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The effective properties of matter of Kaluza–Klein solitons

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The effective 4D properties of matter of 5D solutions of the empty Einstein equations have been calculated by Davidson and Owen. We adopt an alternative approach, which leads to similar asymptotic forms for the density and pressure, but allows us to obtain expressions for these parameters in closed algebraic form. We discuss some possible differences in interpretation of the properties of 5D solitons.

1. Introduction

An important class of solutions in Kaluza–Klein theory is provided by metrics that are static, spherically symmetric and satisfy the empty 5D Einstein equations. These solutions are 5D analogs of the 4D Schwarzschild one. They were derived by Dobiasch and Maison [1], Chodos and Detweiler [2], Gross and Perry [3] and Davidson and Owen [4]. The last two references discussed the effective 4D properties of these 5D solitons, where we use the latter phrase as in ref. [3] rather than black holes as in ref. [4] since some solutions at least of this type do not have event horizons of the sort familiar from the 4D Schwarzschild solution [5–7]. The effective energy-momentum tensor of these solutions was worked out using an approach based on Kac–Moody symmetries [8] by Davidson and Owen [4]. The latter authors found in particular series expressions for the effective 4D density and pressure valid in the asymptotic limit of large (3D) distances. In what follows we will use an alternative approach based on a physical interpretation of the 5D Einstein tensor to obtain the 4D density and pressure in closed algebraic form [9–11]. While our results are elegant mathematically, it is not clear how to interpret any higher-dimensional theory physically [12], and for this reason our discussion will be brief.

2. The effective 4D density and pressure of 5D solitons

For ease of comparison of our results with those of Davidson and Owen [4], we will mainly follow the terminology of the latter reference. Then the metric is

$$ds^2 = -A^2(r)\,dt^2 + B^2(r)\,dx^i\,dx^i + C^2(r)\,d\psi^2\,, \tag{1}$$

where the coordinates are t, x, y, z, ψ and $r^2 = x^2+y^2+z^2$. The use of cartesian spatial coordinates results in somewhat unfamiliar components of the Ricci tensor, but we have confirmed the following:

$$R_{tt} = \frac{-AA'}{B^2}\left(\frac{2}{r} + \frac{A''}{A'} + \frac{B'}{B} + \frac{C'}{C}\right), \tag{2a}$$

$$\begin{aligned} R_{ij} = {} & \left[\frac{1}{r}\left(\frac{A'}{A} + \frac{3B'}{B} + \frac{C'}{C}\right) + \frac{B''}{B} + \frac{B'}{B}\left(\frac{A'}{A} + \frac{C'}{C}\right)\right]\delta_{ij} \\ & + \left[\left(\frac{A''}{A} + \frac{B''}{B} + \frac{C''}{C}\right) \right. \\ & \left. - \left(\frac{1}{r} + \frac{2B'}{B}\right)\left(\frac{A'}{A} + \frac{B'}{B} + \frac{C'}{C}\right)\right]\frac{x^i x^j}{r^2}\,, \end{aligned} \tag{2b}$$

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$$R_{\psi\psi}=\frac{CC'}{B^2}\left(\frac{2}{r}+\frac{A'}{A}+\frac{B'}{B}+\frac{C''}{C'}\right). \tag{2c}$$

Here a prime means the total derivative with respect to r and i, j run from 1 to 3. The one-body problem in 5D involves solutions of the set of equations where the components (2) are set equal to zero. The solutions are

$$A(r)=\left(\frac{ar-1}{ar+1}\right)^{\epsilon k}, \tag{3a}$$

$$B(r)=\frac{1}{a^2r^2}\frac{(ar+1)^{\epsilon(k-1)+1}}{(ar-1)^{\epsilon(k-1)-1}}, \tag{3b}$$

$$C(r)=C_0\left(\frac{ar+1}{ar-1}\right)^{\epsilon}. \tag{3c}$$

Here a is a constant related to the central body, and ϵ, k are arbitrary except for obeying $\epsilon^2(k^2-k+1)=1$. The constant C_0 in (3c) was not set to unity by Davidson and Owen [4] like the equivalent constants in (3a) and (3b). This because they formed an effective 4D metric tensor by writing $g^{\text{eff}}_{\mu\nu}=[C(r)/C_0]g_{\mu\nu}$ as in ref. [8], prior to working out an effective energy–momentum tensor $T^{\text{eff}}_{\mu\nu}=R^{\text{eff}}_{\mu\nu}-\frac{1}{2}g^{\text{eff}}_{\mu\nu}R^{\text{eff}}$, and the size of C_0 is significant in this approach. However, we will not need to be concerned about the size of C_0 below, because we will use a fully covariant 5D approach in which the fifth dimension is treated on the same footing as the other four and which gives results for $T^{\text{eff}}_{\mu\nu}$ that are independent of the size of C_0. To implement this alternative approach we will need to calculate components of the Einstein tensor in 5D, namely $G_{AB}=R_{AB}-\frac{1}{2}g_{AB}R$, and for this we will need the 5D Ricci scalar which can be calculated from (2) and is

$$R=\frac{4A'}{rAB^2}+\frac{2A''}{AB^2}+\frac{2A'B'}{AB^3}+\frac{2A'C'}{AB^2C}+\frac{8B'}{rB^3}$$
$$-\frac{2(B')^2}{B^4}+\frac{4B''}{B^3}+\frac{4C'}{rB^2C}+\frac{2B'C'}{B^3C}+\frac{2C''}{B^2C}. \tag{4}$$

Since we will be concerned with the relation between 5D solitons and the effective 4D density ρ and pressure p of a fluid, we note here that we will adopt units in which the speed of light is 1 and the (4D) gravitational constant is $1/8\pi$.

Then the 4D field equations with matter are $G^{\mu}_{\nu}+T^{\mu}_{\nu}=0$. If we assume as in ref. [4] that T^{μ}_{ν} here is a manifestation of the extra dimension, we can compare the 4D equations with the 5D ones in vacuum $G^{A}_{B}=0$. Most terms are the same, of course. But there are extra parts which depend on the new dimension (i.e., on C), and which we collect here:

$$G^{t}_{t}(C)=-\frac{1}{B^2}\left(\frac{2C'}{rC}+\frac{B'C'}{BC}+\frac{C''}{C}\right), \tag{5a}$$

$$G^{i}_{j}(C)=-\frac{1}{B^2}\left(\frac{C'}{rC}+\frac{A'C'}{AC}+\frac{C''}{C}\right)\delta_{ij}$$
$$+\frac{1}{B^2}\left[\frac{C''}{C}-\left(\frac{1}{r}+\frac{2B'}{B}\right)\frac{C'}{C}\right]\frac{x^ix^j}{r^2}, \tag{5b}$$

$$G^{\psi}_{\psi}(C)=0. \tag{5c}$$

Following previous suggestions, we now simply identify the extra parts of the 5D Einstein tensor with the density and pressure terms of the 4D energy–momentum tensor ([9–11], see also refs. [13,14]). Mathematically, this is well defined, and simply amounts to a classification of terms into those which depend on C and those which do not. Physically, this way of obtaining T^{μ}_{ν} was suggested in connection with an earlier version of Kaluza–Klein theory [9,15]. It avoids the conceptual problems otherwise encountered with this kind of theory [2]. And it is known to work for certain cosmological models ([10,11], see also refs. [16,17]). Specifically, it gives back the usual 4D properties of matter for 5D Friedmann–Robertson–Walker models that reduce to the usual 4D ones on hypersurfaces ψ=constant [11,16]. In the present context, we cannot in general expect to recover known properties of matter because the 5D solutions (3) do not have this property (the 4D Schwarzschild solution is only recovered as an embedding in 5D for $\epsilon\to 0$, $k\to\infty$, $\epsilon k\to 1$). That is, the solutions (3) are essentially 5D ones and we cannot expect to recover known 4D solutions or properties of matter. However, we will find that the method of splitting the 5D Einstein tensor does give expressions for the density and pressure that are functionally similar to those found using the method of Davidson and Owen.

For the density, identifying T^{t}_{t} with $G^{t}_{t}(C)$ of (5a) and using (3) gives after some algebra

$$\rho = \frac{4\epsilon^2 k a^6 r^4}{(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\epsilon(k-1)} . \tag{6}$$

This implies $\rho \to 4\epsilon^2 k/a^2 r^4$ for $r \to \infty$, which involves the same r-dependency as that of the series expression found by Davidson and Owen [4]. Their prescription for recovering the Schwarzschild solution ($\epsilon \to 0$, $k \to \infty$, $\epsilon k \to 1$) implies $\rho \to 0$, as expected. They noted that the solution (3) is invariant under $\epsilon \to -\epsilon$, $a \to -a$, $k \to k$. For (6), this invariance is manifest and means that ρ is unaffected by the signs of ϵ and a. However, it is affected by the sign of k, and we need $k > 0$ for $\rho > 0$. For the pressure, a similar calculation using (5b) and (3) yields the three-tensor

$$p^i_j = \frac{1}{B^2}\left(\frac{2\epsilon a}{r(ar-1)(ar+1)} + \frac{4\epsilon a^2(\epsilon k - \epsilon - ar)}{(ar-1)^2(ar+1)^2}\right)\delta_{ij} - \frac{1}{B^2}\left(\frac{6\epsilon a}{r(ar-1)(ar+1)} + \frac{4\epsilon a^2(2\epsilon k - 3\epsilon - 3ar)}{(ar-1)^2(ar+1)^2}\right)\frac{x^i x^j}{r^2} .$$

This has parts similar to those of the series expression found by Davidson and Owen (ref. [4] eq. (13)). These authors did not comment on it explicitly, but it is clear that their pressure tensor and ours both imply off-diagonal components of the 4D energy–momentum tensor, which therefore in general must be a sum of a material (perfect) fluid and a free electromagnetic field. The terms "density" and "pressure" have therefore to be treated with some caution. The same applies to any relation between them, as Davidson and Owen realized when they put in quotes the phrase "equation of state" (ref. [4] p. 249). However, we need an equation of state for a physical interpretation (see below), and for this we need a scalar p to compare to the ρ of (6). To obtain a scalar pressure, we follow previous usage [4] and take $\frac{1}{3}$ of the trace of the pressure three-tensor to obtain

$$p = \tfrac{1}{3} p^i_i = \frac{4\epsilon^2 k a^6 r^4}{3(ar-1)^4(ar+1)^4}\left(\frac{ar-1}{ar+1}\right)^{2\epsilon(k-1)} . \tag{7}$$

This with $k > 0$ means $p > 0$.

The equation of state obeyed by (6) and (7) is $p = \frac{1}{3}\rho$, which is of course the familiar one for relativistic particles including photons. This is a physically reasonable result, and should be contrasted with the $p = -\frac{1}{3}\rho$ found by Davidson and Owen [4]. Thus it appears that while the two approaches involve functions of the same form mathematically, they differ in their end results physically. Actually, it is not known if either approach is valid physically [12]. But we feel that the approach adopted here is at least as plausible as the one used by Davidson and Owen [4]. It should also be noted that the approach adopted here is consistent, in the following sense. Above, we used the extra part of G^t_t in (5a) to obtain ρ and the extra part of G^i_j in (5b) to obtain p, but passed over the wholly new component G^ψ_ψ because it was zero. This is actually remarkable, because if it had not been zero we would have been faced with the problem of identifying it with an unknown property of matter in the approach adopted here. There is, of course, an extra field equation due to the presence of the fifth dimension, namely $R_{\psi\psi} = 0$ of (2c). This is not trivially satisfied, however, so we should ask about its meaning. It turns out that $R_{\psi\psi} = 0$ is functionally equivalent to $G^t_t + G^i_i = 0$, or $\rho - 3p = 0$ in the present approach. This means that the extra field equation in the 5D case is equivalent to what we normally call the equation of state in the 4D case.

3. Conclusion

We should remind ourselves that the soliton or black-hole solutions of refs. [1–4] are canonical for 5D Kaluza–Klein theory in the same way that the Schwarzschild solution is for 4D general relativity. It is therefore important to explore ways of interpreting the properties of the 5D solutions in a 4D world. One way that was based on symmetry considerations [8] was investigated by Davidson and Owen [4], who found expressions for the effective density and pressure. Another way that is based on a splitting of the Einstein tensor [9–11] has been investigated here, and found to lead to results which are similar mathematically but appear to differ physically. The present discussion has been brief, and has not gone into questions to do with compactification and the nature of the vacuum. Further study of these and related subjects should make it clearer what is the best way to get 4D properties of matter from 5D geometry.

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