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Hence it follows, therefore, that E_R and E' cannot be converted into one another; as we are able to deduce this in the same way for E_R and E'' and also for E' and E'', the property mentioned sub a is proved. At the same time it appears from the deduction that it need not be true for equilibria in unstable condition.

For the equilibria E' and E'' this property follows also at once without calculation viz. from the condition that under constant P and at constant T ξ must be a minimum.

The properties, mentioned sub b and c follow now at once from property a and formula (20).

Leiden, Inorg. Chem. Lab. (*To be continued*).

Mechanics. — *"On the relativity of inertia. Remarks concerning* EINSTEIN'S *latest hypothesis"* [1]) By Prof. W. DE SITTER.

(Communicated in the meeting of March 31, 1917).

If we neglect the gravitational action of all ordinary matter (sun, stars, etc.), and if we use as a system of reference three rectangular cartesian space-coordinates and the time multiplied by c, then in that part of the four-dimensional time-space which is accessible to our observations, the $g_{\mu\nu}$ are very approximately those of the old theory of relativity, viz.:

$$\left.\begin{array}{rrrr} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{array}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The part of the time-space where this is so, I shall call "our neighbourhood". In space this extends at least to the farthest star, nebula or cluster in whose spectrum we can identify definite lines [2]).

How the $g_{\mu\nu}$ are outside our neighbourhood we do not know, and any assumption regarding their values is an extrapolation, whose uncertainty increases with the distance (in space, or in time, or in both) from the origin. How the $g_{\mu\nu}$ are at infinity of space or of time, we will never know. Nevertheless the need has been felt to

[1]) A. EINSTEIN, Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, Sitzungsber. Berlin, 8 Febr. 1917, page 142.

[2]) W. DE SITTER, On EINSTEIN's theory of gravitation and its astronomical consequences (second paper), Monthly Notices R.A.S. Dec. 1916, Vol. LXXVII, p. 182. This limit refers to g_{44} only.

make hypotheses on this subject. The extrapolation, which offers itself most naturally, and which is also tacitly made in classical mechanics, is that the values (1) remain unaltered for all space and time up to infinity. On the other hand the desire has arisen to have constants of integration, or rather boundary-values at infinity, which shall be the same in all systems of reference. The values (1) do not satisfy this condition. The most desirable and the simplest value for the $g_{\mu\nu}$ at infinity is evidently *zero*. EINSTEIN has not succeeded in finding such a set of boundary values [1]) and therefore makes the hypothesis that the universe is not infinite, but spherical: then no boundary conditions are needed, and the difficulty disappears. From the point of view of the theory of relativity it appears at first sight to be incorrect to say: the world *is* spherical, for it can by a transformation analogous to a stereographic projection be represented in a euclidean space. This is a perfectly legitimate transformation, which leaves the different invariants ds^2, G etc. unaltered. But even this invariability shows that also in the euclidean system of coordinates the world, in natural measure, remains finite and spherical. If this transformation is applied to the $g_{\mu\nu}$ which EINSTEIN finds for his spherical world, they are transformed to a set of values which at infinity degenerate to

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2A)$$

It appears, however, that the $g_{\mu\nu}$ of EINSTEIN's spherical world [and therefore also their transformed values in the euclidean system of reference] do not satisfy the differential equations originally adopted by EINSTEIN, viz:

$$G_{\mu\nu} = -\varkappa(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

EINSTEIN thus finds it necessary to add another term to his equations, which then become

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\varkappa(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Moreover it is found necessary to suppose the whole three-dimen-

[1]) l.c. page 148. It will appear below that EINSTEIN's hypothesis is equivalent to a determined set of values at infinity, viz: the set (2A). It is, in fact, evident that, if the universe measured in natural measure be finite, then, if euclidean coordinates are introduced the $g_{\mu\nu}$ must necessarily be zero at infinity, and inversily if the $g_{\mu\nu}$ at infinity are zero of a sufficiently high order, then the universe is finite in natural measure.

sional space to be filled with matter, of which the total mass is so enormously great, that compared with it all matter known to us is utterly negligible. This hypothetical matter I will call the "world-matter".

EINSTEIN only assumes *three*-dimensional space to be finite. It is in consequence of this assumption that in (2*A*) g_{44} remains 1, instead of becoming zero with the other $g_{\mu\nu}$. This has suggested the idea [1]) to extend EINSTEIN's hypothesis to the *four*-dimensional time-space. We then find a set of $g_{\mu\nu}$ which at infinity degenerate to the values

$$\left.\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (2B)$$

Moreover we find the remarkable result, that now no "world-matter" is required.

In order to point out the analogy of the two cases I give the two sets of formulae together. The formulae *A* refer to EINSTEIN's (*three*-dimensional) hypothesis, the formulae *B* refer to the assumption here introduced (*four*-dimensional). I shall use the indices i and j, when they take the values 1, 2, 3 only; μ and ν take the values from 1 to 4. Further Σ is a sum from 1 to 4 and Σ' from 1 to 3; and $\delta_{\mu\mu} = 1$, $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$.

I first take the system of reference used by EINSTEIN. In case *A* we take $x_4 = ct$, in *B* I take, for the sake of symmetry [2]), $x_4 = ict$. In both cases R is the radius of the hypersphere. The $g_{\mu\nu}$ for the two cases are

A	*B*
$g_{ij} = -\delta_{ij} - \dfrac{x_i x_j}{R^2 - \Sigma' x_i^2}$	$g_{\mu\nu} = -\delta_{\mu\nu} - \dfrac{x_\mu x_\nu}{R^2 - \Sigma x_\mu^2}$
$g_{44} = 1$	

In order better to show the spherical character I introduce hyperspherical coordinates by the transformations:

1) The idea to make the four-dimensional world spherical in order to avoid the necessity of assigning boundary-conditions, was suggested several months ago by Prof. EHRENFEST, in a conversation with the writer. It was, however, at that time not further developed.

2) We can also take $x_4 = ct$. Then the four-dimensional world is hyperbolical instead of spherical, but the results remain the same.

$$x_1 = R \sin\chi \sin\psi \sin\vartheta \qquad x_1 = R \sin\omega \sin\chi \sin\psi \sin\vartheta$$

$$x_2 = R \sin\chi \sin\psi \cos\vartheta \qquad x_2 = R \sin\omega \sin\chi \sin\psi \cos\vartheta$$

$$x_3 = R \sin\chi \cos\psi \qquad x_3 = R \sin\omega \sin\chi \cos\psi$$

$$x_4 = R \sin\omega \cos\chi$$

The expression of the line-element then becomes

$$A: \quad ds^2 = -R^2[d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi\, d\vartheta^2)] + c^2dt^2,$$

$$B: \quad ds^2 = -R^2[d\omega^2 + \sin^2\omega\{d\chi^2 + \sin^2\chi(d\psi^2 + \sin^2\psi d\vartheta^2)\}].$$

Finally I perform the "stereographic projection", and at the same time I introduce again rectangular coordinates, by the transformations:

A	B
$r = 2R \tan \frac{1}{2}\chi$	$h = 2R \tan \frac{1}{2}\omega$
$x = r \sin\psi \sin\vartheta$	$x = h \sin\chi \sin\psi \sin\vartheta$
$y = r \sin\psi \cos\vartheta$	$y = h \sin\chi \sin\psi \cos\vartheta$
$z = r \cos\psi$	$z = h \sin\chi \cos\psi$
	$ict = h \cos\chi$
$x^2 + y^2 + z^2 = r^2$	$x^2 + y^2 + z^2 - c^2t^2 = h^2$

It need hardly be pointed out that in A x, y, z and in B x, y, z, ict can be arbitrarily interchanged. I put further

$$\sigma = \frac{1}{4R^2}.$$

The $g_{\mu\nu}$ for the variables x, y, z, ct then become [1])

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[1]) In the system B all $g_{\mu\nu}$ are infinite on the "hyperboloid"

$$1 + \sigma h^2 = 0 \text{ or } 4R^2 + x^2 + y^2 + z^2 - c^2t^2 = 0 \quad . \quad . \quad . \quad (a)$$

This discontinuity is however only apparent. The four-dimensional world, which we have for the sake of symmetry represented as spherical, is in reality hyperbolical, and consists of two sheets, which are only connected with each other at infinity. The formulae embrace both sheets, but only one of them represents the actual universe. The hyperboloid (a) is the limit between the two parts of the euclidean space x, y, z, ct corresponding to these two sheets. It is intersected by the axis of t at the points $ct = \pm 2R$, the distance of which from the origin is, in natural measure, $\int_0^{2R} \frac{c\,dt}{1 - \sigma c^2t^2} = \infty$. The length in natural measure of the half-axis of x is, in both systems, $\int_0^{\infty} \frac{dx}{1 + \sigma x^2} = \pi R.$

$$\begin{array}{c|c} A & B \\ g_{ij} = -\dfrac{\delta_{ij}}{(1+\sigma r^2)^2} & g_{ij} = -\dfrac{\delta_{ij}}{(1+\sigma h^2)^2} \\ g_{44} = 1 & g_{44} = \dfrac{1}{(1+\sigma h^2)^2} \end{array} \Bigg\} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

All $g_{\mu\nu}$ outside the diagonal are zero. If σ is very small the $g_{\mu\nu}$ for moderate values of r and h have very approximately the values (1). At infinity they degenerate to the values (2 A) and (2 B), which have already been given above.

In order to find the relation between σ and λ we must substitute [1]) the values (5) in the equations (4). We must, in doing this, allow for the possibility that it may be found necessary to introduce a "world-matter". We neglect all ordinary matter, and we will suppose the world-matter to be uniformly distributed [2]) over the whole of space, and at rest, so that $T_{44} = g_{44}\varrho$, and all other $T_{\mu\nu} = 0$. The field-equations then become

$$G_{ij} - (\lambda + \tfrac{1}{2}\varkappa\varrho)\, g_{ij} = 0,$$
$$G_{44} - (\lambda + \tfrac{1}{2}\varkappa\varrho)\, g_{44} = -\varkappa\varrho.$$

For the quantities $G_{\mu\nu}$ we find in the two systems

$$\begin{array}{c|c} A & B \\ G_{ij} = 8\sigma g_{ij} & G_{\mu\nu} = 12\sigma g_{\mu\nu} \\ G_{44} = 0, \; g_{44} = 1 & \end{array}$$

From which

$$\begin{array}{c|c} \lambda = 4\sigma & \lambda = 12\sigma \\ \varrho = \dfrac{8\sigma}{\varkappa} & \varrho = 0 \end{array} \Bigg\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The result for A is the same as found by EINSTEIN. For B we have $\varrho = 0$: the hypothetical world-matter does not exist.

Which of the three systems is to be preferred: A with world-matter, B without it, both with the field-equations (4) and at infinity the $g_{\mu\nu}$ (2A) or (2B); or the original system without world-matter, with the field-equations (3) and the $g_{\mu\nu}$ (1), which retain the same values at infinity?

From the purely physical point of view, for the description of

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1) We can, of course, as well take the values in any other system of reference.

2) The meaning is a distribution in which ϱ is constant, ϱ being the density in *natural* measure. The density in coordinate-measure then, of course, is not constant, but (in the system x, y, z, ct) becomes zero at infinity.

phenomena in our neighbourhood, this question has no importance. In our neighbourhood the $g_{\mu\nu}$ have in all cases within the limits of accuracy of our observations the values (1), and the field-equations (4) are not different from (3). The question thus really is: how are we to extrapolate outside our neighbourhood? The choice can thus not be decided by physical arguments, but must depend on metaphysical or philosophical considerations, in which of course also personal judgment or predilections will have some influence.

To the question: If all matter is supposed not to exist, with the exception of one material point which is to be used as a test-body, has then this test-body inertia or not? the school of MACH requires the answer *No*. Our experience however very decidedly gives the answer *Yes*, if by "all matter" is meant all ordinary physical matter: stars, nebulae, clusters, etc. The followers of MACH are thus compelled to assume the existence of still more matter: the world-matter. If we place ourselves on this point of view, we must necessarily adopt the system *A*, which is the only one that admits a world-matter. [1])

This world-matter, however, serves no other purpose than to enable us to suppose it not to exist. Now the formula (6) shows, that if it does not exist ($\varrho = 0$), the field-equations are not satisfied: supposing it not to exist thus appears to be a logical impossibility; in the system *A*, the world-matter *is* the three-dimensional space, or at least is inseparable from it.

We can also abandon the postulate of MACH, and replace it by the postulate that at infinity the $g_{\mu\nu}$, or only the g_{ij} of three-dimensional space, shall be zero, or at least invariant for all transformations. This postulate can also be enounced by saying that it must be possible for the whole universe to perform arbitrary motions, which can never be detected by any observation. The three-dimensional world must, in order to be able to perform "motions", i.e. in order that its position can be a variable function of the time, be thought movable in an "absolute" space of three or more dimensions (*not* the time-space x, y, z, ct). The four-dimensional world requires for its "motion" a four- (or more-) dimensional absolute space, and moreover an extra-mundane "time" which serves as independent variable for this motion. All this shows that the postulate of the

[1]) The hypothesis formerly held by EINSTEIN, and denied by me, that it would be possible, with the equations (3) and by means of very large masses at very large distances, to get values of $g_{\mu\nu}$, which would degenerate to an invariant set at infinity, has now been shown to be untenable by EINSTEIN himself (l. c. page 146).

invariance of the $g_{\mu\nu}$ at infinity has no real physical meaning. It is purely mathematical.

The system A, with at infinity the values (2 A) of the $g_{\mu\nu}$ satisfies this postulate, if it is applied only to the three-dimensional world, and if we do not require invariance for all transformations, but only for those which at infinity have $t' = t$ [1]). If the postulate is applied to the four-dimensional world, and to *all* transformations, then the system B is the only one that satisfies. We thus find that in the system A the time has a separate position. That this must be so, is evident a priori. For speaking of *the* three-dimensional world, if not equivalent to introducing an absolute time, at least implies the hypothesis that at each point of the four-dimensional space there is one definite coordinate x_4 which is preferable to all others to be used as "time", and that at all points and always this one coordinate is actually chosen as time. Such a fundamental difference between the time and the space-coordinates seems to be somewhat contradictory to the complete symmetry of the field-equations and the equations of motion (equations of the geodetic line) with respect to the four variables.

Some features of the systems A and B may still be pointed out. In A the velocity of light is variable [2]), at infinity it becomes infinite. In B it is always and everywhere the same. From the facts that we can identify lines in the spectra of the most distant objects known to us such as the Nubeculae, and that the parallaxes of these objects are not negative, we can conclude that at these distances we have still approximately $g_{ij} = -\delta_{ij}$, $g_{44} = 1$ and consequently that for A σr^2, and for B σh^2 must be very small. In the case A we can derive in this way an upper limit for σ. In B on the other hand we have, in consequence of the constancy of the velocity of light, $h^2 = 0$ for all purely optical observations (if we neglect the influence of matter).

As to the effect of σ on planetary motions: in both cases the

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1) Thus e. g. an ordinary LORENTZ-transformation:

$$x' = \frac{x - qct}{\sqrt{(1-q^2)}} \quad , \quad ct' = \frac{ct - qx}{\sqrt{(1-q^2)}}$$

is not allowed in the system A, but must be replaced by

$$x' = \frac{x - qct}{\sqrt{\left(1 - \frac{q^2}{(1+\sigma r^2)^2}\right)}} \quad , \quad ct' = \frac{ct - \frac{qx}{1+\sigma r^2}}{\sqrt{\left(1 - \frac{q^2}{(1+\sigma r^2)^2}\right)}} .$$

2) In the system x, y, z, ct; in the system χ, ψ, ϑ, ct it is constant.

orbital plane is not disturbed. In case A there are no secular terms depending on σ.

In B the terms produced by σ are of a lower order, in consequence of the fact that all $g_{\mu\nu}$ depend explicitly on the time. The motion of the perihelion is

$$d\bar{\omega} = \frac{3\sigma a^3}{\lambda_0^2} nt - 2\sigma c^2 t^2,$$

and also the other elements have terms with c^2t^2; thus e.g. the parameter of the elliptic orbit is

$$p = p_0 \, e^{-2\sigma c^2 t^2},$$

where $\lambda_0^2 = \varkappa m / 8\pi$, m being the sun's mass, and $e = 2.718 \ldots$ These "perturbations" [1]) being insensible according to our experience, we can here also assign an upper limit to σ.

I shall not attempt to determine this upper limit with any accuracy. For both cases we will be safe in taking e.g. $\sigma < 10^{-24}$ in astronomical units, or $2\sigma < 10^{-50}$ in centimeters [2]). We can, however, do no more than assign an upper limit to σ. To make possible a *determination* of the value of this constant, it would be necessary that it had a measurable effect on some physical or astronomical phenomenon. Now it cannot, of course, be excluded a priori that at some future time observations will be made, or phenomena will be discovered which can be explained with the aid of the constant σ, but so far no such phenomena are known, and there are no indications of anything in that direction. The constant σ only serves to satisfy a philosophical need felt by many, but it has no real physical meaning, though it can be mathematically interpreted as a curvature of space.

Finally we can also reject both systems A and B, and stick to the original field-equations (3) and the values (1) of the $g_{\mu\nu}$, which

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[1]) The terms of the lowest order in the "perturbing forces" are for the two cases:

$$In\ A: \quad S = -3\sigma + \frac{2\sigma}{\lambda_0^2} r(\dot{r}^2 - r^2\dot{\vartheta}^2) \quad , \quad T = \frac{4\sigma}{\lambda_0^2} r^2 \dot{r} \dot{\vartheta}, \quad W = 0,$$

$$in\ B: \quad S = \frac{2\sigma}{\lambda_0^2} r - \frac{2\sigma}{\lambda_0^2} ct\dot{r} \quad , \quad T = -\frac{2\sigma}{\lambda_0^2} ctr\dot{\vartheta}, \quad W = 0.$$

(For the notation see e. g. DE SITTER, *On* EINSTEIN's *theory of gravitation*, M.N. Vol. LXXI, pages 724 sqq.).

The terms with c^2t^2 in the case B arise through the fact that the units both of time and space (in coordinate-measure) depend on the time.

[2]) The density of the world matter in the system A then becomes $\varrho < 3.10^{-17}$ (astronomical units), or $\varrho < 2.10^{-23}$ (C. G. S. units). This corresponds to one star (of the same mass as the sun) in a sphere of one parsec radius.

are not invariant at infinity. Then, of course, inertia is not explained: we must then prefer to leave it unexplained rather than explain it by the undetermined and undeterminable constant λ. It cannot be denied that the introduction of this constant detracts from the symmetry and elegance of EINSTEIN's original theory, one of whose chief attractions was that it explained so much without introducing any new hypothesis or empirical constant.

Postscript.

Prof. EINSTEIN, to whom I had communicated the principal contents of this paper, writes (March 24, 1917): "Es wäre nach meiner Meinung unbefriedigend, wenn es eine denkbare Welt ohne Materie gäbe. Das $g_{\mu\nu}$-Feld soll vielmehr *durch die Materie bedingt sein, ohne dieselbe nicht bestehen können.* Das ist der Kern dessen, was ich unter der Forderung von der Relativität der Trägheit verstehe". He therefore postulates what I called above the logical impossibility of supposing matter not to exist. We can call this the "material postulate" of the relativity of inertia. This can only be satisfied by choosing the system A, with its world-matter, i. e. by introducing the constant λ, and assigning to the time a separate position amongst the four coordinates.

On the other hand we have the "mathematical postulate" of the relativity of inertia, i. e. the postulate that the $g_{\mu\nu}$ shall be invariant at infinity. This postulate, which, as has already been pointed out above, has no real physical meaning, makes no mention of matter. It can be satisfied by choosing the system B, without a world-matter, and with complete relativity of the time. But here also we need the constant λ. The introduction of this constant can only be avoided by abandoning the postulate of the relativity of inertia altogether.

Astronomy. — "*On the Theory of Hyperion, one of Saturn's Satellites.*". By J. WOLTJER JR. (Communicated by Prof. W. DE SITTER).

(Communicated in the meeting of April 27, 1917).

1. Among the peculiar disturbances, which the satellites of Saturn undergo by their mutual attraction, those, produced by Titan in the motion of Hyperion, are of much importance. In this paper I intend to give a short development of the theory of the latter satellite; my dissertation will contain more extensive calculations on this subject.