then oscillations within this family would be observed as massless scalars. Some of these oscillations might involve departures from $SU(2)\times U(1)$ symmetry and could be the desired Higgs bosons.

One might also obtain massless scalars as components of the antisymmetric tensor gauge field. In fact, massless scalars can be obtained in this way, but tend to be neutral under the gauge group.

Regardless of where the scalars come from, why would they be massless? The most plausible explanation would be an unbroken supersymmetry relating the massless bosons to massless fermions. This could involve the possibility discussed above that the equation $\bar{D}_\mu\varepsilon=0$ has a unique non-trivial solution, leaving $N=1$ supersymmetry unbroken. In this case, of course, we must hope to find a non-perturbative mechanism spontaneously breaking the supersymmetry and giving a small vacuum expectation value to the scalar bosons. (Some relevant issues will be discussed in a future paper [21].)

Without understanding the Higgs bosons and the low-energy symmetry breaking, it is of course not possible to predict the quark and lepton masses and mixing angles. If we understood the dynamics that determines the metric of M^{pqr} (assuming that the ground state really is $M^4\times M^{pqr}$), we could predict the strong, weak, and electromagnetic coupling constants, since the gauge fields all arise, by the ansatz of eq. (3), as part of the metric tensor in eleven dimensions, and the gauge field kinetic energy is part of the Einstein action. [The most general $SU(3)\times SU(2)\times U(1)$ invariant metric on M^{pqr} depends on three arbitrary parameters. If we understood the dynamics and could calculate the three parameters, we could predict the SU(3), SU(2), and U(1) coupling constants.] Even though we do not understand this dynamics (see below), it is possible to make a useful comment.

In a theory of this kind, the gauge coupling constants, which are determined by integrating the action over the compact dimensions, would scale as a rather high power of $1/(M_pR)$, where M_p is the Planck mass and R is the radius of the extra dimensions. The fact that the observed gauge coupling constants in nature differ from one by only one or two orders of magnitude shows that R cannot be too much greater than $1/M_{\mathrm{p}}$; the extra dimensions really have a radius not too different from 10^{-33} cm.

The eleven-dimensional supergravity theory has no global symmetry that could be interpreted as baryon number, so in this theory nucleons are almost surely unstable. The mass scale in nucleon decay, however, would probably be $1/R$, which is the mass scale of the heavy quanta in this theory. Since, as just noted, $1/R$ cannot be much less than M_{p}, the nucleon lifetime will probably be very long, perhaps 10^{45} years, which is far too long for nucleon decay to be observable. If the present nucleon decay experiments give a positive result, the approach described in this paper would become significantly less attractive.

It is now time to finally discuss the question of whether one can really sensibly expect $\mathrm{M}^4 \times \mathrm{M}^{pqr}$ to be the ground state of this theory.

The most attractive possibility would be that $\mathrm{M}^4 \times \mathrm{M}^{pqr}$ might be a solution of the classical equations of motion, possibly with a suitable vacuum expectation assumed for $F_{\mu\nu\alpha\beta}$. Unfortunately, a straightforward calculation shows that this is not true (regardless of what vacuum expectation value one assumes). If one arbitrarily adds to the lagrangian a cosmological constant (with a sign corresponding to a positive energy density) then $\mathrm{M}^4 \times \mathrm{M}^{pqr}$ can be a solution. However, local supersymmetry does not permit a cosmological constant in the eleven-dimensional lagrangian.

This problem is not necessarily fatal, since one can always hope that $\mathrm{M}^4 \times \mathrm{M}^{pqr}$, although not a solution of the classical equations of motion, is the minimum of the appropriate effective potential. In eleven-dimensional supergravity, there is no small dimensionless parameter whose smallness could justify the use of the classical field equations as an approximation. So the fact that $\mathrm{M}^4 \times \mathrm{M}^{pqr}$ does not satisfy the classical equations, while not encouraging, is not necessarily critical.

In any case, there is absolutely no obvious reason that $\mathrm{M}^4 \times \mathrm{M}^{pqr}$, rather than the more obvious possibility of eleven dimensional Minkowski space, should be the ground state of this theory.

It will be shown in a separate paper that even when Kaluza–Klein vacuum states are stable classically, they can be destabilized by quantum mechanical tunneling [22]. However, unbroken supersymmetry (plus a technical requirement that the extra dimensions be simply connected; this is not satisfied in the original Kaluza–Klein theory) seems to be a sufficient condition for stability. This is another reason that theories in which $\bar{D}_\mu \varepsilon = 0$ has a solution and there is an unbroken supersymmetry at the energies of compactification would be attractive.

As has been pointed out above, the most serious obstacle to a realistic model of the type considered in this paper is that the fermion quantum numbers do not turn out right. It is conceivable that this problem could be overcome if instead of riemannian geometry one considered geometry with torsion or some generalization of torsion; in such a theory the fermion transformation laws might be different.

How can one obtain torsion in eleven-dimensional supergravity? As has been noted [11], the theory formally contains torsion in the sense that certain fermion bilinears enter, formally, in the way that torsion would appear. Of course, a "torsion" that is bilinear in Fermi fields does not have a classical limit. However, by analogy with QCD, in which $\bar{q}q$ has a vacuum expectation value, one may be willing in supergravity to assume a vacuum expectation value for the "torsion field" $K \sim \bar{\psi}\psi$ (or perhaps for some other bilinears). Perhaps in this way the predictions for fermion quantum numbers can be modified. This possibility is under study.

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Note added in proof

For a recent discussion of Dirac zero modes in Kaluza–Klein theories, see ref. [23].

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