Infinitely Large New Dimensions

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We construct intersecting brane configurations in anit–de Sitter (AdS) space which localize gravity to the intersection region, generalizing the trapping of gravity to any number n of infinite extra dimensions. Since the 4D Planck scale M_{Pl} is determined by the fundamental Planck scale M_* and the AdS radius L via the familiar relation $M_{\mathrm{Pl}}^2 \sim M_*^{2+n} L^n$, we get two kinds of theories with TeV scale quantum gravity and submillimeter deviations from Newton's law. With $M_* \sim$ TeV and $L \sim$ submillimeter, we recover the phenomenology of theories with large extra dimensions. Alternatively, if $M_* \sim L^{-1} \sim M_{\mathrm{Pl}}$, and our 3-brane is at a distance of $\sim 100 M_{\mathrm{Pl}}^{-1}$ from the intersection, we obtain a theory with an exponential determination of the weak/Planck hierarchy.

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Unification of gravity with other forces of nature suggests that the world has more than three spatial dimensions. Since only three of these are presently observable, one has to explain why the additional ones have eluded detection. The conventional explanation is that the dimensions are compactified with tiny radii of order the Planck length $\sim 10^{-33}$ cm, which makes them impossible to probe with currently available energies.

It has recently been pointed out that new dimensions may have a size R much larger than the fundamental Planck length of the theory, perhaps as large as a millimeter [1,2]. This has the effect of diluting the strength of the 4D gravity observed at distances much larger than R. The 4D Planck scale M_{Pl} is determined by the fundamental Planck scale M_* via Gauss's law $M_{\mathrm{Pl}}^2 \sim M_*^{2+n} R^n$ where n is the number of new dimensions. The original motivation was to bring the fundamental gravitational scale close to the weak scale in order to solve the hierarchy problem. These large dimensions are not in conflict with experiment if the standard model fields are confined to a 3-brane in the extra dimensions.

In this scenario, the only reason to compactify the extra dimensions at all is to reproduce 4D Newtonian gravity at long distances. One can wonder if even this is necessary: if gravity itself is somehow "trapped" to our 3-brane, then 4D gravity can be reproduced even if the extra dimensions are infinitely large. A very interesting recent construction by Gogberashvili [3] and by Randall and Sundrum [4] provides an explicit realization of this idea for the case of one extra dimension. Solving Einstein's equations with a 3-brane in $(4 + 1)$ dimensions, together with a bulk cosmological constant, they find a massless 4D graviton localized to the 3-brane. The resulting gravitational potential between any two objects on the brane is inversely proportional to the distance between the objects, and not its square, despite the presence of the infinite fifth dimension.

It is clearly desirable to extend this idea to any number of new dimensions. At first sight, however, the mechanism of [3,4] seems to rely on the peculiar properties of codimension one objects in gravity and seems hard to extend to the case of more dimensions. However, all that seems to be required is the presence of *some* codimension one branes in the system, while our 3-brane can have larger codimension. We are led to consider a system of n mutually intersecting $(2 + n)$ branes in $(3 + n) + 1$ dimensions with a bulk cosmological constant. The branes intersect on three spatial dimensions, where the standard model fields reside. Intuitively, each of the $(2 + n)$ branes has codimension one and tries to localize gravity to itself. Therefore gravity will be localized to the intersection of all the branes. We will now confirm this intuition by explicit calculations.

We begin by deriving the solution describing the intersection of branes. Consider an array of n orthogonal $n + 2$-spatial dimensional branes in $(3 + n) + 1$ dimensions, with a bulk cosmological constant Λ. For simplicity we take the branes to have identical tension σ. The field equations can be derived from the action

$$S = \int_M d^{4+n}x \sqrt{g_{4+n}} \left(\frac{1}{2\kappa_{4+n}^2} R + \Lambda \right) - \sum_{k=1}^{n} \int_{k\mathrm{th\,brane}} d^{3+n}x \sqrt{g_{3+n}}\, \sigma \,. \qquad (1)$$

Here $\kappa_{4+n}^2 = 8\pi / M_*^{n+2}$, where M_* is the fundamental scale of the theory. Note that the measure of integration differs between each brane, and between the branes and the bulk. This will be reflected in the field equations, where ratios $\sqrt{g_{3+n}}/\sqrt{g_{4+n}}$ weigh the δ-function sources. After the standard Euler-Lagrange variational procedure, the field equations are

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$$R^a_b - \frac{1}{2}\,\delta^a_b R = \kappa^2_{4+n}\Lambda\delta^a_b - \frac{\sqrt{g_{3+n}(1)}}{\sqrt{g_{4+n}}}\,\kappa^2_{4+n}\sigma\,\delta(\bar z^1)\mathrm{diag}(1,1,1,1,0,\ldots,1,1) - \cdots - \frac{\sqrt{g_{3+n}(n)}}{\sqrt{g_{4+n}}}\,\kappa^2_{4+n}\sigma\,\delta(\bar z^n)\mathrm{diag}(1,1,1,1,1,\ldots,10)\,, \tag{2}$$

where the coordinates $\bar z^k$ parametrize the extra dimensions. We note that the ratios $\sqrt{g_{3+n}(k)}/\sqrt{g_{4+n}}$ reduce to $\sqrt{g^{kk}}$ for diagonal metrics. In general, they cannot be gauged away.

It is now straightforward to write down the solutions. Away from the branes, the solution in the bulk comprises patches of the $(4+n)$-dimensional anti–de Sitter (AdS) space. Hence if the branes are mutually orthogonal, by symmetry the full solution simply consists of 2^n identical patches of the AdS_{n+4} which fill up the higher-dimensional quadrants between the branes, and are glued together along the branes. To construct it, we start with the Poincaré half-plane parametrization of AdS_{n+4}, given by

$$ds^2_{n+4} = \frac{L^2}{z^2}\,(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + d\vec w^2_{n-1} + dz^2)\,. \tag{3}$$

The length scale L is determined by the bulk cosmological constant as

$$L^2 = \frac{(n+3)(n+2)}{2\kappa^2_{4+n}\Lambda}\,. \tag{4}$$

To make use of the symmetry, it is convenient to find the patch of (3) where the metric is manifestly symmetric under permutations of all extra dimensions. This is most easily accomplished by an $O(n)$ rotation of $\vec w_{n-1}, z$. We transform to new coordinates $\bar z^k$, $k \in \{1,\ldots,n\}$ by a rigid rotation chosen such that $z = \sum_{j=1}^n \bar z^j/\sqrt{n}$. In terms of these coordinates, the metric is

$$ds^2_{n+4} = \frac{nL^2}{(\sum_{j=1}^n \bar z^j)^2}\left(\eta_{\mu\nu}\,dx^\mu\,dx^\nu + \sum_{k=1}^n (d\bar z^k)^2\right). \tag{5}$$

The metric (5) covers a segment of the extra dimensions bounded by the branes. Led by our discussion above, we will take such a cell of AdS_{n+4} with, e.g., all the $\bar z^j > l$ for some l. We then fill out the rest of the space by reflecting this cell in all 2^n distinct ways about its boundaries. The resulting metric is given by replacing $\sum_j \bar z^j$ by $\sum_j |\bar z^j| + l$ in Eq. (5). By rescaling $x, \bar z$, we can set l to any value we wish, and we use this freedom to put the metric in the final form

$$ds^2_{n+4} = \frac{1}{(k\sum_{j=1}^n |\bar z^j| + 1)^2} \times \left(\eta_{\mu\nu}\,dx^\mu dx^\nu + \sum_{k=1}^n (d\bar z^k)^2\right), \tag{6}$$

where $k \equiv (\sqrt{n}\,L)^{-1}$. [For $n = 1$, the solution coincides with that given by [3,4] if we make the coordinate transformation $L\exp(|y|/L) = |z| + L$, and $k = 1/L$.] This choice corresponds to setting the conformal factor in (6) to unity at the intersection $\bar z^k = 0$. Physically this means that the unit of length on the intersection is set by M_*^{-1}. In this metric, each $\bar z^j$ is allowed to vary on the whole real line. The curvature of the space will now have singularities at the seams where we have pasted together the elementary cells, but these will be precisely those dictated by the presence of the branes.

It is straightforward to see this explicitly. The metric is a conformal transformation of flat space $g_{ab} = \Omega^2\eta_{ab}$ where

$$\Omega = \frac{1}{k\sum_j |\bar z^j| + 1} \tag{7}$$

is the "warp factor." We can trivially compute the Einstein tensor $G_{ab} = R_{ab} - 1/2 g_{ab}R$ using the standard relation (in general for $g_{ab} = \Omega^2\tilde g_{ab}$ for D spacetime dimensions)

$$G_{ab} = \tilde G_{ab} + (D-2)(\tilde\nabla_a \log\Omega\,\tilde\nabla_b\log\Omega - \tilde\nabla_a\tilde\nabla_b\log\Omega) + (D-2)\tilde g_{ab}\left(\tilde\nabla^2\log\Omega + \frac{D-3}{2}(\tilde\nabla\log\Omega)^2\right). \tag{8}$$

Using this it is easy to compute the Einstein tensor for our metric, and we find

$$G^a_b = \frac{n(n+2)(n+3)k^2}{2}\,\delta^a_b - \frac{2(n+2)k}{\Omega}\,\delta(\bar z^1)(1,1,1,1,0,1,\ldots,1) - \cdots - \frac{2(n+2)k}{\Omega}\,\delta(\bar z^n)(1,1,1,1,1,1,\ldots,0)\,, \tag{9}$$

which reproduces Eq. (2) if the brane tension σ is chosen to be $\kappa_{4+n}^2 \sigma = 2(n+2)k$. Using Eq. (4), we can rewrite this condition as $\kappa_{4+n}^2 \sigma^2 = \frac{8(n+2)}{n(n+3)}\Lambda$, and recognize it as a single fine-tuning condition equivalent to requiring the vanishing of the 4D cosmological constant at the intersection. One can straightforwardly verify that in a general situation with different brane tensions, and also a tension of a 3-brane at the intersection, there always exist solutions as long as an appropriate generalization of this single fine-tuning condition is satisfied.

In order to demonstrate that we have indeed localized gravity to the intersection, we must look at the linear perturbations about this solution. It is convenient to parametrize the perturbations by replacing $\eta_{\mu\nu}$ with $\eta_{\mu\nu} + h_{\mu\nu}(x, \bar{z})$ in Eq. (6). Again using the conformal transformation Eq. (8) we easily find the linearized field equations for $h_{\mu\nu}$, which are in the gauge $h_\mu^\mu = 0, \partial_\alpha h^{\alpha\mu} = 0$

$$\left[\Box_4 - \nabla_{\bar{z}}^2 + (n+2)\Omega \sum_j \mathrm{sgn}(\bar{z}^j)\partial_j\right] h(x, \bar{z}) = 0, \tag{10}$$

where we have dropped the $\mu\nu$ index on h. The transverse-traceless gauge is invariant under conformal transformations on the background (6), and hence our calculation in the conformal frame exactly reproduces the results in the original frame (6). This immediately shows that the ordinary four-dimensional graviton is present as a massless mode in the theory, corresponding to a $\bar{z}$ independent solution $h(x, \bar{z}) = h(x)$. Indeed, replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}^{(4)}(x)$ in Eq. (6) and inserting into the action, we find

$$S = \int d^n \bar{z}\, \Omega^{2+n} \times \int d^4 x \sqrt{g^{(4)}}\, R^{(4)}, \tag{11}$$

which shows that the four-dimensional graviton couples with strength

$$M_{\mathrm{Pl}}^2 = M_*^{2+n} \int d^n \bar{z}\, \Omega^{2+n} \sim M_*^{2+n} L^n. \tag{12}$$

The exact calculation gives $M_{\mathrm{Pl}}^2 = [2^n n^{n/2}/(n+1)!] M_*^{2+n} L^n$. This relation suggests that L can be interpreted as the effective size of n compact dimensions, even though the extra dimensions are infinitely large. This is indeed a correct interpretation as we will see shortly. For a complete analysis of the effective $(3+1)D$ spectrum, it is convenient to make a change of variables $h = \Omega^{-(n+2)/2}\hat{h}$, in terms of which the linearized equations are

$$\left[\frac{1}{2}\Box_4 + \left(-\frac{1}{2}\nabla_{\bar{z}}^2 + V(\bar{z})\right)\right]\hat{h} = 0, \tag{13}$$

where

$$V(\bar{z}) = \frac{n(n+2)(n+4)k^2}{8}\Omega^2 - \frac{(n+2)k}{2}\Omega \sum_j \delta(\bar{z}^j). \tag{14}$$

In order to determine the spectrum of 4D masses, we set $\hat{h} = e^{ipx}\hat{\psi}(\bar{z})$; the 4D masses are then determined by the eigenvalues of an effective n dimensional Schrödinger equation

$$\left(-\frac{1}{2}\nabla_{\bar{z}}^2 + V(\bar{z})\right)\hat{\psi}_\lambda = \frac{1}{2} m_\lambda^2 \hat{\psi}_\lambda, \tag{15}$$

where λ labels the eigenfunctions. All of the important physics follows from a qualitative analysis of this potential and parallels the story with one extra dimension. The potential has a repulsive piece which goes to zero for $\sum_j |\bar{z}^j| \gg L$, and a sum of attractive δ functions. We already know that the 4D massless graviton corresponds to a bound state with the wave function (numerical factors will be omitted in all that follows)

$$\hat{\psi}_{\mathrm{bound}} \sim \Omega^{(n+2)/2}. \tag{16}$$

Since the potential falls off to zero at infinity, we will also have continuum modes. Since the height of the potential near the origin is $\sim k^2$, the modes with $m^2 < k^2$ will have suppressed wave functions, while those with $m^2 > k^2$ will sail over the potential and will be unsuppressed at the origin. In order to see the physics more explicitly, suppose we place a test mass M on the intersection at $(x = 0, \bar{z} = 0)$, and ask for the gravitational potential $U(r)$ at a distant point on the intersection $(|x| = r, \bar{z} = 0)$. To do this, we simply insert a source $G_{N(4+n)} M \delta^3(x) \delta^n(\bar{z})$ on the right-hand side of Eq. (13), and straightforwardly solve the equation to find

$$\begin{aligned} \frac{U(r)}{M} &= \sum_\lambda G_{N(4+n)} |\hat{\psi}_\lambda(0)|^2 \frac{e^{-m_\lambda r}}{r} \\ &\sim \frac{G_{N(4+n)}}{L^n}\frac{1}{r} + \sum_{\mathrm{continuum}} G_{N(4+n)} |\hat{\psi}_\lambda(0)|^2 \frac{e^{-m_\lambda r}}{r}. \end{aligned} \tag{17}$$

In the second line we have separated the bound-state from the continuum contributions. It is straightforward to evaluate the suppression of $|\psi_\lambda(0)|$ for modes with $mL < 1$, but this is not needed for the discussion of the limiting behavior of $U(r)$. Consider first large distances $r \gg L$. Even with *no* suppression of the continuum modes for $mL < 1$, the continuum sum would yield the $(4+n)$D potential $G_{N(4+n)}/r^{n+1}$, which is subdominant to the term generated by the 4D graviton bound state for $r \gg L$. Therefore, for $r \gg L$,

$$U(r) \sim \frac{G_{N(4)} M}{r}, \qquad G_{N(4)} \sim \frac{G_{N(4+n)}}{L^n}. \tag{18}$$

On the other hand, for distances $r \ll L$, it is the continuum modes with $mL \gg 1$ which dominate, and these have unsuppressed wave functions at the origin. Therefore, for $r \ll L$, we just get the $(4+n)$D potential

$$U(r) \sim \frac{G_{N(4+n)} M}{r^{n+1}}. \tag{19}$$

This point was not discussed in [4], as they were only interested in checking the $r \gg L$ behavior. Of course, a precise treatment is needed to understand the details of the crossover between these limits. However, the qualitative behavior is exactly what we would expect by interpreting L as a "compactification radius." This is in accordance with the intuition that, while the extra dimensions are infinitely large, gravity is localized to a region of size L around the intersection of the branes. The mechanism of localization is realized by the branes repelling all graviton modes with bulk momentum smaller than $1/L$ but greater than zero away from the intersection, to distances of size L. As a result, inside of this region gravity becomes weak and the resulting Planck scale can be many orders of magnitude larger than the fundamental scale. The length scale L is determined by the bulk cosmological constant Λ, and given our ignorance regarding the cosmological constant problem, we do not feel any strong prejudice forcing L to be of the order of the fundamental scale M_*^{-1}. We will simply treat L as a parameter; we know only that L must be smaller than $\sim$1 mm from the present-day gravity measurements.

For $L \gg M_*^{-1}$, the phenomenology seems to be very similar to conventional large extra dimensions of size L, but there is a different theoretical perspective. In particular, the problem of stabilizing the radius at large values is replaced here with explaining the tiny bulk cosmological constant and brane tensions. Furthermore, there do not seem to be any very light moduli fields associated with the large "radius" L.

Another possibility motivated by the first paper in [4] is to stay with $L \sim M_*^{-1}$, and use the factor Ω to exponentially generate the weak scale on our 3-brane, which is placed a distance $O(100)M_*^{-1}$ away from the intersection. Unlike the proposal in the first paper of [4], this can be done with infinitely large extra dimensions. This is because in our case, 3-branes are tiny compared to the $(2 + n)$ branes setting up the gravitational background, so they are just like test particles probing the background geometry. Such a "hybrid" model has interesting phenomenology. Since the bulk is infinitely large, in contrast to the first paper in [4] there is a continuum of graviton modes, and they lead to a correction to the Newtonian potential on our brane $\leq (\mathrm{TeV})^{-(n+2)} r^{-(n+1)}$ at all distances. This correction is irrelevant at large distances, but dominates the Newtonian potential at distances smaller than R where $M_{\mathrm{Pl}}^2 \sim (\mathrm{TeV})^{n+2} R^n$. For $n \geq 2$ this R is less than a millimeter and so the long distance Newtonian gravity is not affected by the presence of the continuum of graviton modes. This framework combines the interesting features of having infinitely large new dimensions, exponential determination of the weak/Planck hierarchy, strong gravity at the TeV scale and possible submillimeter deviations from Newtonian gravity.

In summary, we have shown that gravity can be localized to the intersection of orthogonal $(2 + n)$ branes lying in infinite AdS_{n+4} space. Our solution naturally generalizes the example with one extra dimension of [3,4]. It is therefore possible to mask any number of infinitely large extra dimensions. Furthermore, we pointed out that the curvature L of the bulk AdS space acts as an effective "compactification" scale; the Newtonian potential on the intersection behaves as $1/r$ for $r \gg L$ and $1/r^{n+1}$ for $r \ll L$. Among other things, this could offer new possibilities for constructing theories with submillimeter extra dimensions.

A number of aspects of our setup need to be further elaborated. For instance, in the construction we have assumed that there is no extra tension localized at the intersection of the branes; e.g., we have not included possible 3-brane sources of the form $\delta^n(\bar{z})$ in Einstein's equations. It is important to determine in detail how our solutions are modified in the presence of such sources. There are obvious generalizations to intersecting branes with different tensions or at general angles to each other, and we expect that these will mimic the physics of anisotropic compactification. There are many fascinating questions left to ask, particularly in cosmology where interesting new issues and possibilities have already emerged in the context of submillimeter dimensions [5–8]. It is also interesting to explore the extent to which the (now infinitely large) bulk can be used to address other mysteries of the standard model, perhaps along the lines of [9]. We intend to pursue these issues in future investigations.

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