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A conceptual introduction to the Kaluza–Klein theory

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Abstract A conceptual introduction to the Kaluza–Klein approach to a geometrical unification of gravity and other interactions is presented. After some considerations on the structure of theories of fundamental forces, and having stressed the peculiarity of Einstein's general relativity, close attention is paid to the conceptual features of the five-dimensional Kaluza–Klein theory, which unifies gravitation and electromagnetism in a common geometrical scheme. Finally, with a view towards the possible geometrical unification of all fundamental interactions, some remarks are made about Multidimensional Unified Theories, which are generalised theories of a Kaluza–Klein type.

Riassunto Si presenta una introduzione concettuale all'approccio di Kaluza–Klein per una unificazione geometrica di gravità ed altre interazioni. Dopo alcune considerazioni sulla struttura delle teorie delle forze fondamentali, una volta sottolineata la peculiarità della relatività generale di Einstein, viene prestata una particolare attenzione alle caratteristiche concettuali della teoria 5-dimensionale di Kaluza–Klein, che unifica gravitazione ed elettromagnetismo in uno schema geometrico comune. Infine, in vista di una possibile unificazione geometrica di tutte le interazioni fondamentali, si fanno alcune osservazioni sulle Teorie Unificate Multidimensionali, che sono teorie generalizzate alla Kaluza–Klein.

1. Introduction

The aim of this article is to present a conceptual introduction to the topic of geometrical unification of the fundamental interactions in contemporary theoretical physics. In particular there are at present attempts to work out theories, the so-called Multidimensional Unified Theories (MUTS), in which gravity and other forces can be linked together in a common geometrical scheme. Here we deal mainly with the Kaluza–Klein theory, which is the prototype of modern MUTS.

In the next section we will consider the structure of current theories of the fundamental interactions, stressing their common properties and differences, then moving on to summarise the most significant features of general relativity. Finally, in §4, the structure of the Kaluza–Klein theory is introduced and some of its interesting and deep conceptual consequences are presented: this section constitutes the main part of this article. In conclusion some remarks are made about the extension of the Kaluza–Klein theory performed by MUTS.

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In order to prevent this article from becoming too large, we have assumed the reader is familiar with the conceptual foundations of gauge theories and general relativity; however, some features of these theories are summarised and general references are recommended (some of which have recently appeared in this journal).

2. Fundamental interactions and related theories

After the recent unification of electromagnetic and weak nuclear forces in a unique gauge interaction (the electroweak interaction) by Weinberg and Salam, one can say that today we recognise three distinct fundamental interactions in the world of physical experience (experiments), namely the electroweak force, the strong nuclear force and gravity. Here it is interesting to compare the different theoretical treatments of these distinct interactions (in order to understand how to work out a unified theory).

The theories commonly accepted for the fundamental forces are the Weinberg–Salam theory (WST)

Table 1 The theoretical situation concerning the fundamental interactions.

Interaction	Gauge group	Theory
EM, weak } electroweak	$SU(2) \times U(1)$	Weinberg–Salam } → GUTS
strong	$SU_c(3)$	QCD } → GUTS
gravity	— (Poincaré)	general relativity

for the electroweak interaction, quantum chromodynamics (QCD) for the strong interaction and general relativity for gravity. It is well known that WST and QCD are local gauge theories with gauge groups $SU(2) \times U(1)$ and $SU_c(3)$ of the colour, respectively. General relativity, for its part, was not born as a gauge theory; nevertheless some years ago (see for example Hehl *et al* 1976) it was proved that gravitation may be described in terms of a local gauge theory for the Poincaré group: in particular this gauge approach to gravitation leads to a geometry of spacetime which is more general than that of Einstein's theory†. We can thus affirm that all the fundamental interactions may be described in terms of local gauge theories: the situation is summarised in table 1. Note that there have been attempts to unify electroweak and strong forces in a unique gauge interaction by the so-called Grand Unified Theories (GUTS).

In this brief article we cannot deal with gauge theories in an extensive manner, because the main topic we want to discuss is the Kaluza–Klein theory; moreover a good paper on gauge theories recently appeared in this very journal (Kenyon 1986). For a more technical treatment, refer to (Abers and Lee 1973) or other review articles that have appeared during the last few years. Here we want to stress that WST and QCD are both Yang–Mills-type theories (Yang and Mills 1954), in which particular assumptions (spontaneous symmetry breaking, the Higgs mechanism etc) are made. It should, finally, be remembered that the Yang–Mills theory for the group U(1) represents the theory of electromagnetism; in this framework a local gauge transformation is of the kind

$$\psi \rightarrow \psi' = \psi \exp(\mathrm{i}\lambda(x, t)e) \qquad (1)$$

where e is the charge of the field ψ and λ is a phase change depending on position and time. If we want to work out a local gauge-invariant theory, we must require the invariance of the Lagrangian density $\mathcal{L}(\psi, \partial\psi)$ of the system (that is, in practice, the constancy of the momenta), introducing a gauge-covariant derivative and requiring the transformation

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial\lambda/\partial x^\mu \qquad (2)$$

for the electromagnetic potential.

Even if we have seen that all the interactions are describable by local gauge theories, there are, however, two fundamental differences between general relativity (or, in general, any existing metric theory of gravity) and the other theories, which one should keep in mind when dealing with the problem of the theoretical unification of forces. First of all, WST and QCD are quantum theories, while general relativity is a classical theory and all attempts to work out a consistent quantum theory of gravity have failed. Secondly, while in WST and QCD (as well as in special relativity) spacetime geometry provides a rigid and unchanging background in which bodies and fields move, with general relativity this geometry becomes a dynamical variable of the theory, intrinsically linked to the presence of matter, as one can see in Einstein's field equations:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}. \qquad (3)$$

Simplifying matters as far as possible, in equation (3) we find 'geometry' on the left-hand side (where the metric tensor $g_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R appear) and 'matter' (the energy–momentum tensor $T_{\mu\nu}$) on the right-hand side; G is Newton's constant and in our units $c = 1$. In this way geometry and matter are dynamically correlated: matter (as mass–energy) influences (curves) spacetime and, at the same time, it is influenced by spacetime geometry, as far, for example, as its free falling motion is concerned.

After these simple considerations it becomes clear that there is at present a dichotomy in the theoretical treatment of the fundamental interactions. Leaving aside the obvious difference between the classical and quantum characters of the theories shown in table 1 (one may arrive at a preliminary classical unified theory, only imposing the quantum character at a later stage), we must stress that the really strange result is the success of WST and QCD in a flat spacetime uncorrelated with the presence of

† The resulting theory is known as Einstein–Cartan theory (ECT) and distinguishes itself from general relativity because in ECT one does not require the symmetry of connection coefficients in the lower indices which one imposes in Einstein's theory (see the next section). If, in a coordinate basis, we call the antisymmetric part of the connection 'torsion', one can see that in ECT torsion is correlated with the spin density of matter.

matter, when general relativity requires a curved and dynamical geometry for the universal spacetime (remember that gravitation is the only universal force: it acts on everything). Of course this result is due to the weakness of gravitation in our laboratory situations, so that here spacetime is practically flat to a high degree of accuracy. In spite of this a unified theoretical framework has to be found if we want to establish a field theory in which gravitation and other interactions are linked together on an equal footing: of course it becomes experimentally very important where gravity is strong.

It has been shown that non-metric theories of gravitation, in which one inserts the gravitational field, like any other field, into a flat and constant spacetime, violate the Eötvös experiment (see Schiff 1960, Thorne *et al* 1973, Lightman and Lee 1973). So we can imagine a unified theory as a theory where gravitation and other interactions 'live' in a dynamic geometrical 'scenario' and where matter with all its properties (mass–energy, electric charge, quarks' colour etc) influences (and is influenced by) the geometry of spacetime, sticking to the model provided by Einstein's general relativity. In this sense, as we will see, MUTS simply represent the maximum generalisation of the Einsteinian theory, while the Kaluza–Klein theory gives us a first model suggesting how one can operate this generalisation and also shows the advantages of this treatment.

3. Essential features of general relativity

Before dealing with the Kaluza–Klein theory we need to recall certain features of general relativity. As with gauge theories, we cannot deal with Einstein's theory of gravity extensively in this article; we shall thus only be dealing with those results which are useful for understanding at best the 'conceptual evolution' from general relativity to the Kaluza–Klein theory. A more pedagogical treatment of the Einsteinian theory may be found in an article by Bondi reprinted in this journal (Bondi 1986); see also the bibliography recommended by Pišút in the preface to the same paper.

It is known that in general relativity spacetime is curved: more precisely we can say it is the Riemannian manifold V_4; the fundamental elements of its geometry are the symmetric metric tensor $g_{\mu\nu}(=g_{\nu\mu})$ and the symmetric affine connection $\Gamma^{\lambda}_{\mu\nu}(=\Gamma^{\lambda}_{\nu\mu})$. We want to point out that in a flat spacetime (for example in the Minkowskian one of special relativity) $g_{\mu\nu}$ is reduced to

$$\eta_{\mu\nu}=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & +1 & 0 & 0\\ 0 & 0 & +1 & 0\\ 0 & 0 & 0 & +1\end{pmatrix} \qquad (4)$$

while all the coefficients $\Gamma^{\lambda}_{\mu\nu}$ become zero. In a more general (curved) spacetime the metric tensor $g_{\mu\nu}$ has ten independent components (because of the symmetry condition), while $\Gamma^{\lambda}_{\mu\nu}$ determines the parallel transport in the manifold via the covariant derivative (see, for example, Kenyon 1986). Moreover the treatment of freely falling particles shows that the field determining the gravitational force is precisely the affine connection $\Gamma^{\lambda}_{\mu\nu}$ (Weinberg 1972). See however (Bondi 1986), where it is stressed that the observable of the gravitational field is the curvature tensor.

Now the metric tensor and affine connection are linked together by the following simple relation in a coordinate system x^{μ}:

$$\Gamma^{\lambda}_{\mu\nu}=\tfrac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\nu}}{\partial x^{\mu}}+\frac{\partial g_{\mu\rho}}{\partial x^{\nu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right). \qquad (5)$$

In this way the metric tensor of the theory is correlated with the gravitational field $\Gamma^{\lambda}_{\mu\nu}$: more exactly $g_{\mu\nu}$ represents its potential. When, as has happened in this case, the potential of a physical field is found to be all or part of the components of the metric tensor of the theory, then we say that the field itself is geometrised, because it is nothing but a derivation of the metric and so it is regarded as a purely geometrical entity. In this sense we can affirm that Einstein, with his general relativity, performed a geometrisation of the gravitational field (interaction).

Of course one can also consider other interactions in the framework of general relativity; for example it is possible to insert an electromagnetic field $F_{\mu\nu}$ in the vacuum, reaching the so-called Einstein–Maxwell theory, in which the field equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi G(F_{\mu\rho}F^{\rho}_{\nu}-\tfrac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) \qquad (6)$$

where on the right the source is just the energy–momentum tensor of the electromagnetic radiation. It is clear, moreover, that in this framework our electromagnetic field is not geometrised.

We want to stress two final points about the essential conceptual features of general relativity. But first of all a more technical piece of information: the field equations of the theory (equation (3)) may be derived through the application of a variational principle, requiring the invariance of the action under variation of the metric; in a generalised context this method represents a standard way of acting for the Kaluza–Klein theory and MUTS.

At the basis of Einstein's general relativity there is the principle of equivalence; we can formulate it as the following statement: at every spacetime point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that, within a sufficiently small neighbourhood of this point, the laws of nature take the same form as in special relativity, that is to say as in unaccelerated Minkowskian coordinate systems in the absence of gravitation. Of course in these locally inertial coordinate systems all the coefficients $\Gamma^{\lambda}_{\mu\nu}$ of the connection are equal to zero. It is important to note that the formalisation of the equivalence principle for the problem of freely falling particles in a gravitational field leads to the derivation of their equations of motion: one finds their trajectories are the lines of extremal length in Einstein's curved spacetime, the so called geodesics in V_4. Hence we can reformulate the equivalence principle as follows: in the presence of an arbitrary gravitational field, and in the absence of other interactions, the real trajectory of test particles coincides with the geodesics of the metric of the spacetime V_4. As an example we can affirm that the planets' orbits around the Sun are the projections on the three spatial dimensions of geodesics in the four-dimensional spacetime.

An important consequence for the behaviour of bodies in a gravitational field derives from the following fact: geodesics do not depend on the mass–energy of bodies in free fall. So in the same gravitational field, if one fixes the same initial conditions, everything falls in the same manner and with the same trajectory. As we have seen, the gravitational field is the connection Γ, substantially a derivation of the metric g: it is geometrised. We can say, then, that the motion of test particles is uniquely determined by the topology of the Einsteinian spacetime V_4: it is a really fine result.

4. The Kaluza–Klein theory

At the end of §2 we hinted at the possibility of working out a unified theory in which matter, and its every property, is dynamically correlated with the geometry of universal spacetime. But now, by allowing matter to have other properties besides mass–energy we are 'switching on' other interactions in addition to gravity: for example, as soon as we consider electrically charged matter we are automatically in the presence of an electromagnetic field. So the actual problem is that the geometry of V_4 (the four-dimensional spacetime of general relativity) is completely 'saturated' by the gravitational field and does not allow the insertion of other interactions in a unified geometrical way. The search for a solution to this problem led to extensions of the four-dimensional Riemannian geometry of general relativity being considered; in this section we will be dealing with that generalisation which revealed itself to be the most advantageous and rich in developments: in doing so we shall not follow the story of its birth and growth (see Orzalesi 1983 for this aspect), but describe it in a general geometric language as it is established today.

The basic idea of the so-called Kaluza–Klein theory (Kaluza 1921, Klein 1926) is to add a further

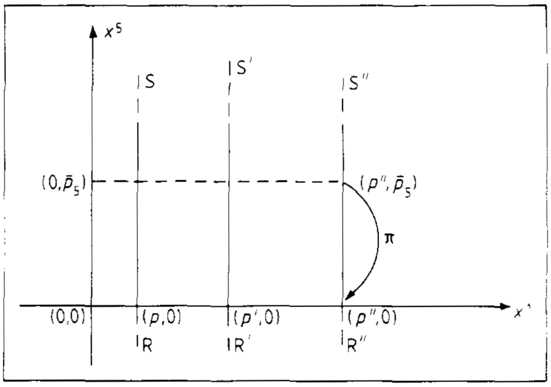

Figure 1 Example of a fibre bundle. π projects every point in V_5 along its own fibre on the point in V_4 where this fibre is 'attached'.

spatial dimension to the spacetime V_4 of general relativity in order to realise the derivation of gravitational and electromagnetic fields from a single universal tensor of the new five-dimensional geometry. If we take the existence of the fifth dimension 'seriously', we have to consider the new five-dimensional extended spacetime thus obtained as the real one and, at the same time, the four-dimensional appearance of the universe has to be explained. In actual fact the extended spacetime in the Kaluza–Klein theory has a particular structure, which represents, however, a simple example of very general kinds of spacetimes: we will try to describe this structure.

If we call V_5 the Riemannian manifold considered as the spacetime of the Kaluza–Klein theory, we can say it is a simple example of a principal fibre bundle (for a rigorous definition see Kobayashi and Nomizu 1963); in particular V_5 is made up by the Riemannian manifold V_4 'plus' another one-dimensional manifold which is also a Lie group—more precisely, the group U(1) of electromagnetism. Moreover there exists a differentiable map π from V_5 to V_4 which is the projection of the extended spacetime onto the usual four-dimensional one and it is possible to introduce a rule of horizontality in vector fields by a bundle connection†. In short V_5 is imaginable as the ordinary spacetime V_4 where at each point $p \in V_4$ one has 'attached' a line, the so-called fibre; figure 1, where the usual four dimensions are reduced to x^1 only, can help to clarify the situation (S, S′, S″, R, R′, R″ $\in V_5$ are the extremal points of the fibres).

We have introduced V_5 as a Riemannian manifold even though we never considered what the metric of the Kaluza–Klein theory is. We can now specify that this metric derives from two constraints (the Ansatz

† Then a horizontal vector field is a field having its fifth component equal to zero, while a vertical field has the four ordinary spacetime components equal to zero.

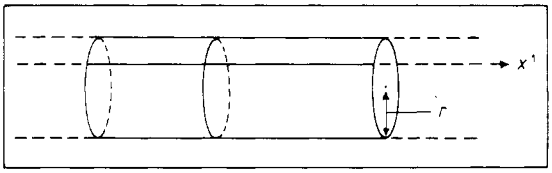

Figure 2 The five-dimensional spacetime of the Kaluza–Klein theory as a hypercylinder.

of Kaluza and Klein): first of all Kaluza required all the components of the five-dimensional metric γ_{MN} not to depend on the fifth dimension; secondly Klein required the component γ_{55} to be a constant†. With these requests, in local coordinates $x^M = (x^\mu, x^5)$ the five-dimensional metric can be written as

$$\gamma_{MN} = \left(\begin{array}{c|c} \dfrac{g_{\mu\nu} + a_\mu a_\nu}{a_\nu} & \dfrac{a_\mu}{1} \end{array} \right) \qquad (7)$$

where $g_{\mu\nu}$ and a_μ depend only on x^μ and not on x^5, so that they are effectively fields in V_4. While $g_{\mu\nu}$ is the metric of general relativity, during the physical development of the theory one can see the four-vector a_μ is algebraically related to the electromagnetic potential A_μ: in fact, if $c = 1$‡,

$$a_\mu = \sqrt{16\pi G} A_\mu. \qquad (8)$$

In this way gravitational and electromagnetic potentials ($g_{\mu\nu}$ and A_μ) are now both a part of the five-dimensional metric; on the basis of what was said in the previous section we can affirm that gravitational and electromagnetic fields (interactions) are geometrised in the Kaluza–Klein theory.

We will before long see what the consequences of this new situation are, but before continuing, we want to stress that, because of Klein's constraint, all the fibres of V_5 become of the same length $L = L_5$, which we can consider as the length of the fifth dimension. So if we suppose $L_5 := \int dx^5 < +\infty$ and we make the identifications $\mathrm{R} = \mathrm{S}$, $\mathrm{R}' = \mathrm{S}'$, ... in figure 1, we can represent our multidimensional universe as a hypercylinder where at each point of the usual spacetime a circle (the fibre) is 'attached'. Figure 2 shows this situation as it can be seen by a space of higher dimensionality: the usual four dimensions are reduced to only x^1 to make the picture possible. In the Kaluza–Klein theory the typical value of the fibre radius is $r \sim 10^{-32}$ cm. This value can be obtained through the insertion of a matter field in the theory; thus we can think of the fifth dimension as really existing and only invisible because of its smallness.

At this point one could make the geometry of V_5 completely explicit, deriving the connection coefficients Γ^P_{MN}, the curvature tensor R^Q_{MNP}, the Ricci tensor R_{MN} and the curvature scalar $R^{(5)}$ in five dimensions: we are not interested in this operation. Instead we want to stress that the Kaluza–Klein theory (or at least its original version) is a theory without matter fields; therefore for 'implementing' physics on the geometrical background described above it is sufficient to write an action for the five-dimensional spacetime continuum and to derive the field equations by the application of the variational principle referred to in the previous section. If we consider $R^{(5)}|\det(\gamma_{MN})|^{1/2}$ as our Lagrangian density in five dimensions (it is the Lagrangian of a general relativity in five dimensions), the action is automatically reducible to the four-dimensional one of the Einstein–Maxwell theory (via the identification set up in equation (8)) and the field equations are

$$R_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu} R^{(5)} = 0 \qquad (9)$$

and

$$R_{\mu 5} - \tfrac{1}{2}\gamma_{\mu 5} R^{(5)} = 0. \qquad (10)$$

When we decompose the five-dimensional geometric objects appearing in equations (9) and (10) making explicit their ordinary spacetime parts, equation (9) becomes the equation (6) of the Einstein–Maxwell theory for an electromagnetic field in the vacuum and equation (10) represents those Maxwell equations which, in the electromagnetic theory, would become inhomogeneous in the presence of charges and currents. The other two Maxwell equations are automatically satisfied because they are an identity for the geometry of the Kaluza–Klein theory.

In such a way we have worked out a theory, the results of which seem equivalent to those of the Einstein–Maxwell theory, i.e. the theory of electromagnetic fields in general relativity. Now, instead, it is important to stress explicitly their differences, at least as far as formal points of view and physical principles are concerned. First of all we can say that in a Kaluza–Klein context the electromagnetic interaction in V_4 comes out of the adopted multidimensional geometry and the choice of an action which is that of a general relativity in five dimensions. In particular, while equation (6) represents the interaction between electromagnetic radiation and four-dimensional geometry in the Einstein–Maxwell theory, now, in our five-dimensional context, the same equation is purely geometric (in more compact terms it is equation (9)) because $F_{\mu\nu}$ is nothing but a derivation of A_μ (which is a part of the metric γ_{MN}):

† Here and afterwards capital italic indices refer to the whole V_5 (that is to say $M, N, \ldots = 1, \ldots, 5$), while Greek ones refer, as usual, to the four dimensions of ordinary spacetime ($\mu, \nu, \ldots = 1, \ldots, 4$).

‡ The identification set up in equation (8) comes out in a natural way only after the insertion of a Lagrangian density in the theory; moreover it will also allow one to affirm that coordinate transformations involving the fifth dimension are essentially the gauge transformations of electromagnetism. These aspects are discussed later in this section.

therefore an electromagnetic field in the vacuum is an expression of our multidimensional geometry.

In addition to this last property about the 'maximum geometrisation' occurring in the field equations of the Kaluza–Klein theory, there are important features suggesting that this theory represents a first step towards a better understanding of the structure of our universe. For example, from considerations about the geometry of Kaluza–Klein extended spacetime one can see there exist only two classes of coordinate transformations on V_5 respecting its fibred structure: the first is the 'translation' of points along V_4 (these are the general coordinate transformations of general relativity); the second is the translation of points along the fibres, that is:

$$\begin{aligned} x^\mu &\rightarrow x'^\mu = x^\mu \\ x^5 &\rightarrow x'^5 = x^5 + \lambda(x^\nu). \end{aligned} \tag{11}$$

It is not difficult to prove that the transformations (11) are precisely those leading to the gauge transformations (2) of the electromagnetic potential A_μ and consequently ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) to those of the electromagnetic field $F_{\mu\nu}$. So, in the Kaluza–Klein theory the gauge transformations of electromagnetism emerge from the geometric structure of the theory as a consequence of spatial translations in the fifth dimension. In this way one internal symmetry and the ordinary spacetime ones are put on the same footing as the effects of spacetime coordinate transformations in a generalised spacetime and they appear to be of a very similar nature.

Of course the trajectory of a charged particle in general relativity is not a geodesic in V_4; in particular it depends on the ratio e/m. The Kaluza–Klein theory leads one to suppose the following generalisation of the equivalence principle holds: in the presence of arbitrary gravitational and electromagnetic fields, and in the absence of other interactions, the real trajectory of test particles coincides with the geodesics of the metric of the extended Kaluza–Klein spacetime V_5. Assuming the validity of this principle we obtain interesting consequences. In short, we re-discover the equation governing the motion of a charged particle in V_4 as a projection of a geodesic in V_5 and at the same time we *derive* the law of charge conservation. In this context the electric charge of a test particle is nothing but the fifth component of its momentum, so that the fifth component of its velocity is just the ratio e/m.

At the end of the previous section we saw that the general-relativity trajectories of freely falling particles do not depend on their mass–energy and, as a consequence, under the same initial conditions everything falls in the same manner in V_4; in particular we can say the motion is uniquely determined by the topology of V_4. With the introduction of an electromagnetic field, the situation seems to be different, because the trajectory of a charged test particle depends on its ratio e/m and so in principle, under the same initial conditions in V_4, there exist infinite distinct trajectories corresponding to the infinite values the ratio e/m can assume for different particles. Geometrically speaking, we are in the following situation: departing from the same initial conditions about the four-position $x^0 = (x_1^0, x_2^0, x_3^0, t^0)$ and the four-velocity v^0 we find different trajectories: the motion is not uniquely determined by the topology of V_4. But here comes the interesting point! In the Kaluza–Klein theory, we have to look at the problem in five dimensions: there e/m is just the fifth component of five-velocity, that is to say it is an initial condition in V_5. On the basis of this crucial observation it is possible to affirm that, initial conditions being equal in V_5, every particle falls in the same manner in fixed gravitational and electromagnetic fields and therefore (these fields are geometrised) the motion of test particles is uniquely determined by the topology of the extended spacetime V_5 of the Kaluza–Klein theory: we have generalised the beautiful result obtained in general relativity. Figure 3, where the usual spacetime is reduced to two spatial dimensions, can help to clarify the situation.

If we now insert a scalar matter field ψ in the theory, other interesting properties may be obtained. First of all, owing to the form of our extended spacetime (see figure 2), ψ is naturally periodic in the fifth dimension, that is

$$\psi(x^\mu, s + L_5) = \psi(x^\mu, s) \tag{12}$$

and therefore it admits a Fourier expansion. On these grounds it is possible to derive a set of Klein–Gordon equations for particles (minimally

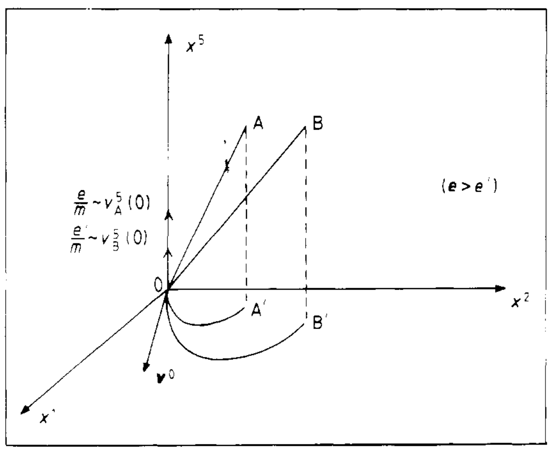

Figure 3 The motion of two charged test particles of equal mass in an arbitrary gravitational field and in a magnetic field directed along the third spatial dimension of V_4. 0A and 0B are geodesics in V_5 (they are distinct because of the two different initial conditions in $v^5(0)$); 0A' and 0B' are their projections on V_4. $v_A^0 = v_B^0 = v^0$ (four-dimensional initial velocity).

coupled to the electromagnetic field) having quantised charges. In this framework the quantisation of the charge is due to the cylindrical geometry of V_5; since V_5 is closed along the fifth dimension, the fifth component of the momentum operator, $p_5 = -\mathrm{i}\, \mathrm{d}/\mathrm{d}s$, has a discrete spectrum. If we remember that the charge of a test particle is nothing but the fifth component of its momentum, we obtain for the spectrum (in units of $(16\pi G)^{-1/2}$, where $c = 1$)

$$p_{5n} = e_n = ne. \tag{13}$$

We cannot go further in this direction without going beyond the aims of this paper; we only want to stress that, together with a 'geometrical explanation' of charge quantisation, the theory leads one to envisage the existence of a tower of particles (with very heavy masses) which are not observed in nature.

Finally, we must stress that there are two generalisations of the Kaluza–Klein theory in the same five-dimensional framework, namely the so-called Einstein–Bergmann theory (Einstein and Bergmann 1938), where Kaluza's constraint is given up, and the Jordan–Thiry theory (Jordan 1948, Thiry 1948), where, instead, Klein's constraint is not imposed.

5. Multidimensional Unified Theories

On the basis of the models given by the Kaluza–Klein theory and its five-dimensional generalisations one can think of trying to unify all the basic interactions in a common geometrical scheme. In particular, extending the dimensionality of the fibres recently resulted in the birth of the so-called Multidimensional Unified Theories (MUTS) (see for example Cho 1975 and Orzalesi 1981). In these theories one tries to reach a unified treatment for gravitation and the Yang–Mills gauge interactions considering a fibre bundle with a non-Abelian Lie group as the space 'added' to the usual spacetime. Today many theoretical physicists think the real dimensionality of our universe is $4 + 7 = 11$, even if, in actual fact, there are still problems concerning the working out of a realistic multidimensional theory of the Kaluza–Klein type (see, for example, Witten 1981). In another context, using superstrings, the dimensionality problem must also be reconsidered (Duff 1985).

At present, we cannot say which is the group leading to a Kaluza–Klein unification of gravity with electroweak and strong nuclear forces. However, we want to stress that the standard theory of gravity and general Yang–Mills interactions is achieved; this is an important step because, as we have already noted, WST and QCD are nothing but theories of the Yang–Mills type in which particular assumptions are made.

MUTS, also called generalised Kaluza–Klein theories, carry good and bad qualities typical of their five-dimensional models, together with some further difficulty due to the multidimensionality of the fibres. In particular the gauge transformations on Yang–Mills potentials (which are present in the D-dimensional metric on the same footing as the gravitational potential $g_{\mu\nu}$) emerge as the effect of spatial translations on the $N = D - 4$ internal dimensions of the extended D-dimensional spacetime: thus internal and spacetime symmetries are now on the same footing and they appear to be of a very similar nature. On the other hand, in MUTS we rediscover the problem concerning the prediction of the existence of particles with very heavy masses which are not observed in nature: for two possible approaches to the solution of this problem see Witten (1981) and Bergia *et al* (1983).

Finally we stress that the modern approach to generalised Kaluza–Klein theories aims to derive (and not to impose) a Kaluza–Klein-type metric as a vacuum solution of field equations under certain initial conditions in a maximally symmetric D-dimensional extended spacetime; the situation is like that of general relativity when one derives the Schwarzschild metric from the field equations under the generic conditions of isotropy and staticity.

What we have presented here is an account of the 'classical' Kaluza–Klein theory and MUTS, both in the sense that, except for the brief mention just made, we have not tried to present the modern approach (spontaneous compactification) and in the sense that we have disregarded quantum aspects, which can give rise to interesting effects (see, for instance, Appelquist and Chodos 1983). Developing these features would have implied a much wider treatment, which would have greatly exceeded the scope of this paper.

In conclusion we can say that MUTS represent the overcoming of the dichotomy concerning the theoretical treatment of the basic forces we described in § 2; geometrically speaking, they give us the possibility of thinking of our universe as a hypercylinder with very small fibres of more than one dimension. Today, research is very active in this field and only the future will tell whether these beautiful ideas about a geometrical unification of fundamental interactions are realistic.

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