Moreover, application of the definition of cavity fields as in Sec. III shows that the right side of Eq. (26) is also the electric field within a small cavity V_δ of static charge distribution ρ. That is, the Maxwellian and cavity electrostatic fields within a static charge distribution are identical for arbitrarily shaped cavities.

V. ARBITRARILY TIME-VARYING FIELDS

Equations (22) give the difference between time-harmonic Maxwellian and cavity fields within current or polarization density for arbitrarily shaped cavities. In addition, Eq. (26) proves that the Maxwellian and cavity electrostatic fields are identical within a continuous distribution of static charge. In Sec. V we derive expressions corresponding to Eqs. (22) for the difference between Maxwellian and cavity fields in source regions of arbitrary time dependence.

The expressions for arbitrary time variation can be derived simply by taking the Fourier transform of Eqs. (22), i.e., inserting $\exp(-i\omega t)$ time dependence and integrating over ω from $-\infty$ to $+\infty$, to get

$$\mathscr{E}(\mathbf{r},t) - \mathscr{E}_c(\mathbf{r},t) = -\mathsf{L}_\delta \cdot \mathscr{P}(\mathbf{r},t)/\epsilon_0, \tag{27a}$$

$$\mathscr{H}(\mathbf{r},t) - \mathscr{H}_c(\mathbf{r},t) = -\mathsf{L}_\delta \cdot \mathscr{M}(\mathbf{r},t). \tag{27b}$$

The script letters in Eqs. (27) denote the time-varying fields and polarization densities.

The only essential difference between these expressions (27) with arbitrary time variation and the time-harmonic expressions (22) is the absence of current density on the right side of Eq. (27a). The current density term that one obtains upon taking the Fourier transform of Eq. (22a) is proportional to the charge separation

$$-\int_{\Delta t} \mathscr{J}(\mathbf{r},t)dt \tag{28}$$

that occurs during the time interval Δt that the current in the cavity is removed.

However, to define cavity fields in a current density $\mathscr{J}(\mathbf{r},t)$ of arbitrary time variation, we have assumed that the current in the cavity is removed instantaneously ($\Delta t \to 0$). Thus the integral in Eq. (28) is zero and only the time-varying electric and magnetic polarization densities are left in Eqs. (27). That is, for arbitrary time variation with instantaneously removed cavities, unlike harmonic time dependence with stationary cavities, the Maxwellian electric field equals the cavity electric field (for arbitrarily shaped cavities) within a source region of volume current density.

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[1]D. Halliday and R. Resnick, *Physics* (Wiley, New York, 1978), 3rd ed., Chaps. 27 and 33.

[2]A. D. Yaghjian, Proc. IEEE **68**, 248 (1980).

[3]A. M. Portis, *Electromagnetic Fields: Sources and Media* (Wiley, New York, 1978).

[4]J. C. Maxwell, *A Treatise on Electricity and Magnetism* (Dover, New York, 1954), republication of 3rd ed. published by Clarendon, 1891.

[5]This difficulty has been pointed out in Refs. 2 and 7, and on p. 40 of Ref. 3.

[6]J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).

[7]H. B. Phillips, *Vector Analysis* (Wiley, New York, 1933), Secs. 55–57.

[8]A. D. Yaghjian, *National Bureau of Standards Technical Note 1000* (Boulder, Colorado, 1978), App. B.

[9]W. Kaplan, *Advanced Calculus* (Addison-Wesley, Reading, MA, 1952), Secs. 4–12.

Kaluza–Klein unified field theory and apparent four-dimensional space-time

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In the 1920s Kaluza and Klein achieved an elegant unified theory of gravitation and electromagnetism by assuming that space-time is really 5-dimensional. Their approach has since been extended to even higher dimensions in an effort to provide a geometrical unification of all the fundamental interactions. Any such scheme must answer the obvious objection emphasized by Einstein: Why, then, does space-time *appear* to be only 4-dimensional? This paper provides a semihistorical introduction to Kaluza–Klein unification on a level accessible to those with a basic knowledge of general relativity and particle physics, and examines the progress made in answering Einstein's, and related, objections.

I. INTRODUCTION: KALUZA–KLEIN UNIFICATION

In 1921, the mathematician Theodor Kaluza published an elegant attempt at a geometrical unification of electromagnetism and gravitation.[1] The crux of Kaluza's effort is the daring suggestion that space-time be considered to be $5 = (1 + 3 + 1)$-dimensional, with a line element given by[2]

$$ds^2 = \gamma_{ij} dx^i dx^j, \quad (i,j = 1,2,3,4,5). \tag{1.1}$$

To account for the fact that physical quantities appear only to change with respect to the usual $4 = (1 + 3)$-dimensional space-time manifold, the metric tensor γ_{ij} is required to satisfy

$$\gamma_{ij,5}=0, \tag{1.2}$$

i.e., it has a vanishing derivative with respect to the newly introduced fifth dimension. Kaluza called this requirement the "cylinder condition." Also, Kaluza took γ_{55} to be a constant normalized to 1. That these choices might lead to a unified geometry is suggested when one considers the Christoffel symbols[3] that result from condition (1.2). In particular, one finds with Kaluza that

$$[\alpha\beta,\delta]=\tfrac{1}{2}(\gamma_{\alpha\beta,\delta}-\gamma_{\beta\delta,\alpha}-\gamma_{\delta\alpha,\beta}),\quad(\alpha,\beta,\delta=1,2,3,4), \tag{1.3}$$

$$[5\beta,\delta]=\tfrac{1}{2}(\gamma_{5\beta,\delta}-\gamma_{5\delta,\beta}). \tag{1.4}$$

Equation (1.3) leads one to hope that the standard field equations of 4-dimensional general relativity will hold on the 4-manifold; Eq. (1.4) suggests setting the metric components $\gamma_{5\alpha}$ proportional to the components A_α of the electromagnetic 4-potential:

$$\gamma_{5\alpha}=\beta A_\alpha. \tag{1.5}$$

Then the electromagnetic field tensor is just

$$F_{\alpha\delta}=(2/\beta)[5\alpha,\delta], \tag{1.6}$$

with Eq. (1.4) thus "explaining" electromagnetic gauge invariance as well.

Indeed, using the above identifications in a weak field approximation (and making an auspicious choice of arbitrary constants), Kaluza shows that the 5-dimensional field equations

$$R_{ij}-\tfrac{1}{2}R\gamma_{ij}=\kappa T_{ij} \tag{1.7}$$

(where T_{ij} is now exclusive of electromagnetism) reduce to the usual Einstein equations when $i,j=\mu,\nu$, and the Maxwell equations when $i,j=\mu,5$ (the 5,5 component yields a trivial identity). Further, geodesics in the 5-manifold correspond to the usual (4-dimensional) paths of charged particles in a combined gravitational and electromagnetic field.[4] Thus at least to a weak field approximation, electromagnetism can be considered as part of the geometrical structure of a 5-dimensional space-time.

Kaluza's original inspiration was given a firmer basis by Oskar Klein in 1926.[5,6] Klein begins with Kaluza's Eqs. (1.1) and (1.2). Klein further requires that the coordinates x^μ characterize the usual 4-space-time; this means that they must transform between frames according to the usual 4-transformation law

$$x^\mu=f^\mu(\hat{x}^\nu). \tag{1.8}$$

Then demanding that $\gamma_{ij,5}$ vanish in all frames implies that x^5 transforms (to within a constant factor) according to[7]

$$x^5=\hat{x}^5+f^5(\hat{x}^\nu). \tag{1.9}$$

Thus the quantity

$$g_{\mu\nu}\equiv\gamma_{\mu\nu}-\beta^2\gamma_{\mu 5}\gamma_{\nu 5} \tag{1.10}$$

is independent of the x^5 coordinate and is a tensor under the transformation (1.8); it can thus be identified with the usual metric tensor for the 4-manifold.[8] Keeping the definition (1.5) for the 4-potential and setting $\gamma_{55}=1$, we have the 5-dimensional metric tensor

$$\gamma_{ij}=\begin{pmatrix} g_{\mu\nu}+\beta^2A_\mu A_\nu, & \beta A_\mu, \\ \beta A_\mu, & 1 \end{pmatrix}. \tag{1.11}$$

It is then straightforward to obtain field equations from the variational principle

$$0=\delta\int d^5x\sqrt{-\gamma}R_5, \tag{1.12}$$

where $\gamma\equiv\det(\gamma_{ij})$ and R_5 is the (5-dimensional) curvature scalar computed from the metric (1.11). Impressively, the usual Einstein and Maxwell equations for gravitation and electromagnetism in four dimensions result, provided we set $\beta=\sqrt{2\kappa}=\sqrt{16\pi G}$.

Finally, consider the behavior of the 4-potential under the particular change of coordinates $(x^\mu,x^5)\to[x^\mu,x^5+f^5(x^\nu)]$. Using Eq. (1.8) and the usual tensor transformation law for the metric, we have

$$\hat{\gamma}_{5\mu}=\gamma_{lm}\frac{\partial x^l}{\partial\hat{x}^5}\frac{\partial x^m}{\partial\hat{x}^\mu}=\gamma_{5\mu}+\gamma_{55}\frac{\partial f^5}{\partial\hat{x}^\mu} \tag{1.13}$$

or

$$A_\mu=\hat{A}_\mu-\frac{\partial f^5}{\partial\hat{x}^\mu}. \tag{1.14}$$

That is, an electromagnetic gauge transformation is now no more than the (purely geometrical) effect of a coordinate change through the fifth dimension.[9,10]

II. THE PERIODIC FIFTH DIMENSION

The "Kaluza–Klein" geometrization of electromagnetism and gravitation just described has been called "the first successful unified field theory."[11] Whether the theory represents more than an elegant curiosity remains unclear, however. Einstein's initial opinion of Kaluza's work was quite enthusiastic; in April 1919 he wrote Kaluza, "The idea that the electric field quantities are mutilated $[\alpha\beta,\delta]$ has also frequently and persistently haunted me. The idea, however, that this can be achieved through a five dimensional cylinder-world has never occurred to me... . I like your idea at first sight very much."[12] Several weeks later Einstein wrote again, remarking that "the formal unity of your theory is startling."[13]

Einstein[14] and others[9,15–19] worked on the Kaluza–Klein approach intermittently in the succeeding decades but with little additional unambiguous success. Einstein tried first to render Kaluza's fifth space-time coordinate less, and then more, physically real.[13,15] With hindsight, perhaps the most important contribution of the latter program was to make precise Klein's earlier suggestion[5,6] that space-time be periodic in the new fifth dimension: 5-space-time is to be thought of as homeomorphic to a "tube," the direct product of 4-space-time by a circle.[9,15,16,19] The components of the 5-metric tensor will therefore be periodic functions of x^5, so they can be written quite generally as[13,20–22]:

$$\gamma_{ij}(x^\mu,x^5)=\sum_{n=-\infty}^{+\infty}\gamma_{ij}^{(n)}(x^\mu)e^{inx^5/r_5}, \tag{2.1}$$

where the fifth dimension is a circle of radius r_5; with coordinate x^5 satisfying

$$0\leqslant x^5<2\pi r_5. \tag{2.2}$$

Thus we are taking the ground state of our 5-space-time to be the direct product $M^4\times S^1$ of 4-dimensional Minkowski space M^4 with the circle S^1. The (at first) apparently more "natural" choice of 5-dimensional Minkowski space M^5 as the "vacuum" of the 5-continuum would seem to lead nowhere; nor does it appear to answer[23] Einstein's ultimate criticism of higher-dimensional unified theories:

"In this case one must explain why the continuum is *apparently* restricted to four dimensions."[24] In the Kaluza–Klein approach, on the other hand, one simply asserts (or argues) that the radius r_5 of the circle is very small, perhaps only a few orders of magnitude larger than the Planck length $(\hbar G/c^3)^{1/2} \approx 1.6\times10^{-33}$ cm. Hence the fifth dimension is not observed in everyday experience—it will only "open up" at probe energies greater than hc/r_5, i.e., at energies almost on the order of the Planck mass.

Both M^5 and $M^4\times S^1$ trivially solve the 5-dimensional vacuum field equations (note that the circle S^1 is "flat," since in one dimension, $R_{abcd}=0$) without a cosmological term.[10,22,25] Classically, either is an equally appropriate choice for the ground state, as both have zero energy (although the two "zero energies" cannot be meaningfully compared).[10,26] The choice $M^4\times S^1$, however, yields a theory where space-time can indeed *appear* to be M^4 with the usual Poincaré (translation and Lorentz) symmetries. The components of the metric field associated with the fifth dimension (the components of the 4-potential) will appear in the 4-dimensional theory as the Maxwell electromagnetic field, with a local $U(1)$ gauge invariance which is the group of rotations of the circle S^1 existing at each "point" of M^4. The usual 4-dimensional theory is called the "dimensional reduction" of the higher-dimensional reality.

III. DIMENSIONAL REDUCTION

To see the appearance of the $\gamma_{5\mu}$ components of the 5-metric as a Maxwell field in M^4, consider standard general relativity in five dimensions, given by varying the Einstein–Hilbert action

$$S=-\frac{1}{16\pi G_5}\int d^5x\sqrt{-\gamma}R_5. \tag{3.1}$$

Here R_5 and G_5 are the curvature scalar and gravitational constant in five dimensions.[27] R_5 is to be computed from the 5-dimensional metric γ_{ij}. The Kaluza–Klein vacuum $M^4\times S^1$ is described by the metric of M^5,

$$\eta_{ij}=\begin{pmatrix}\eta_{\mu\nu}, & 0,\\ 0, & \phi_c\end{pmatrix} \tag{3.2}$$

[where $\eta_{\mu\nu}=\operatorname{diag}(+1,-1,-1,-1)$ is the metric of M^4, each "0" represents a (row or column) 4-vector of zeros, and ϕ_c is a negative constant], but with the additional manifold restriction Eq. (2.2). Note that the fifth dimension enters in a spacelike way ($\phi_c<0$). This must also be the case for the still higher dimensions we will later need to achieve unification with non-Abelian gauge fields: The number of timelike dimensions cannot be allowed to exceed one, or space-time will contain closed timelike lines,[28] which would lead to the usual causal anomalies.[29]

Clearly the interesting case is that for which gauge particles appear in the vacuum M^4. Such particles correspond to a curvature of the 5-dimensional flat space-time $M^4\times S^1$. We must therefore generalize (3.2) by allowing the components of η_{ij} to vary with the coordinates x^i. The 5-metric can then be parametrized in the form

$$\gamma_{ij}(x^\mu,x^5)=\phi^m\begin{pmatrix}g_{\mu\nu}+A_\mu A_\nu\phi, & A_\mu\phi,\\ A_\mu\phi, & \phi\end{pmatrix}, \tag{3.3}$$

where $g_{\mu\nu}$, A_μ, and ϕ may be functions of all five space-time coordinates. ϕ^m, where m is fixed but as yet unchosen, is a "Weyl" or conformal factor whose role will be apparent shortly. Note that (3.3) implies no restriction of the 5-dimensional metric γ_{ij}; clearly we can choose to write any 5-metric in this form. In particular, Eq. (3.3) is more general than the metric (1.11) considered by Klein, as γ_{55} is no longer taken to be constant. Such a generalization of the Kaluza–Klein 5-metric was first considered by Jordan[17] and Thiry[18]; we will see that formally ϕ plays the role of a Brans–Dicke[30] scalar.

Assuming the manifold to be closed in the x^5 direction corresponds to imposing the periodicity requirement (2.1) on the components of γ_{ij}. Thus particles are described by fields periodic in the interval (2.2). In a quantum picture,[10] they can be thought of as small oscillations about the ground state $M^4\times S^1$, and will be given by the different modes in the Fourier decomposition (2.1).

Complete "dimensional reduction" of the 5-dimensional theory (still topologically $M^4\times S^1$), to a theory of gauge particles within a 4-manifold, is achieved by taking the x^5-periodic fields to be only slowly varying in that dimension[15]:

$$2\pi r_5\frac{\partial\gamma_{ij}}{\partial x^5}\ll\gamma_{ij}(x^\mu,x^5). \tag{3.4}$$

Then the γ_{ij} are approximately functions of x^μ only. This corresponds to keeping only the $n=0$ mode $\gamma_{ij}^{(0)}(x^\mu)$ in the decomposition (2.1).

Substituting $\gamma_{ij}=\gamma_{ij}^{(0)}(x^\mu)$ into the action (3.1), we find[31]

$$S=-\frac{1}{16\pi G_5}\int d^5x[-(\det g_{\mu\nu})\phi^{5m+1}]^{1/2}R_5. \tag{3.5}$$

Note that $\gamma_{55}=\phi(x^\mu)$. Then with the distance δ_5 around the fifth dimension given by

$$\delta_5=\int_0^{2\pi r_5}dx^5\sqrt{-\gamma_{55}}=2\pi r_5[-\phi(x^\mu)]^{1/2}, \tag{3.6}$$

Eq. (3.5) becomes an action in only 4 dimensions,

$$S_4=-\frac{1}{16\pi G}\int d^4x\,\phi^{(5m+1)/2}\sqrt{-g}R_5, \tag{3.7}$$

where we have defined the 4-dimensional gravitational constant $G=G_5/2\pi r_5$. It remains to calculate R_5.

Using differential forms, Thiry[18] has calculated the 5-dimensional curvature $\hat{R}_5$ given by the metric

$$\hat{\gamma}_{ij}^{(0)}(x^\mu)=\begin{pmatrix}g_{\mu\nu}+A_\mu A_\nu\phi, & A_\mu\phi,\\ A_\mu\phi, & \phi\end{pmatrix}, \tag{3.8}$$

which is just $\gamma_{ij}^{(0)}(x^\mu)$ from (3.3) in the case $m=0$, i.e., without the Weyl factor. Thiry finds

$$\hat{R}_5=R_4+(2/\sqrt{\phi})(\sqrt{\phi})_{,\mu\nu}g^{\mu\nu}+\tfrac{1}{4}\phi F_{\mu\nu}F^{\mu\nu}, \tag{3.9}$$

where R_4 is the usual 4-curvature scalar computed from $g_{\mu\nu}$. Had we not used a Weyl factor in (3.7), this would result in the action

$$S_4=-\frac{1}{16\pi G}\int d^4x(\sqrt{\phi}R_4+\tfrac{1}{4}\phi^{3/2}F_{\mu\nu}F^{\mu\nu}), \tag{3.10}$$

where we have dropped a total divergence. Setting $\delta S=0$ then yields the field equations of a Brans–Dicke scalar-tensor theory of gravitation.[32] Such a theory in 4-space-time is a step backwards from unification, as gravitational effects are then no longer purely geometrical, but rather are described by a scalar field *within* a Riemannian manifold. In the higher-dimensional case, this criticism is less severe, as the scalar field ϕ is then just a component of the 5-metric: It too is geometrical in origin.

It is now clear why we introduced the Weyl factor ϕ^m in Eq. (3.3). We will show that choosing m properly allows us to eliminate any powers of ϕ multiplying R_4 in (3.10); the variation is then simpler to perform and yields the standard Einstein equations. Of course, since any 5-manifold can be quite generally parametrized as either γ_{ij} [Eq. (3.3)] or $\hat{\gamma}_{ij}$ [like Eq. (3.8)], the physical predictions of "both" theories must be the same when all terms in the action are considered.

To choose m, notice that the metrics $\gamma_{ij}^{(0)}(x^\mu)$ and $\hat{\gamma}_{ij}^{(0)}(x^\mu)$ are related conformally, $\gamma_{ij}^{(0)}(x^\mu) = \phi^m(x^\mu)\hat{\gamma}_{ij}^{(0)}(x^\mu)$. Henceforth we drop the superscript 0 and the argument x^μ. We then use the standard result[25] that two N-dimensional metrics related by the conformal transformation

$$\gamma_{ij} = \Omega^2\hat{\gamma}_{ij}, \quad \gamma^{ij} = \Omega^{-2}\hat{\gamma}^{ij} \tag{3.11}$$

yield curvature scalars related by

$$R_N = \Omega^{-2}\hat{R}_N - 2(N-1)\Omega^{-3}\Omega_{;ij}\hat{\gamma}^{ij} - (N-1)(N-4)\Omega^{-4}\Omega_{,i}\Omega_{,j}\hat{\gamma}^{ij}. \tag{3.12}$$

In our case $N = 5$ and $\Omega^2 = \phi^m$; substituting R_5 from (3.12) into (3.7) leads to the term $\phi^{(5m+1)/2}\times\phi^{-m}\times\hat{R}_5$, which we want to equal $\hat{R}_5$. Thus we take $m = -\frac{1}{3}$.

Inserting these values into (3.11) and (3.12) yields

$$\begin{aligned} R_5 &= \phi^{1/3}(\hat{R}_5 - \tfrac{5}{3}\phi^{-2}\phi_{,i}\phi_{,j}\hat{\gamma}^{ij} + \tfrac{4}{3}\phi^{-1}\phi_{,ij}\hat{\gamma}^{ij}) \\ &= \phi^{1/3}(\hat{R}_5 - \tfrac{5}{3}\phi^{-2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu} + \tfrac{4}{3}\phi^{-1}\phi_{,\mu\nu}g^{\mu\nu}), \end{aligned} \tag{3.13}$$

using $\phi = \phi(x^\mu)$. Substituting $\hat{R}_5$ from (3.9) into (3.13) and the resulting equation into (3.7), we find

$$S_4 = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R_4 + \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} - \frac{13}{6}\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi^2} + \frac{7}{3}\frac{\partial_\mu\partial^\mu\phi}{\phi}\right). \tag{3.14}$$

Dropping the total divergence from the last term gives the result[33]:

$$S_4 = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R_4 + \frac{1}{4}\phi F_{\mu\nu}F^{\mu\nu} + \frac{1}{6}\frac{\partial_\mu\phi\,\partial^\mu\phi}{\phi^2}\right), \tag{3.15}$$

where we recall that R_4 is the usual 4-dimensional curvature scalar computed from $g_{\mu\nu}$ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field tensor. That is, the dimensionally reduced action includes the usual Lagrangian density for 4-dimensional gravitation and electromagnetism,[34] plus a standard kinetic energy term for the scalar field ϕ.

Obviously, the decision to write γ_{ij} in the form (3.3) so that the usual terms would appear in (3.15) benefits from hindsight. Nevertheless, the resulting physics must be independent of the particular parametrization, and is in fact determined by our choice of ground state: Choosing $M^4\times S^1$ rather than M^5 leads to the mode decomposition (2.1) of the metric γ_{ij}; keeping only the zero mode then reduces a purely gravitational 5-dimensional theory to the standard 4-dimensional theories of Maxwell and Einstein. Speaking picturesquely, light is a manifestation of the fifth dimension.[11]

In quantum terms, the five physical degrees of freedom[35] of the metric γ_{ij} are accounted for by a spin-2 graviton (described by $g_{\mu\nu}$), a spin-1 photon (A_μ), and a Brans–Dicke scalar.[21] What of the higher modes? Since R_5 is composed of second derivatives of the metric, it is easy to see that the x^5 dependence of the nth mode in (2.1) will lead *inter alia* to kinetic energy terms with coefficients on the order of n^2/r_5^2. The associated particles will be massive and spin-2.[36] One can think of this "absorption" of the spin-0 and spin-1 degrees of freedom of the zero mode into a single (now massive) spin-2 particle in each nth mode as a Higgs mechanism.[21]

Thus the higher modes produce an infinite number of massive excitations, with the masses on the order of n/r_5. If r_5 is, say, around 10 Planck units, the masses of the associated modes will be about a tenth of the Planck mass $(c\hbar/G)^{1/2} \approx 2\times10^{19}$ GeV, or about 2 μg. Thus such excitations of the ground state $M^4\times S^1$ are so "heavy" that they almost never occur.[11] A different way of saying this is that typical 4-dimensional distances are so much larger than r_5 that the $n\neq0$ modes effectively decouple,[10,22] and Eq. (3.15) should be viewed as an *effective* action for physics at distance scales large enough for this approximation to hold.[20,37] Keeping only the zero mode in (2.1) is then well justified,[38] and the assumption (3.4) need not be independently made.

IV. THE "APPARENT" 4-DIMENSIONAL CONTINUUM

The formally impressive dimensional reduction just described begs several questions. We have already cited[24] Einstein's demand that the theory account for the *apparent* 4-dimensional nature of the continuum. It is comforting to note that, taking for granted the choice of ground state $M^4\times S^1$, one can argue that S^1 should indeed be unobservably small. The broader question of why the universe should look like $M^4\times S^1$ at all is more formidable, and will be addressed in Secs. VI and X.

Our first argument is due originally to Souriau,[9,39] although we shall follow a more recent procedure.[40] Souriau considers a flat 5-dimensional space-time $\eta_{ij} = \mathrm{diag}(+1,-1,-1,-1,-1)$ with small perturbations h_{ij} associated with the electromagnetic field. These are given by $h_{\mu5} = \sqrt{2\kappa}A_\mu$ ($\mu = 1,2,3,4$), and vanish otherwise. The x^5 coordinate is restricted as in (2.2).

Now define a real scalar quantum field ψ upon the underlying space-time. It is reasonable to expect it to satisfy a wave equation

$$\gamma_{ij}(\partial^i\partial^j + \alpha)\psi(x^\mu,x^5) = 0, \tag{4.1}$$

where $\gamma_{ij} = \eta_{ij} + h_{ij}$ and α is some real constant. Now assuming that ψ is periodic in x^5, we can write

$$\psi(x^\mu,x^5) = \sum_n \psi_n(x^\mu)e^{iq_nx^5}, \quad q_n = \frac{n}{r_5}. \tag{4.2}$$

Then (4.1) becomes

$$\gamma_{\mu\nu}\partial^\mu\partial^\nu\psi + 2\gamma_{\mu5}\,\partial^\mu\partial^5\psi + \gamma_{55}\partial^5\partial^5\psi + \alpha\psi = 0 \tag{4.3}$$

or

$$\Box\psi + 2iq_n\sqrt{2\kappa}A_\mu\,\partial^\mu\psi + (q_n^2 + \alpha)\psi = 0. \tag{4.4}$$

This is identical to the Klein–Gordon equation with the "minimal prescription" $\partial_\mu \to \partial_\mu + ie_nA_\mu$ to $O(e_n)$, provided we identify

$$e_n = q_n\sqrt{2\kappa} = (n\sqrt{16\pi G})/r_5, \tag{4.5}$$

$$m_n^2 = \alpha + q_n^2 = \alpha + n^2/r_5^2. \tag{4.6}$$

Setting e_1 equal to the electronic charge, we find the circumference $2\pi r_5 \approx 2.4\times 10^{-31}$ cm, which is quite consistent with the nonobservability of the fifth dimension.

Unfortunately, this procedure faces grave difficulties. The final term in Eq. (4.6) is on the order of 10^{41} MeV^2, so that α must be large, negative, and "fine tuned" to 20 decimal places[40] to put m_1 in the realistic range of 0.5 to 5×10^3 MeV. Yet our discussion of higher order modes in Sec. III suggests the natural choice is $\alpha = 0$. Worse, the fine tuning yields a negative, or tachyonic, mass in Eq. (4.1). Even if we accept this, a final disaster occurs.[39] The energy of a zero mode with momentum $\mathbf{p}$ is $E_0^2 = p^2 + m_0^2 = p^2 + \alpha$. With $\alpha < 0$ fixed by m_1, E_0 will be imaginary for $\mathbf{p}$ small. Thus unless one engages in somewhat exotic modifications,[41] the field theory is classically unstable, as there exist exponentially growing solutions to Eq. (4.1). Despite these problems, however, this approach is worth pursuing: We shall see in Sec. VI that it lends itself to a dynamical argument for the "compactification" of space-time into $1+3$ (large) + 1(small) dimensions.

A simpler argument is the original one by Klein.[6] Klein begins with the line element given by the metric (1.11),

$$ds^2 = (g_{\mu\nu} + \beta^2 A_\mu A_\nu)dx^\mu\, dx^\nu + 2\beta A_\mu\, dx^\mu\, dx^5 + (dx^5)^2, \tag{4.7}$$

where $\beta \equiv \sqrt{16\pi G}$. Then the 5-dimensional Lagrangian L for a particle of mass m and charge q is

$$L = \frac{1}{2}m\left(\frac{ds}{d\tau}\right)^2, \tag{4.8}$$

where $d\tau$ is the differential of proper time. Calculating the conjugate momenta p_μ and p_5 from the usual definition

$$p_i = \partial L/\partial(dx^i/d\tau), \tag{4.9}$$

we easily find the relation

$$p_\mu = mg_{\mu\nu}\frac{dx^\nu}{d\tau} + \beta A_\mu p_5, \tag{4.10}$$

which is just the usual expression for the momentum of a charged particle in an electromagnetic field, provided we identify $p_5 \equiv q/\beta$ (this is similar to Kaluza's[1] unification of energy–momentum–charge into a single "5-vector"). Now q must be an integral multiple of the electronic charge e, so we may write

$$p_5 = Ne/\beta. \tag{4.11}$$

de Broglie's relation for p_5 is $p_5 = 2\pi/\lambda_5$. If we require that a whole number M of wavelengths fit around the fifth dimension, we have $M\lambda_5 = 2\pi r_5$ or

$$p_5 = M/r_5, \quad (\hbar = c = 1). \tag{4.12}$$

Then (4.11) and (4.12) yield

$$r_5 = \beta/e = 1.3\times 10^{-31}\ \text{cm}, \tag{4.13}$$

where we have, with Klein, taken $M = N$. Equation (4.13) is then identical to the $n = 1$ mode of Eq. (4.5). The fifth dimension is indeed unobservably small, and is scaled by the Planck length $\sqrt{G}$.

Of course, if we knew from other considerations the circumference $2\pi r_5$, we could "explain" the numerical value of the charge quantum $e = (16\pi G)^{1/2}/r_5$. Indeed, in more general Kaluza–Klein theories, where additional dimensions allow unification with non-Abelian fields, the relevant coupling constants are again determined by ratios of β and circumferences of the compact dimensions.[42] We shall see such results later (Sec. IX), but first we must extend our discussion to non-Abelian interactions.

V. GRANDER UNIFICATION

In the 1960s it was realized that the Yang–Mills–Utiyama[43,44] non-Abelian generalization of local gauge theories could be unified with gravitation in a Kaluza–Klein-like manner.[45,46] By the 1970s, such accounts had been given a mathematically compelling form.[47–51] One begins with general relativity in $1+3+D$ dimensions, but takes the ground state to be $M^4\times B^D$ (where B^D is a compact space of dimension D), rather than M^{4+D} [$(4+D)$-dimensional Minkowski space]. As in the case $M^4\times S^1$, the symmetries (isometry group) of B^D will manifest themselves as the "internal" gauge symmetries of fields existing "within" M^4. A suitable choice of B^D will allow unification with any particular gauge group. For example, if $B^D = S^2$ [a 2-sphere, for which the group of isometries is SO(3)], then the gauge group will be SO(3) or, up to a local isomorphism, SU(2). As in the 5-dimensional theory, the "lack of direct tactile evidence for the extra dimensions"[45] will be explained by assuming that these higher dimensions have "compactified" to a very small size.

Presumably one ultimately wants to obtain either the gauge group SU(3) ⊗ SU(2) ⊗ U(1) or a GUT such as SU(5). Assuming we want the former, we must require that the symmetry group of B^D contains it at least as a subgroup. Occam's Razor[52] then suggests choosing B^D to be a manifold of minimum dimension containing SU(3) ⊗ SU(2) ⊗ U(1). It is not difficult to show[10] that B^D is then 7-dimensional.[53] Thus 11 is the *minimum* number of dimensions that the full manifold must have in order to accomodate the standard model. The "very intriguing numerical coincidence" that 11 also seems to be the *maximum* number of dimensions for supergravity theories urges that Kaluza–Klein theory in 11 dimensions be given serious consideration.[10]

Before embarking on such a program, several problems must be acknowledged. The first is that Witten[26] has shown that the Kaluza–Klein vacuum $M^4\times S^1$, while classically stable, is unstable against a process of semiclassical barrier penetration. Stability can be achieved, however, either by restricting excitations of $M^4\times S^1$ to ones in which the topology is unchanged, or by introducing elementary fermions into the Lagrangian. The stability of the general case $M^4\times B^D$ is not easy to evaluate.

A second problem which has motivated much recent work[37,54–59] is that the manifold $M^4\times B^D$ is not, in general, a solution to the vacuum Einstein equations in $4+D$ dimensions. This is easy to see.[22,37] The $(4+D)$-dimensional Einstein equations are

$$R_{ij} - \tfrac{1}{2}g_{ij}(R + \Lambda) = 0, \tag{5.1}$$

where Λ is the cosmological constant and i,j now run from 1 to $4+D$. The dimensionally reduced theory has $R_{\mu\nu} = 0$, with $\mu,\nu = 1,2,3,4$ labelling flat Minkowski space M^4. Then $R + \Lambda = 0$. This in turn requires $R_{ij} = 0$ for all indices, in particular $R_{AB} = 0$ for $A,B = 5,\ldots,D$. However, this is impossible if B^D has curvature, as is generally the case for $D > 1$.

A final difficulty that remains is the question of why the universe should treat three of the spatial dimensions so difffrently from the rest. One possible way to address both this and the preceding problem is via higher-dimensional cosmology.

VI. WHERE HAS THE FIFTH DIMENSION GONE?

In this section, we discuss preliminary cosmological models[40,60–62] which show that a $(4+D)$-dimensional universe might indeed expand in three spatial dimensions, while contracting in the others. Thus it is at least possible that a universe in which $3+D$ spatial dimensions entered initially in a symmetrical way might have evolved into its present dimensionally asymmetrical condition.

One evident way around the problem of solving the vacuum equations (5.1) in $N = 4 + D$ dimensions with a curved manifold is to instead solve the equation

$$R_{ij} - \tfrac{1}{2}g_{ij}(R + \Lambda) = 8\pi G_N T_{ij}, \tag{6.1}$$

where G_N is the N-dimensional gravitational constant and $i,j = 1,...,N$. A number of such cosmologies have been explored, with T_{ij} being the energy-momentum tensor of a higher-dimensional pure dust,[60] perfect fluid,[62] or homogeneous Friedman–Robertson–Walker cosmology.[61] Such models allow solutions in which all but three of the spatial dimensions contract to microscopic size. However, in the usual view of general relativity T_{ij} is a nongeometrical entity, and one can argue that its introduction into the N-dimensional field equations defeats the very purpose of Kaluza–Klein unification.[56]

Chodos and Detweiler[40] have instead considered the N-dimensional Kasner[63] solution for the vacuum equations $R_{ij} = 0$. While such a cosmology cannot hope to provide the topologies that a realistic Kaluza–Klein unification will require,[53] it does exhibit in a very simple context some of the essential behavior of the more elaborate models. In N dimensions, the Kasner solution is

$$ds^2 = dt^2 - \sum_{i=2}^{N} \left(\frac{t}{t_0}\right)^{2p_i} (dx^i)^2, \tag{6.2}$$

where the p_i satisfy the constraints

$$\sum_{i=2}^{N} p_i = 1 = \sum_{i=2}^{N} p_i^2. \tag{6.3}$$

We take each spatial dimension to be S^1, i.e., we impose $0 \leqslant x^i < 2\pi r_i$ $(i = 2,...,N)$. Consider for simplicity the 5-dimensional case. Then an auspicious choice of the p_i is

$$p_2 = p_3 = p_4 = \tfrac{1}{2}, \quad p_5 = -\tfrac{1}{2}, \tag{6.4}$$

giving the line element

$$ds^2 = dt^2 - (t/t_0)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] - (t_0/t)(dx^5)^2. \tag{6.5}$$

so at the time $t = t_0$, the universe had no preferred spatial dimensions; we take the distance around each to have been $2\pi r_i = L$. As t increased, the distance around the fifth dimension shrank like $(t_0/t)^{1/2}L$, with the other three growing as $(t/t_0)^{1/2}L$. Is $(t_0/t)^{1/2}L$ in fact now too small for the fifth dimension to be observed? To answer this, first rescale the coordinates so that (6.5) takes the form

$$ds^2 = dt^2 - (t/\tau)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] - (\tau/t)(dx^5)^2. \tag{6.6}$$

where $(t/\tau) \approx 1$, τ being a time characteristic of the present age of the universe. Hence x^5 must now satisfy

$$0 \leqslant x^5 < 2\pi r_5 (t_0/\tau)^{1/2} \equiv 2\pi r_5'(\tau). \tag{6.7}$$

Then the 5-metric (6.6) is essentially η_{ij}, which we can perturb as in our discussion of Souriau's results (Sec. IV). Analogously, Chodos and Detweiler find

$$e_n = \frac{n\sqrt{16\pi G}}{r_5'(\tau)} = \frac{n\sqrt{16\pi G}}{r_5}\left(\frac{\tau}{t_0}\right)^{1/2}, \tag{6.8}$$

or $(t_0/\tau)^{1/2} 2\pi r_5 = 2.4 \times 10^{-31}$ cm. Thus for purely cosmological reasons, the distance around the fifth dimension is at present very small (taking $\tau \approx t > t_0$). It is remarkable that Eq. (6.8) conforms to Dirac's large number hypothesis[64]: The ratio of the electromagnetic to gravitational coupling constants increases with the age of the universe.

The Kasner solution is readily extended to N dimensions.[40] However, neither this model, nor those previously mentioned, nor a higher-dimensional Jordan–Brans–Dicke theory,[65] will *require* a cosmological evolution where only three dimensions remain large: For example, in the Kasner cosmology, the choice of conditions (6.4) was crucial, but in no way required. A convincing treatment of these problems seems to require the introduction of $N = 11$ supergravity.[28,55,65] Before proceeding to this discussion (Sec. X), we first look more closely at Kaluza–Klein theory in N dimensions.

VII. KILLING VECTORS AND NON-ABELIAN GAUGE FIELDS

The Kaluza–Klein theory sketched in Secs. II and III can be thought of as a 5-dimensional Einstein theory with a spacelike Killing vector $K^i(\partial/\partial x^i) = \partial/\partial x^5$. Then M^4 is the space of equivalence classes of the 5-space-time under the group of motions of the Killing vector field. The isometry generated by this field appears as the $U(1)$ gauge symmetry in M^4.

To achieve the "grander unification" of Sec. V we take as ground state the $(N = 4 + D)$-dimensional manifold $M^4 \times B^D$, where B^D admits a group G of isometries generated by D Killing vector fields. These form a basis for the Lie algebra of G:

$$[K_A, K_B] = -f_{AB}^{C} K_C, \tag{7.1}$$

and in this discussion we take them to be complete: Their integral curves span the entire space B^D. Then M^4 is the quotient space $(M^4 \times B^D)/G$ of equivalence classes under G-transformations of $M^4 \times B^D$. In the dimensionally reduced theory, G will appear as a non-Abelian gauge group on M^4: Gauge invariance is really just a (higher dimensional) space-time invariance.

Let $g_{\mu\nu}$ $(\mu,\nu = 1,...,4)$ be the components of the metric tensor of M^4, and ϕ_{AB} $(A,B = 5,...,N)$ be the metric of B^D. Given these, we need to construct the N-dimensional metric γ_{ij} $(i,j = 1,...,N)$. γ_{ij} will have $\tfrac{1}{2}(20 + 9D + D^2)$ components. Knowledge of $g_{\mu\nu}$ and ϕ_{AB} gives $\tfrac{1}{2}(20 + D + D^2)$ of these. By analogy to Eq. (3.3) for the 5-dimensional case, it is natural to write γ_{ij} as

$$\gamma_{ij} = \phi^m \begin{pmatrix} g_{\mu\nu} + B_\mu^A B_\nu^B \phi_{AB}, & B_\mu^C \phi_{CA}, \\ B_\nu^C \phi_{CB}, & \phi_{AB} \end{pmatrix}, \tag{7.2}$$

with the $B_\mu^C \phi_{CA}$ providing the remaining $4D$ components. Here $\phi^m = (\det \phi_{AB})^m$ is a Weyl factor. Our goal is to find the dimensional reduction of the N-dimensional Einstein theory resulting from (7.2), given the imposed symmetries of (7.1). Analogously to the 5-dimensional case (Sec. III), the $\gamma_{\mu A}$ components of the metric (7.2) will become gauge fields in M^4. We proceed by varying the N-dimensional action

$$S = -\frac{1}{16\pi G_N}\int d^N x\sqrt{-\gamma}(R_N + \Lambda_N) \tag{7.3}$$

where $\gamma \equiv \det(\gamma_{ij})$, Λ_N is the cosmological constant in N dimensions, and R_N is the N-dimensional curvature scalar derived from (7.2).

We calculate (7.3) in Sec. IX. In Sec. VIII, we first verify that Eq. (7.2) is in fact a valid form for the N-dimensional metric γ_{ij}.[66] The reader in a hurry can accept (7.2) and omit Sec. VIII. In any case, our treatment will be only a heuristic sketch.[67]

VIII. SUPERSPACE

The manifold $M^4 \times B^D$ can be viewed as a fiber bundle,[68] or "superspace," which is (at least locally) the direct product of the "base" or space-time manifold M^4 and the "fiber" or group manifold B^D. It is reasonable to choose the basis $\{\mathbf{e}_A\}$ for B^D to be isomorphic to the algebra of the group (the Killing vectors, which span B^D). If we assume no *a priori* special properties for the basis $\{\mathbf{e}_\mu\}$ of M^4, we then have the following algebra:

$$[\mathbf{e}_A, \mathbf{e}_B] = -f^{\,C}_{AB}\mathbf{e}_C; \quad [\mathbf{e}_\mu, \mathbf{e}_A] = 0;$$
$$[\mathbf{e}_\mu, \mathbf{e}_\nu] = -F^{\,i}_{\mu\nu}\mathbf{e}_i. \tag{8.1}$$

Of course, we could choose a coordinate basis $\{\mathbf{h}_\mu\}$ for M^4, i.e., $\mathbf{h}_\mu = \partial/\partial x^\mu$. Then $[\mathbf{h}_\mu, \mathbf{h}_\nu] = 0$ and the algebra of the basis $\{\mathbf{h}_i\}$ of $M^4 \times B^D$ becomes:

$$[\mathbf{h}_A, \mathbf{h}_B] = -f^{\,C}_{AB}\mathbf{h}_C; \quad [\mathbf{h}_\mu, \mathbf{h}_A] = 0; \quad [\mathbf{h}_\mu, \mathbf{h}_\nu] = 0. \tag{8.2}$$

We will see shortly that this basis yields the metric (7.2).

We are given that $g_{\mu\nu}$ is the metric of M^4 and ϕ_{AB} is that for B^D, so we know that one possible basis $\{\mathbf{E}_i\}$ for $M^4 \times B^D$ is the one in which the N-dimensional metric tensor γ'_{ij} is block diagonal:

$$\gamma'_{\mu\nu} \equiv \mathbf{E}_\mu \cdot \mathbf{E}_\nu = g_{\mu\nu}, \quad \gamma'_{AB} \equiv \mathbf{E}_A \cdot \mathbf{E}_B = \phi_{AB},$$
$$\gamma'_{A\nu} \equiv \mathbf{E}_A \cdot \mathbf{E}_\nu = 0. \tag{8.3}$$

Take the group manifold basis $\{\mathbf{E}_A\}$ to be $\{\mathbf{h}_A\}$, and choose the projection $\Pi(\mathbf{E}_\mu)$ of $\mathbf{E}_\mu$ onto M^4 to be just $\mathbf{h}_\mu$. Then $\mathbf{E}_\mu$ is in general a linear combination of $\mathbf{h}_\mu$ and the $\mathbf{h}_A$. So projecting down the final commutation relation in (8.1) gives

$$\Pi([\mathbf{E}_\mu, \mathbf{E}_\nu]) = [\mathbf{h}_\mu, \mathbf{h}_\nu] = 0$$
$$= \Pi(-F^{\,i}_{\mu\nu}\mathbf{E}_i) = -F^{\,\alpha}_{\mu\nu}\mathbf{E}_\alpha. \tag{8.4}$$

Thus $F^{\,\alpha}_{\mu\nu} = 0$ and the $\{\mathbf{E}_i\}$ have the algebra

$$[\mathbf{E}_A, \mathbf{E}_B] = -f^{\,C}_{AB}\mathbf{E}_C; \quad [\mathbf{E}_\mu, \mathbf{E}_A] = 0;$$
$$[\mathbf{E}_\mu, \mathbf{E}_\nu] = -F^{\,A}_{\mu\nu}\mathbf{E}_A. \tag{8.5}$$

We can expand the basis vectors $\mathbf{h}_i$ in terms of the $\mathbf{E}_j$:

$$\mathbf{h}_A = h^{\,i}_{A}\mathbf{E}_i, \quad \mathbf{h}_\mu = h^{\,i}_{\mu}\mathbf{E}_i. \tag{8.6}$$

Then we have

$$-f^{\,C}_{AB}\mathbf{h}_C = [\mathbf{h}_A, \mathbf{h}_B] = h^{\,i}_{A} h^{\,j}_{B}[\mathbf{E}_i, \mathbf{E}_j]$$
$$= h^{\,C}_{A} h^{\,D}_{B}[\mathbf{E}_C, \mathbf{E}_D] + h^{\,\mu}_{A} h^{\,\nu}_{B}[\mathbf{E}_\mu, \mathbf{E}_\nu]$$
$$= -h^{\,C}_{A} h^{\,D}_{B} f^{\,E}_{AB}\mathbf{h}_E - h^{\,\mu}_{A} h^{\,\nu}_{B} F^{\,C}_{\mu\nu}\mathbf{h}_C, \tag{8.7}$$

giving

$$h^{\,C}_{A} = \delta^C_A, \quad h^{\,\mu}_{A} = 0. \tag{8.8}$$

Since $\{\mathbf{h}_\mu\}$ is a coordinate basis, we also have

$$h^{\,\nu}_{\mu} = \delta^\nu_\mu. \tag{8.9}$$

Now consider the basis $\{\mathbf{N}^i\}$ dual to $\{\mathbf{h}_i\}$:

$$\langle \mathbf{N}^i | \mathbf{h}_j \rangle = N^{\,i}_{k}\, h^{\,k}_{j} = \delta^i_j. \tag{8.10}$$

Using (8.8) and (8.9) in (8.10), the cases $i,j = A,B; \mu,B; \mu,\nu$; and A,ν yield

$$N^{\,A}_{B} = \delta^A_B, \quad N^{\,\mu}_{B} = 0, \quad h^{\,\mu}_{\nu} = N^{\,\mu}_{\nu} = \delta^\mu_\nu, \quad h^{\,A}_{\nu} = -N^{\,A}_{\nu}. \tag{8.11}$$

Then Eq. (8.6) becomes

$$\mathbf{h}_A = \mathbf{E}_A, \quad \mathbf{h}_\mu = \mathbf{E}_\mu + h^{\,A}_{\mu}\,\mathbf{E}_A. \tag{8.12}$$

Using (8.3) and (8.12), the components of the metric tensor γ_{ij} for the basis $\{\mathbf{h}_i\}$ are therefore

$$\gamma_{\mu\nu} \equiv \mathbf{h}_\mu \cdot \mathbf{h}_\nu = \mathbf{E}_\mu \cdot \mathbf{E}_\nu + h^{\,A}_{\mu}\,\mathbf{E}_A \cdot h^{\,B}_{\nu}\,\mathbf{E}_B = g_{\mu\nu}$$
$$+ h^{\,A}_{\mu} h^{\,B}_{\nu}\phi_{AB} = g_{\mu\nu} + N^{\,A}_{\mu} N^{\,B}_{\nu}\phi_{AB}, \tag{8.13}$$

$$\gamma_{AB} = \mathbf{h}_A \cdot \mathbf{h}_B = \mathbf{E}_A \cdot \mathbf{E}_B = \phi_{AB}, \tag{8.14}$$

$$\gamma_{\mu B} \equiv \mathbf{h}_\mu \cdot \mathbf{h}_B = h^{\,A}_{\mu}\phi_{AB} = -N^{\,A}_{\mu}\phi_{AB}. \tag{8.15}$$

Thus the metric for the "direct product basis" $\{\mathbf{h}_i\}$ does indeed take the form (7.2), with the identification $B^{\,C}_{\mu} = -N^{\,C}_{\mu}$.

IX. REDUCTION FROM N DIMENSIONS

Using the $(N = 4 + D)$-dimensional metric (7.2), we wish to evaluate the action (7.3) to obtain the dimensional reduction of the purely geometrical N-dimensional theory. First we note that (7.2) gives

$$\det(\gamma_{ij}) \equiv \gamma = \phi^{Nm}(\det g_{\mu\nu})(\det \phi_{AB}) = \phi^{Nm+1} g. \tag{9.1}$$

For simplicity, in the remainder of this discussion we drop the Weyl factor, i.e., set $m = 0$. It can be reinserted and chosen as desired[21] to simplify the final form of the dimensionally reduced action. Denoting the coordinates of the space-time manifold M^4 by $\{x^\mu\}$ and those of the D-dimensional manifold B^D by $\{y^A\}$, the action (7.3) can be written

$$S = \int d^4x \mathscr{L}_4(x), \tag{9.2}$$

where the Lagrangian density is

$$\mathscr{L}_4(x) = -\frac{1}{16\pi G_{4+D}} \times \int d^D y\sqrt{-g(x)}\,\sqrt{\phi(y)}\,(R_{4+D} + \Lambda_{4+D}), \tag{9.3}$$

and we have taken $g_{\mu\nu} = g_{\mu\nu}(x)$ and $\phi_{AB} = \phi_{AB}(y)$. Using the metric connection[3] $\Gamma^{\,j}_{ik} = -\gamma^{jl}[ik,l]$, it is straightforward to break the $(4 + D)$-dimensional curvature scalar R_{4+D} into three parts: the 4-curvature $R_4(x)$ calculated from $g_{\mu\nu}$, the D-curvature $R_D(y)$ from ϕ_{AB}, and a mixed term. In fact, one finds[20,42,47,50]

$$\mathscr{L}_4(x) = -\frac{1}{16\pi G_{4+D}} \times \int d^D y \sqrt{\phi(y)}\sqrt{-g(x)}\,[R_4(x) + R_D(y) + \Lambda_{4+D} + \tfrac{1}{4} G^A_{\mu\nu}\, G^B_{\alpha\beta}\, g^{\mu\alpha}(x) g^{\nu\beta}(x) \phi_{AB}(y)], \quad (9.4)$$

where

$$G^A_{\mu\nu} = B^A_{\mu,\nu} - B^A_{\nu,\mu} + B^B_\mu B^A_{\nu,B} - B^B_\nu B^A_{\mu,B}. \quad (9.5)$$

Now we use the assumed symmetries of B^D to convert (9.5) into the usual Yang–Mills "curl." Equation (7.1) is the standard Lie bracket:

$$[K_A(y), K_B(y)]^C \equiv K^D_A K^C_{B,D} - K^D_B K^C_{A,D} = -f^E_{AB} K^C_E(y). \quad (9.6)$$

We write

$$B^A_\mu = K^A_B(y) A^B_\mu(x,y), \quad (9.7)$$

which becomes in a zero-mode approximation:

$$B^A_\mu = K^A_B(y) A^B_\mu(x). \quad (9.8)$$

Then after a little algebra (9.5) takes the form

$$G^A_{\mu\nu} = K^A_B (A^B_{\mu,\nu} - A^B_{\nu,\mu} - f^B_{CD} A^C_\mu A^D_\nu). \quad (9.9)$$

Thus defining the curl

$$F^B_{\mu\nu}(x) \equiv A^B_{\mu,\nu} - A^B_{\nu,\mu} - f^B_{CD} A^C_\mu A^D_\nu, \quad (9.10)$$

the Lagrangian density (9.4) becomes

$$\mathscr{L}_4(x) = -\frac{1}{16\pi G_{4+D}} \times \int d^D y \sqrt{\phi(y)}\sqrt{-g(x)}\,[R_4(x) + R_D(y) + \Lambda_{4+D} + \tfrac{1}{4}\phi_{AB}(y) K^A_C(y) K^B_D(y) F^C_{\mu\nu}(x) \times F^D_{\alpha\beta}(y) g^{\mu\alpha}(x) g^{\nu\beta}(x)]. \quad (9.11)$$

This requires that the 4-dimensional gravitational constant G be defined via

$$\frac{1}{16\pi G} = \frac{1}{16\pi G_{4+D}} \int d^D y \sqrt{\phi(y)}, \quad (9.12)$$

and that $(\Lambda_{4+D})/16\pi G$ be chosen to cancel the integral over $R_D(y)$.[42] We also see that if we normalize the Killing vectors according to

$$\frac{1}{16\pi G_{4+D}} \int d^D y \sqrt{\phi(y)}\, \phi_{AB}(y) K^A_C(y) K^B_D(y) = \delta_{CD}, \quad (9.13)$$

the final term in (9.11) becomes the usual Yang–Mills Lagrangian

$$\mathscr{L}_{YM}(x) = -\tfrac{1}{4}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}^{\mu\nu}. \quad (9.14)$$

Thus (9.11) generalizes the results of Sec. III to provide non-Abelian gauge fields in M^4 via dimensional reduction.

Weinberg[42] has shown that Eq. (9.13) can be used to obtain a formula for the gauge coupling constant resulting from the isometry group of the compact manifold B^D. Each coupling g is proportional to the ratio of $2\pi\sqrt{16\pi G}$ to an appropriate rms circumference of the manifold. For example, if $B^D = S^2$, a 2-sphere of radius r [isometry group SU(2)], he finds

$$g = (\sqrt{16\pi G}/r)\,(\tfrac{3}{2})^{1/2}. \quad (9.15)$$

With $B^D = CP^2$ [4-dimensional complex projective space, isometry group SU(3)], Weinberg is able to derive the coupling ratio $g'/g = 1/\sqrt{3}$ for SU(3). Thus, at least in principle, Kaluza–Klein theory gives a geometrical account of the strengths of the fundamental interactions. To agree with the known values of the coupling constants, the scale r of the manifold B^D must be somewhat larger than the Planck length $\sqrt{G}$.

X. SPONTANEOUS COMPACTIFICATION AND SUPERGRAVITY

So far, we have only been able to argue that $r \approx \sqrt{G}$ by starting from the known values of the couplings g. A satisfying Kaluza–Klein theory would *predict* the values of g by accounting for the magnitude of r from other considerations. Recently, it has been shown[20–22,69–71] that a consideration of the quantum properties of Kaluza–Klein theories may provide such an account.

In this approach, one thinks of the Kaluza–Klein problem as a gravitational version of the Casimir effect in electrodynamics. In 1948, Casimir[72] showed that vacuum fluctuations of the electromagnetic field produce an attractive force between two parallel conducting plates. In five dimensions,[20–22] the Kaluza–Klein "plates" can be thought of as the boundaries of the x^5 coordinate, $x^5 = 0$ and $x^5 = 2\pi r_5$. Then if one does not make the zero-mode approximation in the metric (3.3), one finds an effective potential arising from the massive spin-2 excitations in the "cavity" $0 \leqslant x^5 < r_5$. This potential gives a force tending to make the distance around the fifth dimension contract to the order of the Planck length. At this level, the one-loop approximation used to derive the attractive force breaks down, and one hopes[21] that some sort of stabilization sets in. This analysis can be extended to $D > 1$ extra dimensions,[69,70] where it is found that the compact dimensions either expand or contract depending on their initial conditions.

These calculations proceed from the assumption that the topology of space-time is $M^4 \times B^D$. We are still faced with the objection raised in Sec. V that such a space-time is generally not a solution of the $(4 + D)$-dimensional vacuum Einstein equations (5.1). How to consistently achieve this "spontaneous compactification"[54] of the extra dimensions into a compact manifold B^D is the final problem to be addressed.

As in Sec. VI, we consider modifying the vacuum field equations. One approach is to include additional scalar invariants in the Einstein–Hilbert action. Wetterich[37] has shown that an action containing the invariants R^2, $R_{ij}R^{ij}$, and $R_{ijkl}R^{ijkl}$, in addition to R, yields vacuum field equations that are satisfied by a ground state manifold $M^4 \times S^D$, where S^D is a D-sphere. The addition of the extra invariants may be rationalized by invoking quantum fluctuations, and the usual higher-dimensional action (7.3) or field equations (5.1) can be interpreted as only *effective* equations for long distances. Of course, such a proposal faces all the usual objections to including additional scalars in the action.[73]

If we rule out *ab initio* such additional scalar terms, it seems the only remaining way to achieve compactification is to add a matter scalar to the total action (i.e., add matter fields to the field equations). We are then solving an equation like (6.1) rather than (5.1). It has been shown[54,56] that the addition of such terms does indeed result in field equations which admit solutions like $M^4 \times B^D$. Also, such an approach allows one to build spontaneous symmetry breaking into the theory.[54] Nevertheless, the objection that such an implementation of nongeometrical entities seems to defeat the whole purpose of Kaluza–Klein unification remains.

Such an objection cannot be made in supergravity theories, where bosonic fields other than gravity automatically appear.[28,55,56] The important example is $N = 11$ supergravity,[74] where supersymmetry requires the introduction of an antisymmetric rank-4 Maxwellian gauge field F^{ijkl}. The relevant field equations are

$$R^{mn} - \tfrac{1}{2}\gamma^{mn}R = -8\pi G\theta^{mn}, \tag{10.1}$$

$$\theta^{mn} = F^{m}_{ijk}\, F^{ijkn} - \tfrac{1}{8}F_{ijkl}F^{ijkl}g^{mn}, \tag{10.2}$$

$$(1/\sqrt{|\gamma|})\, \partial_i\, (\sqrt{|\gamma|}\, F^{ijkl}) = 0. \tag{10.3}$$

Freund and Rubin[28] have looked for solutions to these equations, where the 11-dimensional space-time is the product of 4- and 7-dimensional manifolds: $M^{11} = M^4 \times M^7$ (here M^4 and M^7 are not necessarily Minkowski space-times). The 11-dimensional metric γ_{mn} can then be written as

$$\gamma_{mn} = \begin{pmatrix} g_{\mu\nu}, & 0, \\ 0, & \phi_{AB} \end{pmatrix}, \tag{10.4}$$

where μ, $\nu = 1,...,4$ and A, $B = 5,...,11$. In this case, Eq. (10.3) admits the solution

$$F^{ijkl} = (f/\sqrt{|g|})\, \epsilon^{ijkl}. \tag{10.5}$$

Here $\epsilon^{ijkl} = \epsilon^{\mu\nu\alpha\beta}$ if $i = \mu,...,l = \beta$, and 0 otherwise; $\epsilon^{\mu\nu\alpha\beta}$ is totally antisymmetric; and f is a constant. We can define the scalar curvatures of M^4 and M^7 to be $R_4 = g^{\mu\nu}R_{\mu\nu}$ and $R_7 = \phi^{AB}R_{AB}$, so that (10.4) implies $R_{11} = R_4 + R_7$. If we assume that M^4 contains the lone timelike dimension, then contracting (10.1), using (10.2) and (10.5), yields

$$R_4 = \tfrac{8}{3}(8\pi G f^2), \quad R_7 = -\tfrac{7}{3}(8\pi G f^2) \tag{10.6}$$

(had we put the timelike dimension in M^7, the signs would have been reversed). Here a compact Riemannian manifold has negative scalar curvature. Thus $N = 11$ supergravity yields a preferential compactification of M^{11} into seven compact and four "large" space-time dimensions! (or vice versa).

XI. FINAL REMARKS

It is evident that Kaluza–Klein theory provides an elegant geometrical scheme for unifying the fundamental interactions. It faces many problems,[75] however, including a number related to dimensional reduction itself. In the 5-dimensional theory, we have seen (Sec. IV) that assuming the ground state to be $M^4 \times S^1$ can explain why the universe *appears* only 4-dimensional: The circumference of S^1 is on the order of the Planck length divided by the electromagnetic coupling constant. Similar relationships hold in the theory's higher-dimensional extensions (Sec. IX), where the vacuum is taken to be $M^4 \times B^D$. However, cosmological schemes in which a universe begins with such a ground state and evolves to one in which only three spatial dimensions remain macroscopic are not satisfactory (Sec. VI). Worse, $M^4 \times B^D$ is generally not even a solution to the $(4 + D)$-dimensional vacuum field equations. Inserting matter fields into the equations by hand seems to be a self-defeating approach to solving these problems. It may be that supergravity provides an answer.

It was noted in Sec. V that 11 is the minimum number of dimensions needed to yield a reduction to $SU(3) \otimes SU(2) \otimes U(1)$ gauge symmetry "within" M^4. Eleven is also the maximum for supergravity. Further, $N = 11$ supergravity naturally solves the compactification problem, giving a space-time-like $M^4 \times M^7$, with one or the other (M^7, one hopes!) compact. Why the compact space-time should have precisely the isometries needed to give, say, $SU(3) \otimes SU(2) \otimes U(1)$ gauge symmetry to M^4, and how M^7 has evolved, remain unanswered. However, it has been shown[65] that an early universe cosmology for $N = 11$ supergravity admits a solution in which 3-space (in M^4) expands much faster than M^7. One might even hope that quantum effects explain why M^7 contracts to the order of $\sqrt{G}$.

Of course, it is always possible that the numerical "coincidences" which arise in the combination of supergravity with Kaluza–Klein theory, and which seem to point the way towards solving some outstanding difficulties of the latter, are merely fortuitous. Might the appearance of gauge fields via dimensional reduction be so as well? Perhaps, but as Kaluza[1] remarked in 1921, "it appears hard to believe that those relations, hardly to be surpassed in their formal correspondence, are nothing but an alluring play of whimsical chance."

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[1] Th. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. **K1**, 966 (1921). English trans. by C. Hoenselaers in *Unified Field Theories of More than 4 Dimensions*, edited by V. De Sabbata and E. Schmutzer (World Scientific, Singapore, 1983), p. 427.

[2] Our notation is as follows: The usual 4-dimensional manifold has signature $(+,-,-,-)$, and is labelled with greek indices which run from 1–4. It has metric tensor $g_{\alpha\beta}$. Similarly, the 5-manifold $(+,-,-,-,-)$ is labelled by latin indices running from 1–5, with metric γ_{ij}. The obvious summation convention applies for either repeated latin or greek indices. Later, latin indices will run from 1–N dimensions.

[3] We use the notation $[ik,l] = \tfrac{1}{2}(\gamma_{ik,l} - \gamma_{kl,i} - \gamma_{li,k})$ for Christoffel symbols of the first kind; Christoffel symbols of the second kind (the metric connection) are then $\Gamma^j_{ik} = -\gamma^{jl}[ik,l]$.

[4] Showing that the 5-dimensional theory yields this result was suggested by Einstein in a letter to Kaluza in 1919; Einstein made his communication of Kaluza's work to the Prussian Academy for publication in the Sitzungsberichte conditional on it meeting this test; see "Einstein–Kaluza Correspondence," Ref. 1, p. 447.

[5] O. Klein, Z. Phys. **37**, 875 (1926); see Ref. 1, p. 434.

[6] O. Klein, Nature **118**, 516 (1926).

[7] $\gamma_{ij,5}$ would generally vary from frame to frame under arbitrary coordinate transformations. However, with the restrictions (1.8) and (1.9), and

given $\partial\hat{\gamma}_{kl}/\partial\hat{x}^5=0$ in any one frame, we find

$$0=\partial/\partial\hat{x}^5\left(\gamma_{ij}\frac{\partial x^i}{\partial\hat{x}^k}\frac{\partial x^j}{\partial\hat{x}^l}\right)=\frac{\partial\gamma_{ij}}{\partial\hat{x}^5}\frac{\partial x^i}{\partial\hat{x}^k}\frac{\partial x^j}{\partial\hat{x}^l}=\frac{\partial\gamma_{ij}}{\partial x^5}\frac{\partial x^i}{\partial\hat{x}^k}\frac{\partial x^j}{\partial\hat{x}^l},$$

which implies $(\partial\gamma_{ij}/\partial x^5)=0$, since $(\partial x^i/\partial\hat{x}^k)(\partial x^j/\partial\hat{x}^l)\neq 0$ in general. That is, $\gamma_{ij,5}=0$ in all frames, if it vanishes in one.

[8]To motivate Eq. (1.10), notice that it is merely an extension of the usual formula relating the metric of real 3-space to the metric of 4-dimensional space-time. See, e.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), Sec. 84.

[9]J.-M. Souriau, Nuovo Cimento **30**, 565 (1963).

[10]E. Witten, Nucl. Phys. **B186**, 412 (1981).

[11]B. DeWitt, Sci. Am. **249**, 104 (1983).

[12]"Einstein–Kaluza Correspondence," Refs. 4 and 1.

[13]A. Pais, *'Subtle is the Lord...,' The Science and Life of Albert Einstein* (Oxford U. P., New York, 1982).

[14]A comprehensive outline of Einstein's work in this area is given in the biography by A. Pais, Ref. 13, Chap. 17.

[15]A. Einstein and P. Bergmann, Ann. Math. **39**, 683 (1938).

[16]A review of a variety of classical attempts at unification through modified space-time geometry is given by P. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, New York, 1947), Chaps. 16–18.

[17]P. Jordan, Z. Phys. **157**, 112 (1959) and references therein.

[18]Y. Thiry, C. R. Acad. Sci. (Paris) **226**, 216 (1948).

[19]The work of Kaluza, Klein, Jordan, and Thiry is analyzed by A. Lichnerowicz, *Théories Rélativistes de la Gravitation et de l'Electromagnétisme* (Masson et Cie, Paris, 1955), book II.

[20]T. Applequist and A. Chodos, Phys. Rev. Lett. **50**, 141 (1983).

[21]T. Appelquist and A. Chodos, Phys. Rev. D **28**, 772 (1983).

[22]T. Appelquist, Yale preprint YTP83-19 (1983). To appear in *Proceedings of the 1983 Annual Meeting of the Division of Particles and Fields of the American Physical Society*, Blacksburg, Virginia, 15–17 September 1983.

[23]However, as we shall see in Sec. VI, there are cosmologies which locally at first resemble M^5 but then evolve to look like $M^4\times S^1$.

[24]A. Einstein, *The Meaning of Relativity* (Princeton U. P., Princeton, NJ, 1956), 5th ed., Appendix II.

[25]S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* (Cambridge U. P., Cambridge, England, 1973), Sec. 2.6.

[26]E. Witten, Nucl. Phys. **B195**, 481 (1982).

[27]R_5, given by second derivatives of the dimensionless metric γ_{ij}, has units $(\text{length})^{-2}$. For S to be dimensionless (we have $c=\hbar=1$), the units of G_5 must then be $(\text{length})^{-3}$.

[28]P. Freund and M. Rubin, Phys. Lett. **97B**, 233 (1980).

[29]A general discussion is given by L. Sklar, *Space, Time, and Spacetime* (Univ. of Calif. P., Berkeley, 1976), Chap. IV.

[30]C. Brans and R. Dicke, Phys. Rev. **124**, 925 (1961); see also S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Sec. 7.3.

[31]Simple row reduction on the metric (3.3) brings it into the form

$$\phi^m\begin{pmatrix} g_{\mu\nu}, & 0, \\ A_\mu\phi, & \phi \end{pmatrix},$$

which yields the determinant $\phi^{5m}\times\phi\times\det(g_{\mu\nu})$.

[32]Such a calculation is performed in detail by R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1975), Sec. 11.5.

[33]This is indeed the result cited in Refs. 20–22.

[34]This is the "Kaluza–Klein miracle." (Ref. 20).

[35]In n dimensions, the symmetric $n\times n$ metric tensor γ_{ij} has $\frac{1}{2}n(n+1)$ independent components. n of these components are arbitrary, as there are n degrees of freedom to make coordinate transformations without altering the space-time geometry. In addition, the n components γ_{i0} of the metric can be determined via the field equations from the initial conditions of specifying the γ_{kl} $(k,l\neq 0)$ on a spacelike hypersurface. Thus there are $\frac{1}{2}n(n+1)-n-n$ degrees of freedom, or $15-10=5$ in five dimensions. See Ref. 25, Sec. 3.4, or Ref. 21.

[36]A. Salam and J. Strathdee, Ann. Phys. **141**, 316 (1982), Appendix V.

[37]C. Wetterich, Phys. Lett. **113B**, 377 (1982).

[38]Nevertheless, the massive excitations are not always negligible: Their contribution to scattering amplitudes is of the same order as the contributions from graviton exchange (Ref. 36).

[39]S. Unwin, Phys. Lett. **103B**, 18 (1981).

[40]A. Chodos and S. Detweiler, Phys. Rev. D **21**, 2167 (1980).

[41]The instabilities can be eliminated if we are willing to admit the "perverse structure" of "twisted" real scalar field configurations. See Ref. 39.

[42]S. Weinberg, Phys. Lett. **B125**, 265 (1983).

[43]C. Yang and R. Mills, Phys. Rev. **96**, 191 (1954).

[44]R. Utiyama, Phys. Rev. **101**, 1597 (1956).

[45]B. DeWitt, *Dynamical Theory of Groups and Fields* (Blackie & Son, London, 1965), prob. 77. Published originally in *Lectures at 1963 Les Houches School, Relativity, Groups and Topology,* edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).

[46]A. Trautman, Rep. Math. Phys. **1**, 29 (1970), Sec. 7; R. Kerner, Ann. Inst. Henri Poincaré **9**, 143 (1968).

[47]Y. Cho and P. Freund, Phys. Rev. D **12**, 1711 (1975).

[48]Y. Cho, J. Math. Phys. **16**, 2029 (1975).

[49]Y. Cho and P. Jang, Phys. Rev. D **12**, 3789 (1975).

[50]L. Chang, K. Macrae, and F. Mansouri, Phys. Rev. D **13**, 235 (1976).

[51]C. Orzalesi, Fortschr. Phys. **29**, 413 (1981).

[52]For the record, William of Ockham (1300?–1349) was the English philosopher who warned against invoking moonmen in order to explain lunar features: "Beings ought not to be multiplied," he admonished, "except out of necessity," See J. Jackson, *Pictorial Guide to the Planets* (Thomas Crowell, New York, 1965).

[53]Witten (Ref. 10) gives the following construction: The circle S^1 has U(1) symmetry, the 2-sphere S^2 is the lowest-dimensional space with SU(2) symmetry, and the space of lowest dimension (4, in fact) with symmetry group SU(3) is the complex projective space CP^2. So $CP^2\times S^2\times S^1$ is a 7-manifold with SU(2) ⊗ SU(3) ⊗ U(1) symmetry. There are also many other 7-manifolds which have this symmetry.

[54]E. Cremmer and J. Scherk, Nucl. Phys. **B108**, 409 (1976).

[55]F. Englert, Phys. Lett. **B119**, 339 (1982).

[56]C. Tze, Phys. Lett. **B128**, (1983), and references therein.

[57]M. Awada and D. Toms, Phys. Lett. **B135**, 283 (1984).

[58]M. Duff, B. Nilsson, and C. Pope, Nucl. Phys. **B233**, 433 (1984).

[59]For a review of much of this work, see E. Cremmer, in *Supergravity 1981*, edited by S. Ferrara and J. Taylor (Cambridge U. P., Cambridge, England, 1982).

[60]R. Bergamini and C. A. Orzalesi, Phys. Lett. **B135**, 38 (1984).

[61]S. Randjbar-Daemi, A. Salam, and J. Strathdee, Phys. Lett. **B135**, 388 (1984).

[62]D. Sahdev, Phys. Lett. **B137**, 155 (1984).

[63]E. Kasner, Am. J. Math. **43**, 217 (1921). See also Ref. 8, Sec. 117.

[64]P. Dirac, Proc. R. Soc. London Ser. A **165**, 199 (1938); **365**, 19 (1979).

[65]P. Freund, Nucl. Phys. **B209**, 146 (1982).

[66]Section VIII derives from the presentation in Ref. 50.

[67]A mathematically rigorous discussion is given in Ref. 48. Other derivations are in Refs. 47, 49, and 59.

[68]An introduction to fiber bundles can be found in B. Schutz, *Geometrical Methods of Mathematical Physics* (Cambridge U. P., Cambridge, England, 1980), Secs. 2.9–2.11 and 6.14, and Ref. 25, Sec. 2.9.

[69]T. Inami and O. Yasuda, Phys. Lett. **133B**, 180 (1983).

[70]T. Appelquist, A. Chodos, and E. Myers, Phys. Lett. **127B**, 51 (1983).

[71]A Chodos and E. Myers, Yale preprint YTP83-18 (1983).

[72]H. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).

[73]Briefly, adding such scalars as $R_{ij}R^{ij}$ to the action would yield field equations involving the fourth derivative of the metric tensor. To determine the evolution of the metric from such equations, one would first have to specify the metric up through its third derivative as initial conditions. Yet all other equations of physics seem to be no higher than first or second order. See Ref. 25, Sec. 3.4.

[74]E. Cremmer, B. Julia, and J. Scherk, Phys. Lett. **76B**, 409 (1978).

[75]In addition to those seen here, Kaluza–Klein theory also seems not to provide a realistic fermion spectrum. See Ref. 10.