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Particle masses and the cosmological constant in Kaluza-Klein theory

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Abstract

A Machian interpretation of Kaluza-Klein gravity is outlined, wherein the fifth coordinate is related to mass. It is shown that with an appropriate form for the metric, the usual law of inertial motion is obtained together with a cosmological variation of particle masses. This variation, however, is slow enough to be compatible with observation. When the 5D theory is reduced to 4D general relativity, a cosmological constant appears which is of acceptable size.

1. Introduction

The suggestion that spacetime should be extended from 4 to 5 dimensions as a means of unifying gravity and electromagnetism was made by Kaluza [1] and Klein [2] in 1921 and 1926, respectively. In recent years, the wish to develop a unified theory that includes the other interactions has led to a rapid growth in the literature of such theories, including 10D string theory and 11D supergravity. In most versions of Kaluza-Klein theory, the extra dimensions are assumed to be rolled-up or compactified lengths. However, such theories have well-known problems (for reviews of modern Kaluza-Klein theory, see Refs. [3-6]). This has led to an examination of non-compact Kaluza-Klein theories, which may help towards an understanding of the size of the cosmological constant and the masses of elementary particles [7-11]. The smallness of the latter compared to the Planck mass has led to the idea of gravitational bags, where an extra dimension is small at most places but large in certain others, allowing for the existence of relatively light states [12-15]. There has also been recent interest in the idea that the extra dimension of basic Kaluza-Klein theory is not a simple length [16-19]. It may have the appropriate dimensions, but at least for soliton and cosmological solutions, the extra dimension and curvature in it are known to be related to the existence of mass-energy in ordinary 4D spacetime [17,18]. In fact, any solution of the apparently empty equations of 5D Kaluza-Klein theory can be expressed as a solution with matter of the 4D Einstein equations of ordinary general relativity [19]. The situation appears to be that Kaluza-Klein theory is mathematically viable but has not met with universal acceptance because of uncertainties about the nature and effects of its extra dimensions.

We aim in what follows to address this issue. We will take the basic version of Kaluza-Klein theory with

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one extra dimension, and highlight the effects of the latter by using appropriate forms for the metric, certain coordinate transformations, and relevant solutions of the geodesic equation. We will show that 5D Kaluza-Klein theory can be cast in a form which is Machian [20]: it is compatible with known physics, but also provides a deeper rationale for mass.

2. A spacetime-mass manifold

In Newtonian mechanics, space and time are the arena for the interactions between particles, whose main characteristics are their masses. The lack of a direct connection between the extrinsic properties of a particle (its position and velocity) and its intrinsic property (its mass) was noted by Mach [20]. Over the years, there have been various attempts to form a connection between space, time and mass, but none has been completely successful. Einstein was of course motivated by Mach's ideas, but even in general relativity spacetime remains separate as a concept from the masses of the particles that move through it. In this section, we wish to attempt a unification of these concepts via the idea of a spacetime-mass manifold.

That such a unification is possible in principle can be appreciated on dimensional grounds. Thus depending on whether we are in the microscopic or macroscopic regime, the rest mass m of a particle can be parametrized using the constants provided to us by nature as a length: h/mc or Gm/c^2, respectively. (Here h is Planck's constant, G is Newton's constant and c is the speed of light; we will keep these constants explicit in what follows to aid physical interpretation.) These scales become comparable in the regime of quantum gravity [21], but we will use the latter parametrization in what follows because we are attempting to construct a classical (as opposed to quantum) theory of mass. Our theory is based on the commonly-understood form of Mach's principle, wherein the local mass m is dependent on the averaged-out properties of the rest of the matter in the universe. Thus we imagine that mass is defined at every point of spacetime by a length parameter $\ell = Gm/c^2$. And following earlier work which shows that matter can be related to an extra dimension [17–19], we imagine that ℓ is in fact the extra coordinate in a 5D Kaluza-Klein theory. It should be possible to write the metric of this model in the local limit as

$$\eta_{\alpha\beta}dx^\alpha dx^\beta \pm d\ell^2 , \tag{1}$$

where $x^\alpha = (ct, \boldsymbol{x})$, Greek indices run from 0 to 3, and the signature of Minkowski metric is +2. Let us first observe that the classical equations of physics remain invariant if $(x^\alpha; \ell) \to (\sigma x^\alpha; \sigma\ell)$; here σ is a scale factor which may be identified with a change of the basic classical units of measurement. If electromagnetic phenomena are included, then the electric charge must also be scaled by σ and the electromagnetic field by σ^{-1}. It is important to note that the fundamental constants of classical physics, i.e. G and c, are thereby unaffected. Secondly, imagine a "boost" or a "rotation" in the (t, m)-plane such that the metric (1) is invariant. In this way the observed masses of the particles may be made to vary arbitrarily with time. This is contrary to experience, and leads us to infer that only certain choices of the ℓ-coordinate can correspond *directly* to mass. It does not appear possible to consider arbitrary coordinate transformations in the 5D Lorentzian manifold and still interpret the fifth coordinate as representing mass.

In order to arrive at a satisfactory interpretation of the fifth coordinate, we adopt an approach based on the equations of motion of free test particles. In Minkowski spacetime, the equation of motion of a free particle of mass m can be obtained from

$$\delta \int -mc\, ds = 0 , \tag{2}$$

where $-ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$, and $s/c = \tau$ has the interpretation of proper time along the trajectory. Thus the associated Lagrangian is $\mathcal{L}_4 = -mc^2(1 - v^2/c^2)^{1/2}$ in this case. Let us now generalize this action principle by introducing the fifth coordinate such that $\delta \int {}^{(5)}ds = 0$, where

$$-{}^{(5)}ds^2 = \frac{\ell^2}{L^2}\left[\eta_{\alpha\beta}dx^\alpha dx^\beta \pm f^2(\ell)d\ell^2\right] , \tag{3}$$

and L is a constant length. When ℓ is constant, ${}^{(5)}ds$ reduces to the Minkowski metric up to a multiplicative constant. Thus dimensional considerations make it necessary to introduce the universal length L. The function $f(\ell)$ is real and dimensionless but otherwise totally arbitrary. The corresponding Lagrangian for free particle motion in the 5D manifold is

$$\mathcal{L}_5 = -mc^2 \left[1 - \frac{v^2}{c^2} \mp f^2(\ell)\frac{\dot{\ell}^2}{c^2}\right]^{1/2} , \quad (4)$$

where $m = c^2\ell/G$ and $\dot{\ell} = d\ell/dt$. It will now be shown that $\mathcal{L}_5$ has essentially the same physical content as $\mathcal{L}_4$ insofar as geodesic motion in spacetime is concerned. It follows from the equations of motion that

$$\boldsymbol{P} = \Gamma m \boldsymbol{v} \quad (5)$$

is the conserved linear momentum with $\Gamma = (1 - u^2/c^2)^{-1/2}$, $u^2 = v^2 \pm f^2\dot{\ell}^2$, and

$$\frac{d}{dt}\left(\Gamma m f^2 \dot{\ell}\right) = mf\frac{df}{d\ell}\dot{\ell}^2\Gamma \mp \frac{c^4}{G}\frac{1}{\Gamma} . \quad (6)$$

It is a consequence of Lagrange's Eqs. (5), (6) that $\Gamma\ell$ is a constant of the motion; this constant may be physically interpreted in terms of the particle's total energy. It is simpler, however, to find the conserved energy of the particle from the expression for the Hamiltonian and the result is

$$E = \Gamma m u^2 + \frac{mc^2}{\Gamma} . \quad (7)$$

It follows from Eq. (7) that $E = \Gamma mc^2$. Thus Γm and $\boldsymbol{v}$ are constants along the path and Eq. (6) can be written as

$$f^2\ddot{\ell} + f\frac{df}{d\ell}\dot{\ell}^2 = \mp\omega^2\ell , \quad (8)$$

where $\omega = c^5/GE$; the first integral of Eq. (8) is, in fact, $E = \Gamma mc^2$. Hence $\boldsymbol{P} = E\boldsymbol{v}/c^2$ and the action principle with Lagrangian (4) implies inertial motion of particles just as in Minkowski spacetime. It is useful to define the *proper mass* m_0 as

$$\left(\frac{m_0 c^3}{E}\right)^2 \equiv c^2 - v^2 = \omega^2\ell^2 \pm f^2\dot{\ell}^2 , \quad (9)$$

so that $E = \gamma m_0 c^2$ and $\boldsymbol{P} = \gamma m_0 \boldsymbol{v}$, where γ is the Lorentz factor $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. *The mass m varies along the path while the proper mass m_0 remains the same.* The requirement that proper mass m_0 should turn out to be a (positive) real constant would impose limitations on the function $f(\ell)$ if the lower sign were chosen in Eq. (9). However, if the mass of a particle is determined from local momentum conservation as in Newtonian mechanics [20], then in terms of spacetime physics the Lagrange function $\mathcal{L}_5$ is essentially equivalent to $\mathcal{L}_4$.

It is a remarkable fact that geodesic motion in accordance with metric (3) in the spacetime-mass manifold reduces to inertial motion in spacetime together with a temporal variation of the fifth coordinate $\ell = \ell(t)$ given by Eq. (9). The issue of how the mass coordinate ℓ actually enters physical measurements is beyond the scope of this investigation; however, it is necessary to ensure that any variation of mass with the proper time of an observer is not in conflict with observation.

To show this, we note that the mass coordinate ℓ is arbitrary in the 5D theory like the spacetime coordinates x^α in the 4D theory and that this is reflected in the arbitrariness of $f(\ell)$. It is this latter function which has to be chosen to be compatible with observation. Since the geodesic equation in the spacetime-mass manifold should reduce to the equation of motion of a test particle in spacetime, it is natural to start with the fundamental Newtonian equation of motion of a particle under gravitational attraction, namely,

$$\frac{d}{dt}(m\boldsymbol{v}) = -m\nabla\Phi , \quad (10)$$

where Φ is the Newtonian gravitational potential. This equation may be written in geodesic form ($i = 1, 2, 3$)

$$\frac{d^2x^i}{dt^2} + \partial_i\Phi + \frac{1}{m}\frac{dm}{dt}\frac{dx^i}{dt} = 0 , \quad (11)$$

which suggests that a natural fifth coordinate representing mass – that would be on a par with the spacetime coordinates – would be proportional to $\mu : -\infty \to +\infty$ such that $d\mu = dm/m$. Let us therefore set $f(\ell) = L/\ell$ in Eq. (3) so that $f(\ell)d\ell = Ld\mu$; the natural fifth coordinate is then $L\ln(\ell/L)$. Eq. (9) can be easily integrated in this case. If the upper sign is chosen in the metric (3), the result is $\ell = \ell_0/\cosh\left[\Omega_\gamma(t - t_0)\right]$, where t_0 is the epoch at which the coordinate mass of the particle is the proper mass $m(t = t_0) = m_0$. Here $\Omega_\gamma = \omega\ell_0/L = c/\gamma L$. Similarly, if the lower sign in Eq. (3) is chosen, $\ell = \ell_0/\cos\left[\Omega_\gamma(t - t_0)\right]$. It is reasonable to assume that $L = GM/c^2$, where M is the total proper mass-energy content of the universe. It follows that $T_\gamma = 2\pi/\Omega_\gamma = 2\pi GM\gamma/c^3 \gtrsim 10^{11}$ yr. This ensures that masses only vary slowly, in agreement with observational data [22,23]. Of the two possible metrics in Eq. (3), we choose the first alternative ("upper sign") in the rest of this paper; therefore, we can safely proceed on the assumption that in every local spacetime region the metric can be taken to be

$$-{}^{(5)}ds^2 = \frac{\ell^2}{L^2}\left(\eta_{\alpha\beta}dx^\alpha dx^\beta\right) + d\ell^2 , \qquad (12)$$

for a particle of proper mass m_0. Particle masses can monotonically decrease in this picture; however, no conflict with observations is expected since the effect would be only of second order in $\epsilon = 2\pi(t - t_0)/T_\gamma \ll 1$. It is interesting to consider the limiting case of $m_0 \to 0$ and $v/c \to 1$. In this case $\ell \to 0$ and hence the propagation of rays of radiation is confined to the spacetime domain.

The fifth coordinate of the flat metric in its Cartesian form (1) clearly has little to do with mass. On the other hand, the preceding considerations provide the motivation to look for a form of the flat metric in which the fifth coordinate would be closely related to mass. It is therefore interesting to consider a flat 5D manifold in spherical coordinates,

$$-c^2dT^2 + dR^2 + R^2d\Omega^2 + d\Psi^2 , \qquad (13)$$

and to show explicitly how one can bring out the mass coordinate as in Eq. (3). Let us consider the coordinate transformation $(R, \Psi) \to (r, \psi)$, where

$$R = r\psi/L \quad \text{and} \quad \Psi = \psi\left(1 - r^2/L^2\right)^{1/2} , \qquad (14)$$

for $r \leq L$. The flat 5D metric takes the form

$$-c^2dT^2 + \frac{\psi^2}{L^2}\left(\frac{dr^2}{1 - r^2/L^2} + r^2d\Omega^2\right) + d\psi^2 , \quad (15)$$

which for constant ψ reduces to a static $k = 1$ Robertson-Walker spacetime. Next, let us choose

$$cT = \ell \sinh\left(\frac{ct}{L}\right) \quad \text{and} \quad \psi = \ell \cosh\left(\frac{ct}{L}\right) , \qquad (16)$$

so that under the transformation $(T, \psi) \to (t, \ell)$ the metric (15) becomes

$$\frac{\ell^2}{L^2}\left[-c^2dt^2 + \cosh^2\left(\frac{ct}{L}\right)\left(\frac{dr^2}{1 - r^2/L^2} + r^2d\Omega^2\right)\right] + d\ell^2 , \qquad (17)$$

which for constant ℓ is an expanding $k = 1$ Robertson-Walker spacetime. For $r \ll L$ and $ct \ll L$, this metric reduces to (12) once spherical coordinates are transformed to Cartesian coordinates.

The metrics (3) and (12) which we have used in this section are generalizations of 4D Minkowski spacetime; our purpose has been to illustrate how mass may be introduced into Kaluza-Klein theory. However, we have not so far considered the field equations which any metric should satisfy; and we should in this regard clearly generalize our treatment to take into account the curvature associated with matter by the replacement $\eta_{\alpha\beta} \to g_{\alpha\beta}(x^\mu; \ell)$. We now proceed to do this, paying particular attention to the cosmological constant since it has been a subject of controversy in Kaluza-Klein theory.

3. Field equations and the cosmological constant

Let us consider a spacetime-mass manifold with metric $-{}^{(5)}ds^2 = \hat{g}_{AB}dx^A dx^B$, where $x^A = (ct, \boldsymbol{x}; \ell)$ and $A, B = 0, 1, 2, 3, 5$. In the case where there is no explicit cosmological constant, we already know that the source-free Kaluza-Klein equations ${}^{(5)}R_{AB} = 0$ contain the source-full Einstein equations $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ [19]. Here $T_{\mu\nu}$ is an effective 4D energy-momentum tensor, representing the induction of matter in 4D via the fifth dimension. We now wish to examine the effective 4D physics in the case where there is an explicit cosmological constant. That is, we wish to compare ${}^{(5)}R_{AB} = 0$ with

$${}^{(4)}R_{\mu\nu} - \tfrac{1}{2}\,{}^{(4)}R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \qquad (18)$$

so as to identify the effective 4D cosmological constant Λ.

The spacetime-mass manifold is assumed to have a smooth but otherwise arbitrary metric tensor $\hat{g}_{AB}$ satisfying the Kaluza-Klein equations. The metric form ${}^{(5)}ds$ is invariant under the group of general coordinate transformations; therefore, it is possible to choose *canonical coordinates* in the spacetime-mass manifold such that $\hat{g}_{\mu 5} = 0$ and $\hat{g}_{55} = 1$. That is, the five arbitrary functions associated with the freedom in the choice of coordinates may be so chosen that the metric takes the form $-{}^{(5)}ds^2 = \hat{g}_{\mu\nu}(x; \ell)dx^\mu dx^\nu + d\ell^2$. It is then possible to define the spacetime metric tensor to be $g_{\mu\nu}(x; \ell)$ such that $\hat{g}_{\mu\nu} = (\ell/L)^2 g_{\mu\nu}$; hence,

$$-{}^{(5)}ds^2 = \frac{\ell^2}{L^2}(g_{\alpha\beta}dx^\alpha dx^\beta) + d\ell^2 , \tag{19}$$

where $g_{\alpha\beta}(x;\ell)$ refers to the 4D spacetime. In the canonical system of coordinates, the mass coordinate lines are geodesics normal to the hypersurfaces $\ell =$ constant. Therefore, starting from a suitable initial spacetime hypersurface, we can choose the mass coordinate lines to be the congruence of geodesic lines normal to the initial hypersurface. Along each such line, the spacetime-mass interval corresponds to the canonical mass coordinate. The geometric construction of the canonical coordinate system is thus straightforward and can be described as the 5D analogue of the construction of the synchronous coordinate system in 4D spacetime [24]. Once a canonical mass coordinate ℓ has been chosen, the corresponding spacetime coordinates could still be subjected to arbitrary transformations that are, however, independent of ℓ.

A canonical coordinate system for the flat spacetime-mass manifold has already been given in Eq. (17). The spacetime part is independent of ℓ and is a $k = 1$ Robertson-Walker metric that satisfies the gravitational field equations for a perfect fluid with $\rho = -p = 3c^4/8\pi GL^2$. Alternatively, one may recognize the spacetime part as the de Sitter metric with a cosmological constant $\Lambda = 3/L^2$. That is, the flat 5D spacetime-mass manifold corresponds to the empty expanding de Sitter spacetime. It turns out that whenever, the spacetime part of the 5D metric in canonical coordinates is independent of ℓ, the Kaluza-Klein equations reduce to the vacuum gravitational field equations with a cosmological constant. To demonstrate this fact, consider the metric in canonical coordinates as in Eq. (19); the components of the 5D Ricci tensor can be written as

$${}^{(5)}R_{\mu\nu} = {}^{(4)}R_{\mu\nu} - S_{\mu\nu} , \tag{20a}$$

$${}^{(5)}R_{\mu 5} = A_{\mu\ ;\alpha}^{\ \alpha} - \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial \ell} , \tag{20b}$$

$${}^{(5)}R_{55} = -\frac{\partial A^\alpha_{\ \alpha}}{\partial \ell} - \frac{2}{\ell}A^\alpha_{\ \alpha} - A_{\alpha\beta}A^{\alpha\beta} , \tag{20c}$$

where $S_{\mu\nu}$ is a symmetric tensor given by

$$S_{\mu\nu} = \frac{\ell^2}{L^2}\left[\frac{\partial A_{\mu\nu}}{\partial \ell} + \left(\frac{4}{\ell} + A^\alpha_{\ \alpha}\right) A_{\mu\nu} - 2A_\mu^{\ \alpha} A_{\nu\alpha}\right] + \frac{1}{L^2}\left(3 + \ell A^\alpha_{\ \alpha}\right) g_{\mu\nu} . \tag{21}$$

Here ${}^{(4)}R_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho}$ are, respectively,the 4D Ricci tensor and the connection coefficients constructed via $g_{\alpha\beta}$. Moreover

$$A_{\alpha\beta} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial \ell} , \tag{22}$$

where $A_\alpha^{\ \beta} = g^{\beta\delta}A_{\alpha\delta}$, and the semicolon in Eq. (20b) represents the usual covariant differentiation in 4D. It follows from Eqs. (18), (20a) and the Kaluza-Klein field equations that

$$T_{\mu\nu} = \frac{c^4}{8\pi G}\left[S_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}(S - 2\Lambda)\right] . \tag{23}$$

This result is general, insofar as $g_{\mu\nu}$ can depend on all 5 coordinates.

If the 4D metric $g_{\alpha\beta}$ does not depend on the fifth coordinate ℓ, then $A_{\alpha\beta} = 0$ and this ensures that ${}^{(5)}R_{\mu 5}$ in Eq. (20b) and ${}^{(5)}R_{55}$ in Eq. (20c) vanish. Furthermore, the 4D Ricci tensor is then ${}^{(4)}R_{\mu\nu} = S_{\mu\nu} = 3L^{-2}g_{\mu\nu}$. It is possible to satisfy Eq. (23) by setting $T_{\mu\nu} = 0$ in this case, and the cosmological constant is then

$$\Lambda = \frac{3}{L^2} . \tag{24}$$

With L as above, this gives $\Lambda = 3c^4/G^2M^2 \approx 10^{-56}\mathrm{cm}^{-2}$, in agreement with the standard interpretation of cosmological data. Thus we find that the cosmological constant appears naturally from the reduction of the 5D empty equations when the spacetime metric is independent of the fifth coordinate, and that it has an acceptable size.

An immediate consequence of this result is that from any vacuum solution of Einstein's equations with cosmological constant (24) we can directly obtain a corresponding 5D solution of the Kaluza-Klein equations. A well-known example is the Schwarzschild-de Sitter solution, from which we get an exact 5D Ricci-flat metric

$$-{}^{(5)}ds^2 = \frac{\ell^2}{L^2}\left[-c^2F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2\right] + d\ell^2 , \tag{25}$$

where $F(r)$ is given by

$$F(r) = 1 - \frac{r_g}{r} - \frac{r^2}{L^2} . \tag{26}$$

Here r_g is the constant gravitational radius of the source and is related to the proper mass of the Schwarzschild spacetime. In this way, 5D "black hole" solutions [25] can be recovered.

Another way to satisfy Eq. (23) is to set $\Lambda = 0$. It then follows from ${}^{(4)}R_{\mu\nu} = S_{\mu\nu} = 3L^{-2}g_{\mu\nu}(x)$ that

$$T_{\mu\nu} = -\frac{3c^4}{8\pi GL^2}\, g_{\mu\nu}\,, \tag{27}$$

which can represent a perfect fluid with an equation of state $p = -\rho$. The energy density of the fluid is given by $\rho = 3c^4/8\pi GL^2$. An interesting example of this situation is provided by the standard inflationary cosmological model with a spatially flat ($k = 0$) Friedmann-Lemaître-Robertson-Walker spacetime. The associated 5D exact solution of the Kaluza-Klein equations is

$$-{}^{(5)}ds^2 = \frac{\ell^2}{L^2}\left[-c^2dt^2 + e^{2Ht}\left(dr^2 + r^2 d\Omega^2\right)\right] + d\ell^2\,, \tag{28}$$

where the Hubble constant is $H = c/L$. The 4D part of the flat metric (28) describes an exponentially expanding space and is locally equivalent to the metric of de Sitter spacetime.

Finally, it is interesting to discuss the geodesic equation for the metric form (12) with the Minkowski metric replaced by $g_{\alpha\beta}(x;\ell)$. The considerations of the previous section involved the *local* trajectory of a free test particle in the spacetime-mass manifold (3). We now proceed to extend those results to the whole trajectory. Using

$${}^{(5)}ds^2 = \left[\frac{\ell^2}{L^2} - \left(\frac{d\ell}{ds}\right)^2\right] ds^2\,, \tag{29}$$

where $-ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, it is possible to reduce the 5D geodesic equation to the 4D equation of motion

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = F^\mu\quad. \tag{30a}$$

Here c^2F^μ is the force experienced by a test particle per unit proper mass and is given by

$$F^\mu = -\left(g^{\mu\alpha} + \frac{1}{2}\frac{dx^\mu}{ds}\frac{dx^\alpha}{ds}\right)\frac{d\ell}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial\ell}\,. \tag{30b}$$

It is important to note that *the acceleration is not orthogonal to the velocity of the particle*. That is, $g_{\mu\nu}F^\mu dx^\nu/ds \neq 0$ since $g_{\mu\nu}$ depends explicitly upon $\ell(s)$. Furthermore, the geodesic equation for the fifth coordinate reduces to

$$\frac{d^2\ell}{ds^2} - \frac{2}{\ell}\left(\frac{d\ell}{ds}\right)^2 + \frac{\ell}{L^2} = \frac{1}{2}\left[\frac{\ell^2}{L^2} - \left(\frac{d\ell}{ds}\right)^2\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{\partial g_{\alpha\beta}}{\partial\ell}\,. \tag{31}$$

If the spacetime metric is independent of ℓ, so $\partial g_{\alpha\beta}/\partial\ell = 0$, then the path of the test particle in 4D is a *geodesic* given by Eq. (30a) with $F^\mu = 0$. Moreover, Eq. (31) can be solved completely in this case and we choose

$$\ell = \ell_0/\cosh\left(\frac{s - s_0}{L}\right)\,, \tag{32}$$

where s/c is the proper time along the geodesic path of the particle and s_0/c is the epoch at which the mass coordinate of the test particle equals its proper mass. These results, when restricted to a local spacetime region, are in complete agreement with those derived in the preceding section.

4. Discussion

Following Mach's basic insight, we have adopted an approach to Kaluza-Klein theory in which the fifth dimension represents the mass generated by the matter in the universe. Generalizing the action principle for the motion of a test particle in Minkowski spacetime, we have studied in detail the geodesics of the 5D metric (3) in which the 4D metric is Minkowskian. This 5D metric is conformally flat but not Ricci flat, so that the Kaluza-Klein equations are not in general satisfied in this case. However, the metric form (3) is only intended to represent the manifold around any local event in spacetime. It has been demonstrated that geodesic motion in this 5D metric actually gives the same results as inertial motion in 4D Minkowski spacetime. In addition, the relative rate of change of the mass coordinate is fixed by the constant L which is independent of space and time. There are, of course, observational limits on the variation with time of the rest masses of particles and the strength of gravity [22,23]. These indicate that any such variation would have to be very slow and of cosmological origin. Therefore L is chosen to be of the order of the Hubble radius (c/H). A judicious choice of the 5D metric form as in Eq. (12) can give a variation compatible with observational limits. Thus we can assume that the mass coordinate of a free particle (with proper mass m_0) varies with the proper time τ along its path as $m_0/m = \cosh[c(\tau - \tau_0)/L]$, where τ_0 is the epoch at which the mass coordinate equals m_0. (This assumes that the extra dimension of Kaluza-Klein theory is spacelike.) It follows that for $\tau - \tau_0 \ll L/c$, $m^{-1}(dm/d\tau) \simeq -c^2(\tau - \tau_0)/L^2$; hence, $|m^{-1}(dm/d\tau)| \lesssim 10^{-18}\mathrm{yr}^{-1}$ consistent with observations that have extended over a period of time, $\tau - \tau_0$, that is less than a century. Replacing the Minkowski metric with $g_{\alpha\beta}(x^\mu; \ell)$, we arrive at the metric form (19) in canonical coordinates in which the fifth dimension is related to mass via $\ell = Gm/c^2$. The form of the metric (19) in the canonical coordinate system is invariant under 4D coordinate transformations that are independent of ℓ but not under arbitrary 5D transformations, so we must restrict ourselves to transformations in ordinary spacetime to maintain the interpretation of the fifth coordinate as mass. In regard to field equations, we have adopted the view [17–19] that matter in 4D is the result of Kaluza-Klein theory in 5D. For the metric (19), we have used the empty 5D Kaluza-Klein equations to find an effective source for the 4D Einstein equations. An immediate consequence of this reduction is that an effective cosmological constant $\Lambda = 3/L^2$ appears when the 4D metric is independent of the fifth coordinate, i.e., $\partial g_{\alpha\beta}/\partial\ell = 0$. This suggests that in our canonical coordinate system the condition $\partial g_{\alpha\beta}/\partial\ell = 0$ defines a Kaluza-Klein vacuum. Thus the introduction of any vacuum Einstein spacetime (with cosmological constant $\Lambda = 3/L^2$) in the metric form (19) generates a solution of the Kaluza-Klein equations in canonical coordinates.

Space and time are *classical* manifestations of extension and movement of physical systems.

The concept of spacetime-mass manifold, via the introduction of a fifth dimension $\ell = Gm/c^2$, constitutes an attempt at a classical unification of space, time, and matter. This classical matter involves classical fields as well as particles. Classical particles follow geodesics of the spacetime-mass manifold; however, this would not be the case for elementary particles. Only in the correspondence limit $(m \to \infty)$ could one ascribe a trajectory to an elementary particle; in this limit the ratio of the Compton wavelength of the particle (h/mc) to its gravitational length (Gm/c^2) tends to zero. Moreover, the inertial properties of an elementary particle are determined by its mass as well as spin; our theory only involves classical particle masses since the spin degrees of freedom can be neglected in the correspondence limit. In the absence of a proper quantum theory of gravitation, the relationship between our classical theory and the quantum description of the gravitational field cannot be elucidated at present. Furthermore, our theory needs to be supplemented with further hypotheses in order to be viable even at the classical level since the reduction of the spacetime-mass manifold to the spacetime is not unique. For instance, a flat spacetime-mass manifold can correspond to a spacetime with metric $g_{\mu\nu} = (L/\ell)^2\eta_{\mu\nu}$ as well as the de Sitter spacetime.

The results we have derived above indicate to us that it is possible, at least in principle, to realize Mach's principle in a 5D Kaluza-Klein theory in which the extra coordinate is related to mass. However, more work remains to be done to see how far this idea is viable. The non-uniqueness of the canonical coordinate system, the properties of solutions of the 5D field equations with $\partial g_{\alpha\beta}/\partial\ell \neq 0$, and the consequences of extending our approach to include electric charge require further investigation.

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