General Relativity and Gravitation, Vol. 26, No. 6, 1994

Kaluza–Klein Theory and Machian Cosmology

Paul S. Wesson[1,2] **and J. Ponce de Leon**[1,3]

Received July 13, 1993

We interpret the 15 equations of Kaluza–Klein gravity as 10 Einstein equations, 1 wave equation and 4 equations of motion. An exact cosmological solution of the apparently empty 5D field equations describes a 4D fluid with an effective density and pressure induced by the curvature associated with the fifth dimension. The rest mass of a particle in the fluid depends on the global solution and changes slowly with time. This approach to Kaluza–Klein theory in general results in Machian cosmologies.

1. INTRODUCTION

Kaluza–Klein theory in more than 4 dimensions provides a possible scheme for the unification of the interactions of physics [1–5]. In its original and simplest form, it is a 5D theory that unifies classical gravity and electromagnetism insofar as its field equations for vacuum give back those of Einstein and Maxwell (with one equation essentially rendered impotent by a condition on the metric: see Refs. 1,2). However, even if the electromagnetic potentials are set to zero, Kaluza–Klein theory in the absence of other restrictions is a theory of gravity with 15 field equations. It is our purpose in what follows to suggest a new way of interpreting these equations.

We will adopt the approach, recently investigated in several contexts

[1] Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
[2] GP-B, Department of Physics, Stanford University, Stanford, California 94305, USA
[3] Theoretical Physics Lab., University of Puerto Rico, Rio Piedras, Puerto Rico 00931, USA

0001-7701/94/0600-0555$07.00/0

(see below), that the field equations for apparently empty 5D space are in fact equations for gravity *with matter* in 4D space. That is, we will adopt the view that geometrical curvature in 5D induces matter in 4D. This idea is not entirely new (Ref. 6, p.129, Ref. 7), but should not be confused with related work in supersymmetry and superstrings [3,4]. The idea here is that there is a *classical* 4D energy-momentum tensor which derives its existence and form from the geometry of an exact 5D solution; and that the phenomenological properties of matter such as the density and pressure are determined by such a solution, as is their equation of state. This approach works well in a number of situations. For example, there is a well-known class of solutions in 5D which are the analogs of the Schwarzschild solution in 4D [8–10]. These 5D apparently empty solutions have effective pressures and densities in 4D that correspond to radiation, with the appropriate equation of state [11–15]. The viability of these solutions has been tested astrophysically [16–18]. Unfortunately, they have limited physical application because they do not involve the fifth dimension in a very significant way (the equation of state is restricted to be radiation-like because there is no dependence of the metric on the fifth coordinate: see below). Therefore attention has focused recently on a class of cosmological Kaluza–Klein solutions, which are simple enough to handle mathematically while general enough to be interesting physically [19–21]. These are analogs of the 4D Friedmann–Robertson–Walker (FRW) solutions with flat space sections, and have acceptable properties (see Ref. 20 and below). However, one can ask not only about exact solutions but about the general reduction of apparently empty 5D equations to 4D ones with an effective energy-momentum tensor. This has been investigated at length from the mathematical side [22], and if the metric coefficients are allowed to depend on the fifth coordinate then it can be shown that there is sufficient algebraic flexibility to ensure that the apparently empty Kaluza–Klein equations can always be reduced to Einstein's equations with an effective and well-behaved energy-momentum tensor. It should be noted that this result depends on putting the fifth coordinate on the same footing as the others of spacetime, and does not necessarily presume that the extra dimension is compactified to an unobservably small size. The former property leads one to infer that the fifth coordinate is related to rest mass in cosmology [20], while the latter property helps alleviate some problems of Kaluza–Klein theory as applied to particle physics [23–27]. In what follows, we wish to build on previous algebraic work [22,28], and suggest how the 15 field equations of 5D Kaluza–Klein theory might be related to the 4D physics of gravitating matter, rest mass and motion.

Our account will be brief, but we will make contact with areas of physics where there is extensive literature, notably as regards wave equations for particles [29–32] and a possible change in the strength of gravity over cosmological time [33–38]. We will also make contact with the extensive literature on other versions of 5D general relativity [39–48]. Of these latter, the best developed is that of projective relativity [40–42,45–48]. Some of the relations in this theory are similar to ones derived below. But our treatment is different in that we do not use a projection tensor, do not introduce an explicit energy-momentum tensor, and regard the constants of physics as merely dimensional conversion factors [11–22,32]. In this last regard, we will streamline the working by absorbing dimensional parameters such as the speed of light, the gravitational constant and Planck's constant via a choice of units that renders $c = 1$, $8\pi G = 1$ and $h/2\pi = 1$ [32]. Our conclusion will be that this approach to Kaluza–Klein gravity leads to self-consistent Machian models in which the local properties of a particle depend on a global cosmological solution. However, more work is needed before we can say if this holds also in the non-cosmological case.

2. ON THE PHYSICAL INTERPRETATION OF KALUZA–KLEIN EQUATIONS

We take a 5D line element $ds^2 = g_{ab}dx^a dx^b$ where the metric coefficients can depend in any way on the 5 coordinates x^a $(a = 0, 1, 2, 3, 4)$. However, it is convenient to use the degrees of freedom connected with a choice of coordinates to set the row and column bordering the 4×4 block $g_{\alpha\beta}$ to zero via $g_{\alpha 4} = 0$ ($\alpha = 0, 1, 2, 3$: see Ref. 22).[4] [5] Also, we write $g_{44} = -\phi^2$ where $\phi = \phi(x^a)$ is a scalar function, which along with the other g_{ab} is to be determined by field equations.

The latter are given in terms of the 5D Ricci tensor by

$$R_{ab} = 0. \tag{1}$$

[4] In principle, we could use the remaining degree of coordinate freedom to put a constraint on g_{44} such as $g_{44} = -1$. This would "compress" the physics into the 4×4 block $g_{\alpha\beta}$, which if dependent on x^4 would still not be identical to general relativity. However, we do not wish to do this, as we anticipate that the resulting form of the field equations would be difficult to interpret.

[5] We take this opportunity to correct some minor misprints in Ref. 22: eq. (18) there is correct, but holds also with $a \to \nu$, $d \to \mu$ in the products of Christoffel symbols, and is more useful in this form; eq. (19) needs asterisks over $g_{\lambda\beta}$ and $g_{\lambda\beta,\alpha}$ in the middle line; and eq. ($A2$) needs the asterisk removing over $g_{44,\mu}$ in $\Gamma^4_{4\mu}$. These are merely cosmetic changes, and all major results in Ref. 22 are correct.

Thus we are considering "vacuum" in 5D. However, the 15 independent components of (1) contain terms which depend on the fifth dimension via g_{44} and partial derivatives of the $g_{\alpha\beta}$ with respect to x^4. (Henceforth we will denote $\partial/\partial x^4$ by an asterisk, $\partial/\partial x^\alpha$ by a comma, and the covariant derivative by a semicolon as usual.) If we regard these extra terms as relating to matter, and move them to the right-hand side of the first 10 components of (1), a somewhat tedious algebraic manipulation [22] shows that the equations can be written this way:

$$G_{\alpha\beta} = T_{\alpha\beta}$$

$$G^{\alpha\beta} \equiv {}^{(4)}R_{\alpha\beta} - \frac{{}^{(4)}Rg_{\alpha\beta}}{2}$$

$$T_{\alpha\beta} \equiv \frac{\phi_{,\alpha;\beta}}{\phi} + \frac{1}{2\phi^2}\left\{\frac{\overset{*}{\phi}\overset{*}{g}_{\alpha\beta}}{\phi} - \overset{**}{g}_{\alpha\beta} + g^{\lambda\mu}\overset{*}{g}_{\alpha\lambda}\overset{*}{g}_{\beta\mu} - \frac{g^{\mu\nu}\overset{*}{g}_{\mu\nu}\overset{*}{g}_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\left[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + (g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2\right]\right\} . \quad (2)$$

That is, the equations of general relativity are satisfied in terms of an Einstein tensor defined with the conventional 4D Ricci tensor and scalar (indicated by a preceding superscript), and an effective 4D energy-momentum tensor. The latter involves derivatives w.r.t. x^4 in general. If there is no dependency on the extra coordinate it can be shown using (3) below that $T = T_{\alpha\beta}g^{\alpha\beta} = 0$, which implies a radiation-like equation of state. But for other equations of state we need the extra coordinate to play a role.

The extra metric coefficient $g_{44} = -\phi^2$ can be elucidated by expanding the component of (1) that reads $R_{44} = 0$. This can be written

$$\Box\phi = \frac{1}{\phi}\left\{\frac{\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{4} + \frac{g^{\lambda\beta}\overset{**}{g}_{\lambda\beta}}{2} - \frac{\overset{*}{\phi}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2\phi}\right\} \quad (3)$$

$$\Box\phi \equiv g^{\mu\nu}\phi_{,\mu;\nu} .$$

That is, the scalar wave equation is recovered, with a source if there is dependency on x^4 but no source if the extra coordinate does not play a role. In this regard, we note that the 4D scalar curvature is given by ${}^{(4)}R = -[\overset{*}{g}^{\mu\nu}\overset{*}{g}_{\mu\nu} + (g^{\mu\nu}\overset{*}{g}_{\mu\nu})^2]/4\phi^2$, and so is finite or zero depending on whether there is x^4-dependency or not.

The other 4 components of (1) read $R_{4\alpha} = 0$. They may be manipulated into the form

$$P^{\beta}_{\alpha;\beta} = 0$$
$$P_{\alpha\beta} \equiv \frac{1}{2\phi}\{\overset{*}{g}_{\alpha\beta} - g_{\alpha\beta}g^{\mu\nu}\overset{*}{g}_{\mu\nu}\}. \quad (4)$$

That is, there is a 4×4 symmetric tensor that obeys 4 relations that look like conservation laws. The tensor $P_{\alpha\beta}$ has no analog in general relativity since it vanishes if there is no dependency on the extra coordinate. We will make a tentative identification of this tensor below, but note here that $P_{\alpha\beta}$ of (4) can hardly be related to $T_{\alpha\beta}$ of (2): they have different algebraic forms, and physically $P_{\alpha\beta}$ has dimensions L^{-1} while $T_{\alpha\beta}$ has dimensions L^{-2}. Thus while $T_{\alpha\beta}$ relates to the properties of the gravitating fluid that is described by (1), $P_{\alpha\beta}$ appears to require a different physical interpretation.

Such an interpretation is greatly aided by using exact solutions that are algebraically simple but include x^4-dependency. A one-parameter class of such solutions is given by

$$ds^2 = \psi^2 dt^2 - t^{2/\alpha}\psi^{2/(1-\alpha)}(dx^2 + dy^2 + dz^2) - \alpha^2(1-\alpha)^{-2}t^2 d\psi^2. \quad (5)$$

Here α is a constant, and we have assigned the coordinates in the normal way and written $x^4 = \psi$. On hypersurfaces ψ = constants, (5) reduces to the standard FRW metric with flat 3D space sections . The solution (5) was first found by Ponce de Leon [28] in earlier work. That it is indeed a solution of (l) may be verified by computer. (This shows that (5) is actually flat in 5D, though it has non-trivial implications in 4D and a conventional curvature scalar ${}^{(4)}R = 6(\alpha-2)/\alpha^2 t^2\psi^2$ which is in general non-zero.) Alternatively, it may be verified that (5) is a solution of (1) by substituting it into the set (2),(3),(4) .

As regards these equations, (2) gives the components of the effective energy-momentum tensor in 4D as

$$T^0_0 = \frac{3}{\alpha^2 t^2 \psi^2}, \qquad T^1_1 = T^2_2 = T^3_3 = \frac{(3-2\alpha)}{\alpha^2 t^2\psi^2}. \quad (6)$$

For a perfect fluid, these give the effective density and pressure ($\rho = T^0_0$, $p = -T^1_1$ etc.). In terms of the time coordinate $T = \psi t$ of a locally inertial observer with metric (5), both decrease as $1/T^2$. The equation of state between them is

$$p = \frac{(2\alpha-3)}{3}\rho. \quad (7)$$

The class of solutions (5) thus includes two cases relevant to the early universe and the late universe. The former has $p = \rho/3$, $\alpha = 2$ and an expansion factor proportional to $t^{1/2}$. The latter has $p = 0$, $\alpha = 3/2$ and an expansion factor proportional to $t^{2/3}$. The metric (5) has other acceptable features [20], so we feel justified in regarding it as the standard Kaluza–Klein cosmology.

Let us now move from the gravity equations (2) to the wave equation (3). The former set of 10 equations successfully links geometry with the *macroscopic* properties of matter (such as the density and pressure), and it seems to us reasonable to expect that the other 5 equations (3),(4) will in an analogous way elucidate the *microscopic* properties of matter (such as the rest mass of a particle in the fluid). We therefore propose that (3) may in fact be the quantum wave equation for a particle in the fluid. A problem with this is that if we wish to match (3) to known physics, the latter has several different wave equations (see Refs. 29–32[6]). However, this problem with uniqueness may merely reflect the fact that we are using the simplest Kaluza–Klein theory (one extra dimension with no electromagnetic potentials). If so, the philosophy of our approach may be valid, and the logical match is between (3) and the simplest relativistic quantum wave equation, namely that of Klein–Gordon: $\Box\phi = m^2\phi$. (Here $\Box$ is defined as in (3), and we have used the comma-goes-to-semicolon rule to go from flat to curved space: see for example Ref. 31.) If we match the equations, the mass of a particle in general is given by

$$m^2 = \frac{1}{\phi^2}\left\{\frac{\overset{*}{g}^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{4} + \frac{g^{\lambda\beta}\overset{**}{g}_{\lambda\beta}}{2} - \frac{\overset{*}{\phi}g^{\lambda\beta}\overset{*}{g}_{\lambda\beta}}{2\phi}\right\}, \tag{8}$$

[6] Wave equations for particles with spin and charge are given in Ref. 32, while the Klein–Gordon equation and other scalar wave equations are discussed there and in Refs. 29–31. Of the latter, we note that an equation of the form $\Box\phi = m^2\phi + 4\lambda\phi^3$ [29], with a self-coupling term involving a scale λ, would in our approach necessitate the use of a 5D solution describing the internal structure of the particle, and this is beyond the scope of the present paper. Similarly, an equation of the form $\Box\phi = C(x^\alpha)\phi$ [30], with a scalar field $C(x^\alpha)$, would go beyond our approach, though may be compatible with it. However, an equation like $\Box\phi = {}^{(4)}R\phi/6$ [29,30], which incorporates conformal invariance via the (4D) scalar curvature, is perfectly compatible with our approach. This can be seen by noting from the main text that ${}^{(4)}R = 6(\alpha-2)/(\alpha t\psi)^2$, which is zero for $\alpha = 2$, and is therefore compatible with the equation of state $p = \rho/3$ of (7), which means that the fluid consists of photons with zero rest mass. We note also that while our match of (3) to the simple Klein–Gordon equation $\Box\phi = m^2\phi$ implies that $|g_{44}|$ is the square of the quantum mechanical wave function, other identifications (e.g. $|g_{44}| = \phi$) lead to relations quite similar to (9), insofar as the constant is different but the mass still depends on $1/\psi t$.

and for the cosmological solutions (5) this gives

$$m = \left(\frac{3}{\alpha}\right)^{1/2} \frac{1}{\psi t} . \tag{9}$$

We interpret this to mean that the absolute value of the mass of a particle is set by ψ, but that all particles decrease their mass at the same rate in inverse proportion to t. This kind of behaviour is reminiscent of certain 4D theories in which the strength of gravity changes with time because G and/or m depend on the epoch [33–38]. However, in addition to this behaviour we see from (9) that m depends on α and thereby on the global fluid. (From (9) and (6) we have $m = (\alpha\rho)^{1/2}$ actually.) Generally, it is clear from (8) that the rest mass of a local particle is determined by the global solution for the universe. Thus in this approach to Kaluza–Klein gravity, cosmology is Machian.

The final 4 equations of the theory are the conservation laws for the 4-tensor $P_{\alpha\beta}$ of (4). Having accounted for the macroscopic fluid via (2) and the rest mass of a particle in it via (3), a natural assumption is that the conserved quantities associated with the particle should figure in $P_{\alpha\beta}$. Algebraically, it transpires that an acceptable identification of the $P_{\alpha\beta}$ of (4) requires two scalar functions, as did the $T_{\alpha\beta}$ of (2). In the previous case they were the density ρ and pressure p of the fluid. In the present case, since we are discussing the origin of mass and there are two logically distinct kinds of this [20,32], we tentatively take the functions to be the inertial mass $m_{\rm i}$ and the gravitational mass $m_{\rm g}$ of the particle. And if k is a constant and $v^\alpha \equiv dt^\alpha/ds_4$ are 4-velocities, we suggest looking at

$$P_{\alpha\beta} = k(m_{\rm i} v_\alpha v_\beta + m_{\rm g} g_{\alpha\beta}). \tag{10}$$

This has the algebraic properties we need, and implies that the 4 field equations $P^\beta_{\alpha;\beta} = 0$ will be equations of motion (see below). We can see that (10) makes general physical sense, since in most situations $m_{\rm i} \simeq m_{\rm g}$, and (10) includes kinetic energy (from the diagonal elements of the first term), gravitational binding energy (from the second term), and linear momenta (from the zeroth row or column). However, we need to ensure that (10) is acceptable in the specific case of the cosmological solutions (5), which means that the components of (10) should match those of (4), which are

$$P^0_0 = -\frac{3}{\alpha\psi t} , \qquad P^1_1 = P^2_2 = P^3_3 = -\left(\frac{3-\alpha}{\alpha}\right)\frac{1}{\psi t} . \tag{11}$$

These components are reproduced if (10) takes the form

$$P^{\nu}_{\mu} = -\left(\frac{\alpha}{3}\right)^{1/2} (m_{\mathrm{i}} v^{\nu} v_{\mu} + m_{\mathrm{g}} \delta^{\nu}_{\mu}), \tag{12}$$

with masses given by

$$m_{\mathrm{i}} = \left(\frac{3}{\alpha}\right)^{1/2} \frac{1}{\psi t}, \qquad m_{\mathrm{g}} = \left(\frac{3-\alpha}{\alpha}\right) \left(\frac{3}{\alpha}\right)^{1/2} \frac{1}{\psi t}. \tag{13}$$

It should be noted that m_{i} here is the same as m in the wave-derived relation (9). This agrees with our expectation that the wave function of quantum theory should involve inertial mass. Also, $m_{\mathrm{i}} \simeq m_{\mathrm{g}}$ when $\alpha = 3/2$, which by (7) is the condition for the pressure of the cosmological fluid to be zero. This agrees with our expectation that the mass of a particle should be unique when it is embedded in a background whose inertial mass density is the same as its gravitational mass density [20,32]. In fact most situations involve $p \ll \rho$, so $m_{\mathrm{i}} \simeq m_{\mathrm{g}} = m$ and (10) gives $P_{\alpha\beta} \simeq km(v_\alpha v_\beta + g_{\alpha\beta})$. However, the field equations (4) still read $P^{\beta}_{\alpha;\beta} = 0$, and since $m = m(\psi, t)$ an important question is whether these laws of motion are in fact acceptable.

We proceed to investigate this, but note that it is closely connected with another issue. Real particles have motions close to those given by the 4D geodesic equation, $dv^\gamma/ds_4 + \Gamma^{\gamma}_{\alpha\beta} v^\alpha v^\beta = 0$, which follows from the assumption that the 4D interval is a minimum. Whereas we have equations of motion derived from the field equations (4). This is an economy of assumptions, but raises the question of whether our relations

$$(m v^\alpha v^\beta + m g^{\alpha\beta})_{;\beta} = 0 \tag{14}$$

are compatible with those of geodesic motion. We have considered the latter (see Appendix), and proceed to show that the two things are indeed compatible, at least in the cosmological case. Thus we expand (14) to give

$$v^{\alpha}{}_{;\beta} v^\beta + v^\alpha v^{\beta}{}_{;\beta} = -\frac{m_{,\beta}}{m} (v^\alpha v^\beta + g^{\alpha\beta}). \tag{15}$$

We can contract this with v_α where as usual we assume that the 4-velocities have inverses defined by $1 = g_{\alpha\beta} v^\alpha v^\beta = v_\beta v^\beta$, whose covariant derivative yields $v_\alpha v^{\alpha}{}_{;\beta} = 0$. Using this in the contracted form of (15) shows that $v^{\beta}{}_{;\beta} = -2m_{,\beta} v^\beta / m$ $(= -2\dot{m}/m$ where $\dot{m} \equiv dm/ds_4)$. Eliminating the divergence in (15) causes this to become

$$v^{\alpha}{}_{;\beta} v^\beta = \frac{m_{,\beta}}{m} (v^\alpha v^\beta - g^{\alpha\beta}). \tag{16}$$

The left-hand side of this is in fact equal to the usual 4D geodesic equation and is zero if the motion is geodesic, while the right-hand side is the projection of the mass change into the surface orthogonal to v^α. For the class of cosmological solutions (5), and indeed for any spatially homogeneous cosmology, the mass m depends only on the time. Also, in (5) the coordinates are comoving (see Appendix). Then (16) yields

$$v^\alpha{}_{;\beta} v^\beta = 0 \qquad (\alpha = 0, 1, 2, 3), \tag{17}$$

confirming that the motion is indeed geodesic.

3. CONCLUSION

We have taken the 5D Kaluza–Klein field equations for apparently empty space (1) and cast them into the form of 10 equations for general relativity with matter (2), a scalar wave equation (3), and 4 conservation relations (4). The last 5 equations have no analogs in 4D general relativity. To help interpret them, we have looked at a class of cosmological solutions (5) that has properties of matter (6) and an equation of state (7) that give good models for the early and late universe. The density and pressure of the macroscopic fluid are the result of curvature associated with the fifth dimension, and adopting the same viewpoint for the microscopic level suggests to us that the rest mass of a particle in the fluid is governed by the Kaluza–Klein wave equation according to (8) in general and (9) for the cosmological solutions. This mass changes slowly as the universe evolves, and depends on the global solution in a Machian manner. Our last 4 equations suggest to us that a particle in the fluid has conserved quantities involving mass and velocity that can in general be put into the form of the 4-tensor (10). Its components for the cosmological solutions are given by (11), which match the general expression if a constant is identified as in (12) and the inertial and gravitational masses of the particle are as given in (13). These masses are the same in the usual case where the pressure is negligible, which results in the equations of motion (14). For the cosmological solutions at least, these are seen by equations (15)–(17) to imply geodesic motion.

Our results are preliminary, in the sense that we have used the simplest Kaluza–Klein theory (one extra dimension with no electromagnetic potentials) and the simplest instructive solutions (cosmological ones with dependence on the fifth coordinate). We do not know if our interpretations and results are robust to changing the dimensionality of the theory or the complexity of the solutions, and urge further work on these things. However, it seems to us that the general idea of interpreting matter and mass as a consequence of the curvature of an extra dimension is a promising way of incorporating Mach's principle into physics.

ACKNOWLEDGEMENTS

Thanks for comments go to P. Lim and H. Liu. This work was supported in part by NSERC and in part by NASA.

APPENDIX

Geodesic motion in the 5D metric of (5) can be investigated by defining 5-velocities $u^a dx^a/ds_5$ and solving the 5D geodesic equation

$$\frac{du^c}{ds_5} + \Gamma^c_{ab} u^a u^b = 0. \tag{A.1}$$

We find that we can (as in 4D) choose the space components to be comoving with the fluid via $u^1 = u^2 = u^3 = 0$. However, the zeroth and fourth components of $(A.1)$ then read

$$\frac{du^0}{ds_5} + \frac{2}{\psi} u^0 u^4 + \frac{\alpha^2}{(1-\alpha)^2} \frac{t}{\psi^2} u^4 u^4 = 0 \tag{A.2}$$

$$\frac{du^4}{ds_5} + \frac{2}{t} u^0 u^4 + \frac{(1-\alpha)^2}{\alpha^2} \frac{\psi}{t^2} u^0 u^0 = 0. \tag{A.3}$$

Solutions of these must be compatible with the condition

$$1 = \psi^2 u^0 u^0 - \frac{\alpha^2 t^2}{(1-\alpha)^2} u^4 u^4 \tag{A.4}$$

set by the metric, and we find

$$u_0 = \frac{\alpha}{(2\alpha - 1)^{1/2}} \psi, \qquad u_4 = \frac{\alpha}{(2\alpha-1)^{1/2}} t. \tag{A.4}$$

Thus $u_4 \neq 0$ in general, and a particle cannot be comoving in the extra dimension. This could actually have been inferred from the symmetry between the first and last parts of the metric, but it is helpful to have the explicit result for u_4, since it shows that the "motion" in the extra dimension is a free translation with a speed fixed by the constant α. Also, the big bang in 4D is when u_4 is momentarily zero.

It is natural to ask how 5-velocities relate to the conventional 4-velocities $v^\alpha \equiv dx^\alpha/ds_4$. This is easily worked out by noting that the velocities are related by $u^\alpha = v^\alpha(ds_4/ds_5)$, and that the intervals are related in general by $ds_5{}^2 = ds_4{}^2 + g_{44}d\psi^2$. For the metric (5), $ds_4/ds_5 = \alpha(2\alpha-1)^{-1/2}$ and

$$v^\alpha = \frac{(2\alpha-1)^{1/2}}{\alpha}u^\alpha. \qquad (A.6)$$

If a particle is comoving in 3D, so u_0 is given by $(A.5)$, then $(A.6)$ gives $v^0 = 1/\psi$. This may seem strange, as in 4D the zeroth component of the 4-velocity is related to the energy of a particle, and in the comoving case to its rest mass. However, $v_0 = g_{00}v^0 = \psi$, so we recover $v^0 v_0 = 1$ as usual.

REFERENCES

1. Kaluza, T. (1921). *Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys. Math.*, **K1** 33, 966.
2. Klein, O. (1926). *Z. Phys.* **37**, 895.
3. Collins, P. D. B., Martin, A. D., and Squires, E. H. (1989). *Particle Physics and Cosmology* (Wiley, New York).
4. Appelquist, T., Chodos, A., and Freund, P. G. O. (1987). *Modern Kaluza–Klein Theories* (Addison-Wesley, Menlo Park).
5. De Sabatta, V., and Schmutzer, E. (1983). *Unified Field Theories of More than 4 Dimensions Including Exact Solutions* (World Scientific, Singapore).
6. Einstein, A. (1956). *The Meaning of Relativity* (Princeton University Press, Princeton).
7. Salam, A. (1980). *Rev. Mod. Phys.* **52**, 525.
8. Gross, D. J., and Perry, M. J. (1983). *Nucl. Phys.* **B226**, 29.
9. Davidson, A., and Owen, D. A. (1985). *Phys. Lett.* **B155**, 247.
10. Sokolowski, L., and Carr, B. (1986). *Phys. Lett.* **B176**, 334.
11. Wesson, P. S. (1992). *Phys. Lett.* **B276**, 299.
12. Liu, H., and Wesson, P. S. (1992). *J. Math. Phys.* **33**, 3888.
13. Wesson, P. S., Liu, H., and Lim, P. (1993). *Phys. Lett.* **B298**, 69.
14. Liu, H., Wesson, P. S., and Ponce de Leon, J. (1993). *J. Math. Phys.* **34**, 4070.
15. Ponce de Leon, J., and Wesson, P. S. (1993). *J. Math. Phys.* **34**, 4080.
16. Wesson, P. S., and Lim, P. (1992). *Astron. Astrophys.* **261**, 373.
17. Lim, P., and Wesson, P. S. (1992). *Astrophys. J. Lett.* **397**, 91.
18. Wesson, P. S. (1994). *Astrophys. J. Lett.* **420**, 49.
19. Wesson, P. S. (1992). *Mod. Phys. Lett.* **A7**, 921.
20. Wesson, P. S. (1992). *Astrophys. J.* **394**, 19.
21. Wesson, P. S., Ponce de Leon, J., Lim, P., and Liu, H. (1993). *Int. J. Mod. Phys.* **D2**, 163.
22. Wesson, P. S., and Ponce de Leon, J. (1992). *J. Math. Phys.* **33**, 3883.
23. Wetterich, C. (1984). *Nucl. Phys.* **B242**, 473.
24. Wetterich, C. (1985). *Nucl. Phys.* **B255**, 480.
25. Gell-Mann, M., and Zwiebach, B. (1984). *Phys. Lett.* **B147**, 111.

26. Gell-Mann, M., and Zwiebach, B. (1985). *Nucl. Phys.* **B260**, 569.
27. Randjbar-Daemi, S., and Wetterich, C. (1986). *Phys. Lett.* **B166**, 65.
28. Ponce de Leon, J. (1988). *Gen. Rel. Grav.* **20**, 539.
29. Callan, C. G., Coleman, S., and Jackiw, R. (1970). *Ann. Phys. (NY)* **59**, 42.
30. Price, R. H. (1972). *Phys. Rev. D***5**, 2419.
31. Bekenstein, J. D. (1972). *Phys. Rev. D***5**, 1239.
32. Wesson, P. S. (1992). *Space Sci. Rev.* **59**, 365.
33. Wesson, P. S. (1978). *Cosmology and Geophysics* (Oxford University Press, New York).
34. Dirac, P. A. M. (1974). *Proc. Roy. Soc. Lond.* **A338**, 439.
35. Hoyle, F., and Narlikar, J. V. (1974). *Action at a Distance in Physics and Cosmology* (Freeman, San Francisco).
36. Canuto, V., Adams, P. J., Hsieh, S.-H., and Tsiang, E. (1977). *Phys. Rev. D***16**, 1643.
37. Will, C. M. (1991). In *Gravitation. A Banff Summer Institute*, R. Mann and P. S. Wesson, eds. (World Scientific, Singapore).
38. Will, C. M. (1992). *Int. J. Mod. Phys.* **D1**, 13.
39. Bergmann, P. G. (1948). *Ann. Math. (Princeton)* **49**, 255.
40. Jordan, P. (1955). *Schwerkraft und Weltall* (Vieweg, Braunschweig).
41. Jordan, P. (1957). *Z. Phys.* **149**, 329.
42. Jordan, P. (1959). *Z. Phys.* **154**, 312.
43. Souriau, J.-M. (1958). *Comptes Rend. Acad. Sci. (Paris) II* **247**, 1559.
44. Souriau, J.-M. (1963). *Nuovo Cimento* **30**, 565.
45. Schmutzer, E. (1968). *Relativistische Physik* (Teubner, Leipzig).
46. Schmutzer, E. (1990). *Astron. Nachr.* **311**, 329.
47. Lessner, G. (1982). *Phys. Rev. D***25**, 3202.
48. Lessner, G. (1991). *Gen. Rel. Grav.* **23**, 897.