

Kaluza–Klein Theories: An Overview

The aim of this Comment is to provide for the nonexpert a descriptive overview, colored with a certain amount of personal prejudice, of the basic ideas underlying the Kaluza–Klein approach to unification, and a very brief survey of some of the more recent developments. The treatment is nonmathematical in the extreme; it is hoped that the references are sufficiently comprehensive (although they are nowhere near complete) that the interested reader can delve into those aspects of the subject that seem most worthy of further attention.

I. HISTORY

If we disregard a precocious attempt by Nordstrom in 1914,¹ the history of Kaluza–Klein theories begins, properly enough, with Kaluza,² whose paper entitled “Zum Unitätsproblem der Physik” was communicated to the Prussian Academy by Einstein in 1921. Klein³ and others⁴ kept the idea alive during its “classical” period, which extended roughly into the mid-sixties or early seventies. All of this work had as its goal the unification of gravity with electromagnetism at the classical level by assuming the formal existence of a single extra spatial dimension.

It was realized, as early as 1964 by DeWitt,⁵ that by adding more than one extra dimension one could encompass non-Abelian gauge theories as well. In the late seventies, spurred by recent progress in higher-dimensional supergravity theories,⁶ and by the fact that the grand-unified scale was only a few orders of magnitude above the Planck length, the Kaluza–Klein idea suddenly acquired widespread

notoriety, with the result that by now the diligent researcher who wishes to become acquainted with the field is faced with a reading list of several hundred papers, all written within the last few years.

II. BASIC STUFF⁷

Since there is, as yet, no compelling experimental evidence in favor of the Kaluza-Klein idea, the decision to work on it requires a certain a priori belief in the likelihood that it is correct. As with all acts of faith, the leap is made easier by the existence of a miracle or two. In this case, the miracle comes about in the following way.

Let us assume, as the simplest nontrivial example, that we are studying general relativity in five space-time dimensions. The action is

$$S_5 = -\frac{1}{16\pi G_5} \int d^5x \sqrt{|g|} R \quad (1)$$

where g_{ab} is the five-dimensional metric, R is the curvature scalar: $R = g^{AB}R_{AB}$ and G_5 is the five-dimensional version of Newton's constant, with dimension of (length)³. We take the topology of our five-dimensional manifold to be not that of five-dimensional Minkowski space, M^5 , but rather that of $M^4 \otimes S^1$, where S^1 is a circle of some as yet unspecified radius r . It is then convenient to write the metric in the following form:

$$g_{AB} = \phi^{-1/3} \begin{bmatrix} g_{\mu\nu} + A_\mu A_\nu \phi & A_\mu \phi \\ A_\nu \phi & \phi \end{bmatrix}. \quad (2)$$

Note that there is no loss of generality in so doing. This is not an Ansatz, but rather a choice of parametrization for g_{AB} , which is arbitrary.

Furthermore, the assumed topology allows us to make a Fourier expansion of each component of the metric in the coordinate x^5 :

$$g_{AB}(x^\mu, x^5) = \sum_{n=-\infty}^{\infty} g_{AB}^{(n)}(x^\mu) e^{inx^5/r}. \quad (3)$$

The standard dimensional reduction occurs if we now assume that $\partial/\partial x^5$ is a Killing vector, i.e., that the metric is independent of x^5 . This amounts to keeping only the $n = 0$ mode in the expansion of Eq. (3). Upon substituting the $n = 0$ piece of g_{AB} into the action Eq. (1) and integrating over x^5 , one finds

$$S = -\frac{1}{16\pi G} \int d^4x |\det\{g_{\mu\nu}\}|^{1/2} \left[R^{(4)} + \frac{1}{4} \phi F_{\mu\nu} F^{\mu\nu} + \frac{1}{6} \frac{\partial_\mu \phi \partial^\mu \phi}{\phi^2} \right]. \quad (4)$$

The appearance of the $U(1)$ gauge term involving $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the simplest example of the Kaluza-Klein "miracle." A pleasing corollary is that, as can easily be shown, the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \lambda$ is induced by the following special case of a five-dimensional coordinate transformation:

$$x'^\mu = x^\mu \quad (5a)$$

$$x'^5 = x^5 + \lambda(x^\mu). \quad (5b)$$

The generalization of this discussion to the non-Abelian case, while nontrivial, is straightforward.⁸ One assumes a background manifold of the form $M^4 \times B$ where B is compact, Riemannian, and admits a set of Killing vectors K_i^a , $i = 1, \dots, n$ generating the algebra of some n -dimensional non-Abelian group G . One finds, in complete analogy to the five-dimensional case, that dimensional reduction generates a term proportional to the combination $F_{\mu\nu}^i F^{\mu\nu i}$ where $F_{\mu\nu}^i$ is the field strength appropriate to the group G , and that the non-Abelian gauge transformation can be viewed as the following coordinate transformation on $M^4 \times B$:

$$x'^\mu = x^\mu \quad (6a)$$

$$x'^a = x^a + \epsilon^i K_i^a \quad (6b)$$

with

$$K_i^a \frac{\partial}{\partial x^a} \epsilon^j = 0. \quad (6c)$$

This generalized “miracle” leads one to what might be called the central dogma of Kaluza–Klein theory: the origin of gauge theories lies in a higher-dimensional version of general relativity (perhaps including supergravity). To the true believer, this is a statement not only of possibility but of necessity: gauge theories must never be put in by hand; they must always emerge through the process of dimensional reduction. A theory given, for example, by

$$S = -\frac{1}{16\pi G_D} \int d^D x \sqrt{|g|} \left[R + \frac{1}{4} F_{AB}^\alpha F^{AB\alpha} \right], \quad (7)$$

where the gauge field F_{AB}^α is inserted by hand in the higher dimensional space, would be anathema (unless, of course, the theory can be shown to derive from pure gravity in a yet higher dimensional space).

Before closing this section, it is instructive to study the dimensionally reduced action, Eq. (4), a little further. By inspection, we see that the field $\ln \phi / \phi_c$ plays the role of a standard scalar field. It is convenient to choose ϕ_c , which is the constant background value of ϕ , equal to unity; the parameter r then truly represents the radius of the fifth dimension. In the early days of Kaluza–Klein, this extra scalar field tended to be an embarrassment (it means that the reduced theory is really a scalar-tensor theory of gravity). Nowadays one recognizes that scalar fields often play important roles in the spontaneous breakdown of gauge theories; furthermore, quantum corrections are likely to give the scalars a mass, thereby removing their long-range gravitational effects.

One also sees from Eq. (4) that it is really $\tilde{A}_\mu = (1/\sqrt{16\pi G})A_\mu$ that plays the role of the usual gauge field. This information is important in extracting the value of the charge q associated with the $U(1)$ gauge field. Let us consider a scalar matter field χ minimally coupled to the five-dimensional Kaluza–Klein metric g_{AB} :

$$g^{AB} \nabla_A \nabla_B \chi = 0. \quad (8)$$

Here the covariant derivative ∇_A is to be evaluated using the metric g_{AB} . In order to endow χ with charge, it is necessary for it to depend on x^5 . The simplest possibility is

$$\chi = \chi_0(x^\mu)e^{(ix^5)/r}, \quad (9)$$

i.e., we keep only the $n = 1$ mode in its Fourier expansion. Furthermore, to isolate the effects of the gauge field it is convenient to set $\phi = 1$ and $g_{\mu\nu} = \eta_{\mu\nu}$ in the g_{AB} of Eq. (2). Doing all this, one finds that $\chi_0(x^\mu)$ obeys the usual Klein–Gordon equation for a charged particle, provided that the charge is identified as

$$q = \frac{\hbar}{c} \frac{(16\pi G)^{1/2}}{r}. \quad (10)$$

[N.B.: Of course, there is no \hbar in Eq. (8); \hbar enters only when one seeks to extract a charge from the coefficient of \tilde{A}_μ .] Now there is no phenomenological basis for identifying q with the observed electric charge (for example, the five-dimensional model has the property that any particle with charge also has a mass of order the Planck mass). Nevertheless, all gauge couplings we know about fall in the range

$$\frac{1}{100} \lesssim q \lesssim \frac{1}{10}, \quad (11)$$

and applying this estimate to Eq. (10) leads us to conclude that r should be taken at most a few orders of magnitude above the Planck length of 1.6×10^{-33} cm. The fact that the gauge coupling constants are proportional to the ratio of the Planck length to the size of the extra dimensions generalizes to the non-Abelian case as well⁹; it is a comforting consistency of Kaluza–Klein theories that the smallness of the extra dimensions, which explains why they are not seen, follows from the requirement that the gauge couplings are not too much smaller than unity.

III. SOME PERSPECTIVES ON DIMENSIONAL REDUCTION

In the case of the five-dimensional model, dimensional reduction works so well that it is easy to overlook the conceptual difficulties

that can infect the process when $D > 5$. The main problem is that, if one assumes a manifold of the form

$$M = (\text{Minkowski space}) \otimes B, \quad (12)$$

where B is Riemannian and compact, and admits a set of non-Abelian Killing vectors, then M cannot be a solution of the classical Einstein equations with or without cosmological constant. And yet it is precisely this form of M that is required if straightforward dimensional reduction is to work as described in the previous section.

Frequently in the literature, one finds that such an M is chosen anyway; one integrates out the extra dimensions and works only with the dimensionally reduced theory from then on. The unwary reader never has time to ask whether the solutions of the reduced theories are also solutions of the D dimensional equations of motion. (They are not, and adding a cosmological constant or a simple conformal factor will not help either.)

There are at least three possible attitudes toward this situation: (i) Dimensional reduction is a device whose sole object is to generate the effective four-dimensional theory; it is then immaterial whether the higher dimensional equations are satisfied. This is fine as far as it goes—for example, this strategy worked well in the construction of $N = 8$ supergravity in four dimensions.⁶ However, all pretense of unification is abandoned at the outset. (ii) The extra dimensions do exist, but one must invoke extra matter fields to achieve spontaneous compactification (i.e., to have a solution to the higher-dimensional equations with the assumed symmetry and topology).¹⁰ This treats unification perhaps slightly more seriously, but there is still the danger that the matter fields, if introduced ad hoc, will violate the central dogma of the previous section. (iii) If one takes the extra dimensions completely seriously, then one must begin with a purely geometrical theory in the higher dimensional space, and find a solution to either the classical or perhaps the quantum-corrected equations of motion that exhibits spontaneous compactification. As remarked above, pure gravity at the classical level does not work; supergravity in 11 dimensions, which has been quite extensively studied,^{11,12} seems to work only if the space-time part of the manifold is not Minkowski space but is anti-deSitter space. Furthermore, the length scale associated

with the curvature in the anti-deSitter space is of the same order of magnitude as that in the internal space—an interesting world, but not the one we live in.

IV. QUANTUM EFFECTS

A possibly important modification of this picture is wrought by the inclusion of one-loop quantum effects. In particular, the fact that the extra dimensions are compact leads, via the Casimir effect, to corrections to the classical equation of motion. These have been computed for the original five-dimensional model,¹³ including finite temperature effects¹⁴; the case in which the compact manifold is a d -dimensional torus has been studied as well.¹⁵ The effect of matter fields has also been investigated.¹⁶ It has been shown, in fact, that matter fields can lead to corrected equations of motion for which the manifold $M^4 \times S_N$ is a solution.¹⁷ (For technical reasons, the computation is restricted to N odd.) Furthermore, the solution fixes the size of the N -sphere relative to the Planck length. Thus in these models the gauge coupling constant of the dimensionally reduced theory is a predicted number. (Presumably this is the coupling constant measured at the Kaluza–Klein scale. In a realistic model, it would be necessary to use the renormalization group to obtain a coupling constant defined at laboratory energies in order to compare with experiment.)¹⁸

While these results are encouraging, there are a number of cautionary remarks to be made:

(i) The matter fields which have so far been shown to produce the quantum compactification tend to give very small contributions to the Casimir energy. This means that a huge number of fields (10^4 or 10^5) are needed to give a reasonable value for the compactification scale. Not only do the matter fields violate the central dogma, but their number has to be unrealistically large.

(ii) Thus it is of interest to compute the Casimir energy due to the gravitational field itself on the background geometry $M^4 \times S^N$ to see whether it too is unexpectedly very small. This is technically much more involved than the matter-field case, but progress is being made on this problem.^{19,20} A further difficulty with the gravitational case, though, is precisely the absence of a tunable parameter such as the number of fields. In the matter-field case, the largeness of this

parameter can be used to guarantee the reliability of the loop expansion¹⁷; in the case of gravity, the one-loop approximation can only be justified post hoc if the compactification scale turns out to be sufficiently bigger than the Planck length, L_p , because at distances well above L_p , higher order quantum effects should be weak. A consoling thought is that if a realistic model is ever found, in order for the gauge coupling to be phenomenologically acceptable, the scale of the extra dimensions must, in fact, turn out to be significantly bigger than L_p .

(iii) As always, any quantum gravitational effects must be viewed with suspicion because of the absence of a consistent theory of quantum gravity. In the present instance, the higher-loop contributions would not only be prohibitively difficult to compute, but would also be rendered meaningless by the nonrenormalizability of the theory. Nevertheless, the Casimir effect in Kaluza–Klein theories does represent a rare example where quantum gravity is expected to play a physically important role.

V. COSMOLOGY

In four-dimensional general relativity the subject of cosmology is approached through the study of time-dependent solutions to the equations of motion which are assumed to describe evolving universes. This same philosophy can be applied to Kaluza–Klein theories. The earliest such study²¹ involved a solution of the five-dimensional model in which one dimension is predicted to shrink with time while the other three spatial dimensions expand. This was generalized to the case of supergravity in eleven dimensions,²² which had the advantage of providing a natural understanding of why three spatial dimensions would expand and seven contract (as opposed to, say, four and six). One limiting feature of these models is that as the size of the internal dimensions changes with time, so do the gauge coupling constants. The possible time variation of fundamental constants is severely restricted by observation.²³ A possible way out is to find a model in which the extra dimensions do not contract but rather remain fixed at some (presumably small) scale.²⁴ Alternatively, one may speculate that as the contracting extra dimensions approach the Planck scale L_p , quantum effects along the lines of those described in the previous section become dominant and freeze the extra dimensions at some fixed size relative to L_p .

These days when one thinks of cosmology, the question of inflation naturally pops up. Some exploratory work has been done, either to see whether Kaluza–Klein theories can accommodate an inflationary phase, or to see if some of the desirable consequences of inflation, such as entropy production, can be viewed as a by-product of dimensional reduction.^{25,26} It is probably too soon to know whether a convincing scenario along these lines can be constructed.

The importance of these cosmological studies is that they seem to offer the best, indeed perhaps the only, hope at present for deducing some observational consequences that may ultimately differentiate Kaluza–Klein theories from other potential avenues of unification.

VI. A CLOUD ON THE HORIZON

The sun does not always shine in the extra dimensions. Perhaps the most serious problem is the absence of a phenomenologically realistic embodiment of the Kaluza–Klein idea. The situation is potentially more difficult than merely finding the right model in a large haystack of possibilities. There is a theorem²⁷ to the effect that it is impossible to have a model in which the massless fermions fall into chiral representations of the gauge group. (The theorem is not quite that absolute, but it is nearly so.) Barring the simple but inelegant possibility that right-handed partners of the observed left-handed leptons and quarks will start showing up at the next generation of accelerators (or at least at some energy insignificant compared to the compactification scale), one is faced with a real quandary. This has prompted work in which one is led to relax some of the standard assumptions of Riemannian geometry in the higher dimensional space.²⁸

VII. CONCLUSIONS

This review has concentrated on issues relating to the basic features of Kaluza–Klein theories in general. There has recently been a great deal of work on the eleven-dimensional version of supergravity^{11,12} which, in view of the intense scrutiny it has received, probably deserves more discussion than the brief mention we have given it. Also, one should be aware that most of the literature on Kaluza–Klein theories is much more mathematical than might be inferred from the present article; in particular, the geometry of fiber bundles²⁹ is ex-

tensively used.³⁰ Whether this kind of treatment illuminates or obscures the physics will depend on the background of the individual reader.

As of this writing, it is not possible to pronounce on the final place of the Kaluza–Klein idea in physics. There is no doubt that to many physicists it has great aesthetic appeal, but we must wait to see whether the ultimate description of Nature will involve extra dimensions of space or whether the Kaluza–Klein idea, as elegant as it appears, will turn out to have been just another pretty face that stands out for a moment in the crowd and then is gone.

ALAN CHODOS

*Department of Physics,
Yale University,
New Haven, Connecticut 06511*

References

1. G. Nordstrom, *Z. Phys.* **15**, 504 (1914).
2. Th. Kaluza, *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.* 966 (1921).
3. O. Klein, *Z. Phys.* **37**, 895 (1926).
4. A. Einstein and P. Bergmann, *Ann. Math* **39**, 683 (1983); A. Einstein, V. Bergmann and P. Bergmann, in *Theodor von Karman Anniversary Volume*, Pasadena (1941), p. 212; A. Einstein and W. Pauli, *Ann. Math.* **44**, 131 (1943); Y. Thiry, *J. Math Pure Appl.* **30**, 275 (1951); J.-M. Souriau, *Nuovo Cimento* **30**, 565 (1963); W. Thirring, *Acta Physica Austriaca Suppl.* IX, 256 (1972).
5. B. DeWitt in “Dynamical Theory of Groups and Fields” (Gordon and Breach, New York, 1965); R. Kerner, *Ann. Inst. Henri Poincaré* **9**, 143 (1968); A. Trautman, *Rep. Math. Phys.* **1**, 29 (1970).
6. E. Cremmer and B. Julia, *Phys. Lett.* **80B**, 48 (1978); *Nucl. Phys.* **B159**, 141 (1979).
7. More detailed versions of some of the results in this section can be found in: E. Witten, *Nucl. Phys. B* **186**, 412 (1981); T. Appelquist and A. Chodos, *Phys. Rev. D* **28**, 772 (1983). See also A. Salam and J. Strathdee, *Ann. Phys.* **141**, 316 (1982).
8. Y. M. Cho, *J. Math. Phys.* **16**, 2029 (1975); Y. M. Cho and P. G. O. Freund, *Phys. Rev. D* **12**, 1711 (1975); Y. M. Cho and P. S. Jang, *ibid.* **D 12**, 3789 (1975); L. N. Chang, K. I. Macrae, and F. Mansouri, *ibid.* **13**, 235 (1976).
9. S. Weinberg, *Phys. Lett.* **125B**, 265 (1983). See also M. J. Duff, C. N. Pope, and N. P. Warner, *Phys. Lett* **130B**, 254 (1983).
10. J. F. Luciani, *Nucl. Phys. B* **135**, 111 (1978); T. Horvath, L. Palla, E. Cremmer, and J. Scherk, *Nucl. Phys. B* **127**, 57 (1977); S. Randjbar-Daemi, A. Salam, and J. Strathdee, *Nucl. Phys. B* **214**, 491 (1983).
11. P. G. O. Freund and M. A. Rubin, *Phys. Lett.* **97B**, 233 (1980); F. Englert, *Phys. Lett.* **119B**, 339 (1982); B. Biran, F. Englert, B. de Wit, and H. Nicolai, *Phys. Lett.* **124B**, 45 (1983); F. Englert, M. Rooman, and P. Spindal, *Phys. Lett.* **127B**, 47 (1983).

12. M. J. Duff, Nucl. Phys. B **219**, 389 (1983); M. J. Duff, B. W. Nilsson and C. N. Pope, Phys. Rev. Lett. **50**, 2043 (1983); D. N. Page, Phys. Rev. D **28**, 2976 (1983); M. A. Awada, M. J. Duff, and C. N. Pope, Phys. Rev. Lett. **50**, 294 (1983).
13. T. Appelquist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983); T. Appelquist and A. Chodos, Phys. Rev. D **28**, 772 (1983); A. Chodos, "Quantum Aspects of Kaluza-Klein Theories," in *An Introduction to Kaluza-Klein Theories*, edited by H. C. Lee (World Scientific, Singapore, 1984).
14. M. A. Rubin and B. Barkan-Roth, Nucl. Phys. B **226**, 444 (1983).
15. T. Appelquist, A. Chodos, and E. Myers, Phys. Lett. **127B**, 51 (1983); T. Inami and O. Yasuda, Phys. Lett. **133B**, 180 (1983).
16. M. A. Rubin and B. Roth, Phys. Lett. **127B**, 55 (1983); K. Tsokos, Phys. Lett. **126B**, 451 (1983).
17. P. Candelas and S. Weinberg, Texas Preprint UTTG-6-83 (1983). See also Ya. I. Kogan and N. A. Voronov, JETP Lett. **38**, 311 (262) (1983).
18. S. Weinberg in *Proceedings of the Fourth Workshop on Grand Unification*, Birkhauser (Boston, 1983).
19. A. Chodos and E. Myers, *Annals of Physics* (to be published).
20. M. A. Rubin and C. R. Ordonez, in preparation.
21. A. Chodos and S. Detweiler, Phys. Rev. D **21**, 2167 (1980).
22. P. G. O. Freund, Nucl. Phys. B **209**, 146 (1982).
23. W. Marciano, Phys. Rev. Lett. **52**, 489 (1984).
24. S. Randjbar-Daemi, A. Salam, and J. Strathdee, Phys. Lett. **135B**, 388 (1984).
25. Q. Shafi and C. Wetterich, Phys. Lett. **129B**, 387 (1983).
26. E. Alvarez and M. Belen-Gavela, Phys. Rev. Lett. **51**, 931 (1983); D. Sahdev, Phys. Lett. (to be published); S. M. Barr and L. S. Brown, University of Washington preprint 40048-03, p. 4; R. B. Abbott, S. M. Barr, and S. D. Ellis, University of Washington preprint 40048-03, p. 4 (1984); Y. Okada, University of Tokyo preprint UT-429 (1984).
27. E. Witten, "Fermion Numbers in Kaluza-Klein Theory," Princeton preprint (October 1983).
28. S. Weinberg, "Quasi-Riemannian Theories of Gravitation in More than Four Dimensions," Texas preprint UTTG-1-84 (1984); C. Wetterich, "Dimensional Reduction of Fermions in Generalized Gravity," Bern preprint BUTP-84/5 (1984).
29. A fiber bundle is something that has been coughed up by a large cat.
30. For example, C. A. Orzalesi, Fortschr. der Physik, **29**, 413 (1981); R. Coquereaux, "Multi-dimensional Universes, Kaluza-Klein, Einstein Spaces and Symmetry Breaking," Marseille, CPT-83/P-1556 (December 1983); Y. M. Cho, Ref. 8.