Probing the origin of inertia behind spacetime deformation

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To investigate the origin and nature of inertia, we introduce a new concept of hypothetical 2D, so-called, *master-space* (MS), subject to certain rules. The MS, embedded in the background 4D-spacetime, is an indispensable individual companion to the particle of interest, without relation to every other particle. We argue that a deformation/(distortion of local internal properties) of MS is the origin of inertia. With this perspective in sight, we construct the alternative *relativistic theory of inertia* (RTI), which allows to compute the *relativistic inertial force* acting on an arbitrary point-like observer due to its *absolute acceleration.* We go beyond the hypothesis of locality with an emphasis on distortion of MS, which allows to improve essentially the standard metric and other relevant geometrical structures related to the noninertial reference frame of an arbitrary accelerated observer. We compute the inertial force exerted on the photon in a gravitating system in the semi-Riemann space. Despite the totally different and independent physical sources of gravitation and inertia, this approach furnishes justification for the introduction of the principle of equivalence. Consequently, we relate the inertia effects to the more general post-Riemannian geometry. We derive a general expression of the relativistic inertial force exerted on the extended spinning body moving in the Rieman-Cartan space.

Keywords: Inertia, Spacetime Deformation, Principle of Equivalence, Noninertial Frames, Post-Riemannian Geometry

I. INTRODUCTION

The current observations made in the Earth-Moon-Sun system [1]-[5], or at galactic and cosmological scales [6]-[11], probe more deeply the *weak* principle of equivalence (PE), which establishes the independence of free-fall trajectories of the internal composition and structure of bodies. The inertia effects in fact are of vital interest also for the phenomenological aspects of the problem of neutrino oscillations, see e.g. [12]-[22]. All this has evoked the study of the inertial effects in an accelerated and rotated frame of stationary laboratories on Earth. As long as all relevant length scales in feasible experiments are very small in relation to the huge acceleration lengths of the tiny accelerations we usually experience, the curvature of the wordline could be ignored and the differences between observations by accelerated and comoving inertial observers will also be very small. Therefore, it is a long-established practice in physics to use the hypothesis of locality for extension of the Lorentz invariance to accelerated observers in Minkowski spacetime. This in effect replaces the accelerated observer by a continuous infinity of hypothetical momentarily comoving inertial observers along its wordline. In this line, in 1990, Hehl and Ni proposed a framework to study the relativistic inertial effects of a Dirac particle [23]. Ever since this question has become a major preoccupation of physicists, see e.g. [24]-[45]. Even this works out, it is still reminds us of a puzzling underlying reality of inertia. Despite our best efforts, all attempts to obtain a true knowledge of the geometry related to the noninertial reference frames of an arbitrary observer seem doomed, unless we find *a physical principle the inertia might refer to*, and that a working alternative *relativistic theory of inertia* (RTI) is formulated. Otherwise one wanders in a darkness. However, it seems that the inertia displays no any physical characteristics of gravitation, because there are many controversies to question the validity of such a description [46]-[50]. For example, the experiments by [48]-[50] tested the key question of anisotropy of inertia stemming from the idea that the matter in our galaxy is not distributed isotropically with respect to the earth, and hence if the inertia is due to gravitational interactions, then the inertial mass of a body will depend on the direction of its acceleration with respect to the direction towards the center of our galaxy. If the nuclear structure of Li^7 is treated as a single $P_{3/2}$ proton in a central nuclear potential, the variation Δm of mass with direction, if it exists, was found to satisfy $\frac{\Delta m}{m} \leq 10^{-20}$. This proves that there is no anisotropy of mass which is due to the effects of mass in our galaxy. Moreover, unlike gravitation, a curvature arisen due to acceleration of coordinate frame of interest, i.e. a "fictitious gravitation" which can be globally removed by appropriate coordinate transformations, relates to this coordinate system itself and does not affect the other systems or matter fields all at once.

In a recent paper [51], we construct the two-step spacetime deformation theory. Thereby, through a choice of the *world-deformation tensor*, $\widetilde{\Omega}$, which we have at our disposal, in general, we have a way to deform the spacetime displayed a different post Riemannian spacetime structures as its corollary. This allows to construct a consistent Einstein-Cartan theory, with the *dynamical torsion.* It is the purpose of the present paper to carry out some details of this program to probe the origin and nature of the phenomenon of inertia. We ascribe the inertia effects to the geometry itself but as having a nature other than gravitation. We propose a new concept of hypothetical 2D, so-called, *master-space* (MS), subject to certain rules. The MS, embedded in the background 4D-space, is an indispensable individual companion to the particle of interest, without relation to the other matter. Namely, the particle has to live with MS-companion as an intrinsic property. This together with the idea that the inertia effects arise as a deformation/(distortion of local internal properties) of MS, are the highlights of the RTI. This allows to compute the *relativistic inertial force* acting on an arbitrary observer due to its *absolute acceleration*. The hypothesis of locality represents strict restrictions, because it approximately replaces the distorted MS, by the flat MS. We might have to go beyond the hypothesis of locality with an emphasis on distortion of MS. This we might expect will essentially improve the standard metric, etc., related to the noninertial system of an arbitrary observer in Minkowski spacetime . We will proceed according to the following structure. In section 2, we explain our view of what is the MS, and lay a foundation of the *relativistic law of inertia* (RLI). In section 3, a general deformation/distortion of MS is described. In section 4, starting with the Minkowski background space M_4, we construct the RTI. In section 5, in the framework of a distortion of MS, we compute the improved metric and other relevant geometrical structures in noninertial system of an arbitrary accelerating and rotating observer in Minkowski spacetime. The case of semi-Riemann background space V_4 is dealt with in section 6, where we give justification for the introduction of the PE on the theoretical basis. In section 7, we relate the RTI to more general post-Riemannian geometry. The concluding remarks are presented in section 8. We will be brief and often suppress the indices without notice. Unless otherwise stated we take natural units, $h = c = 1$.

II. THE HYPOTHETICAL MS-COMPANION

The MS is the 2D Minkowski space, M_2:

$$M_2 = R^1_{(+)} \oplus R^1_{(-)}. \tag{1}$$

The ingredient 1D-space R^1_A is spanned by the coordinates η^A, where we use the "naked" capital Latin letters $A, B, ... = (\pm)$ to denote the world indices related to M_2. The metric in M_2 is

$$\overline{g} = \overline{g}(\overline{e}_A, \overline{e}_B)\, \overline{\vartheta}^A \otimes \overline{\vartheta}^B, \tag{2}$$

where $\overline{\vartheta}^A = d\eta^A$ is the infinitesimal displacement. The basis $\overline{e}_A$ at the point of interest in M_2 consists of two real *null vectors*:

$$\overline{g}(\overline{e}_A, \overline{e}_B) \equiv <\overline{e}_A, \overline{e}_B> = {}^*o_{AB}, \quad ({}^*o_{AB}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

The norm, $i\overline{d} \equiv d\hat{\eta}$, given in this basis reads $i\overline{d} = \overline{e}\overline{\vartheta} = \overline{e}_A \otimes \overline{\vartheta}^A$, where $i\overline{d}$ is the tautological tensor field of type (1,1), $\overline{e}$ is a shorthand for the collection of the 2-tuplet $(\overline{e}_{(+)}, \overline{e}_{(-)})$, and $\overline{\vartheta} = \begin{pmatrix} \overline{\vartheta}^{(+)} \\ \overline{\vartheta}^{(-)} \end{pmatrix}$. We may equivalently use a temporal $q^0 \in T^1$ and a spatial $q^1 \in R^1$ variables $q^r(q^0, q^1)(r = 0, 1)$, such that

$$M_2 = R^1 \oplus T^1. \tag{4}$$

The norm, $i\overline{d}$, now can be rewritten in terms of displacement, dq^r, as

$$i\overline{d} = d\hat{q} = e_0 \otimes dq^0 + e_1 \otimes dq^1, \tag{5}$$

where e_0 and e_1 are, respectively, the temporal and spatial basis vectors:

$$\overline{g}(\overline{e}_r, \overline{e}_s) \equiv <\overline{e}_r, \overline{e}_s> = o_{rs}, \quad (o_{rs}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

The MS is assumed to be embedded in the background 4D space and the motion of the individual particle is fully depends on the properties of MS-companion. In fact, we assume the particle has to be moving simultaneously in the parallel *individual* M_2 space and the ordinary 4D background space (either Minkowskian or Riemannian). Let us, first, concentrate our attention on non-accelerated observer, who for the position of a free test particle in the flat MS uses the inertial coordinate frame $S_{(2)}$, such that

$$v^{(\pm)} = \frac{d\eta^{(\pm)}}{dq^0} = \frac{1}{\sqrt{2}}(1 \pm v_q), \quad v_q = \frac{dq^1}{dq^0} = const. \tag{7}$$

Suppose the position of this particle in the 4D space M_4 is specified by the coordinates $x^l(s)$ $(l = 0, 1, 2, 3)$ with respect to the axes of the inertial system $S_{(4)}$. We may adjust the systems $S_{(2)}$ and $S_{(4)}$ in such a way as the spatial axis $\vec{e}_q \equiv e_1$ of $S_{(2)}$ lies along the velocity $\vec{v} = \vec{e}_v|\vec{v}|$ $(\vec{e}_q \,||\, \vec{e}_v)$, while the time axis $\vec{e}_0 \equiv e_0$ of S_2 be the time axis of a comoving inertial frame S_4, such that the time coordinates in the two systems are taken the same, $q^0 = x^0 = t$, and that $v_q \geq 0$.
We now define the concepts of *absolute* and *relative* states of the ingredient spaces R^1_A. The measure for these states is the very magnitude of the velocity components v^A of the particle:

Definition:

$$\text{\textit{The ingredient space} } R^1_A \text{ \textit{of the individual MS-companion of the particle}}$$
$$\text{\textit{is said to be in}} \begin{cases} \text{\textit{absolute (abs) state if}} \quad v^A = 0, \\ \text{\textit{relative (rel) state if}} \quad v^A \neq 0. \end{cases}$$

Therefore, the MS can be realized either in the *semi-absolute* state (rel, abs), or (abs, rel), or in the *total relative* state (rel, rel). It is remarkable that the *total-absolute* state, (abs, abs), which is equivalent to the unobservable Newtonian *absolute* two-dimensional spacetime, cannot be realized because of the relation $v^{(+)} + v^{(-)} = \sqrt{2}$. An existence of the *absolute* state of the R^1_A is an immediate cause of the light traveling in empty space R^1 along the q-axis with a maximal velocity $v_q = c$ (we re-instate the factor (c)) in the $(+)-$direction corresponding to the state $(v^{(+)},\, 0) \Leftrightarrow$ (rel, abs), and in the $(-)-$direction corresponding to the state $(0,\, v^{(-)}) \Leftrightarrow$ (abs, rel). The *absolute* state of R^1_A manifests its *absolute* character in the important for SR fact that the resulting velocity of light in the empty space R^1 is the same in all inertial frames $S_{(2)}$, $S'_{(2)}$, $S''_{(2)}$,..., i.e., in empty space light propagates independently of the state of motion of the source- if $v^A = 0$ then $v^{A\prime} = v^{A\prime\prime} = ... = 0$. Since the v^A is the very key measure of a deviation from the *absolute* state, we might expect that this has a substantial effect in an alteration of the particle motion under the unbalanced force. This observation allows us to lay forth the foundation of the fundamental RLI as follows:

Conjecture (RLI-Conjecture): *The non-zero local rate $\varrho(\eta, m, f)$ of instantaneously change of a constant velocity v^A (both magnitude and direction) of a massive (m) test particle under the unbalanced net force (f) is the immediate cause of a deformation/(distortion of the local internal properties) of MS: $M_2 \to \widetilde{\mathcal{M}}_2$.*
We can conclude therefrom that, unless MS is flat, a free particle in 4D background space in motion of uniform speed in a straight line tends to stay in this motion and a particle at rest tends to stay at rest. In this way, the MS-companion, therefore, abundantly serves to account for the state of motion of the particle in the 4D background space. In going into practical details, the function $\varrho(\eta, m, f)$ will be determined in section 4.

III. THE GENERAL SPACETIME DEFORMATION/DISTORTION-COMPLEX

Based on the work [51], we now extend the geometrical ideas of the spacetime deformation as applied to the 2D deformation $M_2 \to \widetilde{\mathcal{M}}_2$. To start with, let V_2 be 2D semi-Riemann space, which has at each point a tangent space, $\breve{T}_{\breve{\eta}}V_2$, spanned by the anholonomic orthonormal frame field, $\breve{e}$, as a shorthand for the collection of the 2-tuplet $(\breve{e}_{(+)},\, \breve{e}_{(-)})$, where $\breve{e}_a = \breve{e}_a{}^{\tilde{A}}\, \breve{e}_{\tilde{A}}$, with the holonomic frame is given as $\breve{e}_{\tilde{A}} = \breve{\partial}_{\tilde{A}}$. Here, we use the first half of Latin alphabet $a, b, c, ... = (\pm)$ to denote the anholonomic indices related to the tangent space, and the capital Latin letters with an over $'\,\tilde{}\,'$ - $\tilde{A}, \tilde{B}, ... = (\pm)$, to denote the holonomic world indices related either to the space V_2 or $\widetilde{\mathcal{M}}_2$. All magnitudes related to the space, V_2, will be denoted with an over $'\,\breve{}\,'$. These then define a dual vector, $\breve{\vartheta}$, of differential forms, $\breve{\vartheta} = \begin{pmatrix} \breve{\vartheta}^{(+)} \\ \breve{\vartheta}^{(-)} \end{pmatrix}$, as a shorthand for the collection of the $\breve{\vartheta}^b = \breve{e}^b{}_{\tilde{A}}\, \breve{\vartheta}^{\tilde{A}}$, whose values at every point form the dual basis, such that $\breve{e}_a \,\rfloor\, \breve{\vartheta}^b = \delta^b_a$, where $\rfloor$ denoting the interior product. Namely, this is a C^∞-bilinear map $\rfloor : \Omega^1 \to \Omega^0$ with Ω^p denotes the C^∞-modulo of differential p-forms on V_4. In components $\breve{e}_a{}^{\tilde{A}}\, \breve{e}^b{}_{\tilde{A}} = \delta^b_a$. On the manifold, V_2, the tautological tensor field, $i\breve{d}$, of type (1,1) can be defined which assigns to each tangent space the identity linear transformation. Thus for any point $\breve{\eta} \in V_2$, and any vector $\breve{\xi} \in \breve{T}_{\breve{\eta}}V_2$, one has $i\breve{d}(\breve{\xi}) = \breve{\xi}$. In terms of the frame field, the $\breve{\vartheta}^a$ give the expression for $i\breve{d}$ as $i\breve{d} = \breve{e}\breve{\vartheta} = \breve{e}_{(+)} \otimes \breve{\vartheta}^{(+)} + \breve{e}_{(-)} \otimes \breve{\vartheta}^{(-)}$, in the sense that both sides yield $\breve{\xi}$ when applied to any tangent vector $\breve{\xi}$ in the domain of definition of the frame field. We may consider general transformations of the linear group, $GL(2, R)$, taking any base into any other set of four linearly independent fields. The notation, $\{\breve{e}_a,\, \breve{\vartheta}^b\}$, will be used below for general linear frames. The holonomic metric can be defined in the semi-Riemann space, V_2, as

$$\breve{g} = \breve{g}_{\tilde{A}\tilde{B}}\, \breve{\vartheta}^{\tilde{A}} \otimes \breve{\vartheta}^{\tilde{B}} = \breve{g}(\breve{e}_{\tilde{A}},\, \breve{e}_{\tilde{B}})\, \breve{\vartheta}^{\tilde{A}} \otimes \breve{\vartheta}^{\tilde{B}}, \tag{8}$$

with components, $\breve{g}_{\tilde{A}\tilde{B}} = \breve{g}(\breve{e}_{\tilde{A}}, \breve{e}_{\tilde{B}})$ in the dual holonomic base $\{\breve{\vartheta}^{\tilde{A}}\}$. The anholonomic orthonormal frame field, $\breve{e}$, relates $\breve{g}$ to the tangent space metric, ${}^{*}o_{ab}$, by ${}^{*}o_{ab} = \breve{g}(\breve{e}_a, \breve{e}_b) = \breve{g}_{\tilde{A}\tilde{B}}\, \breve{e}_a{}^{\tilde{A}}\, \breve{e}_b{}^{\tilde{B}}$, which has the converse $\breve{g}_{\tilde{A}\tilde{B}} = {}^{*}o_{ab}\, \breve{e}^a{}_{\tilde{A}}\, \breve{e}^b{}_{\tilde{B}}$ because $\breve{e}_a{}^{\tilde{A}}\, \breve{e}^a{}_{\tilde{B}} = \delta^{\tilde{A}}_{\tilde{B}}$. With this provision, we build up a general *distortion-complex* (DC), yielding a distortion of the flat space M_2, and show how it restores the *world-deformation tensor* $\widetilde{\Omega}$, which still has to be put in [51] by hand. The DC-members are the invertible distortion matrix D, the tensor Y and the *flat-deformation tensor* Ω. Symbolically,

$$DC \,\sim\, (\breve{D},\, \breve{Y},\, \Omega) \rightarrow \widetilde{\Omega}.$$

The following two steps went into the principle foundation of a *distortion of local internal properties of MS.*
1) We assume that the linear frame $(\overline{e}_A;\, \overline{\vartheta}^A)$, at given point $(p \in M_2)$, is undergone the *distortion* transformations, conducted by $(\breve{D},\, \breve{Y})$ and $(D,\, Y)$, respectively, relating to V_2 and $\widetilde{\mathcal{M}}_2$, recast in the form

$$\breve{e}_{\tilde{A}} = \breve{D}^B_{\tilde{A}}\, \bar{e}_B, \quad \breve{\vartheta}^{\tilde{A}} = \breve{Y}^{\tilde{A}}_B\, \bar{\vartheta}^B, \quad e_{\tilde{A}} = D^B_{\tilde{A}}\, \bar{e}_B, \quad \vartheta^{\tilde{A}} = Y^{\tilde{A}}_B\, \bar{\vartheta}^B. \tag{9}$$

2) We write the norm, $d\widetilde{\eta} \equiv id$, of the infinitesimal displacement, $d\widetilde{\eta}^{\tilde{A}}$, on the general smooth differential 2D-manifold, $\widetilde{\mathcal{M}}_2$, in terms of the spacetime structures of V_2:

$$id = e\,\vartheta = \widetilde{\Omega}_{\tilde{A}}^{\ \tilde{B}}\, \breve{e}_{\tilde{B}} \otimes \breve{\vartheta}^{\tilde{A}} = \Omega_b{}^a\, \breve{e}_a \otimes \breve{\vartheta}^b = e_{\tilde{C}} \otimes \vartheta^{\tilde{C}} = e_a \otimes \vartheta^a = \Omega_A{}^B\, \bar{e}_B \otimes \bar{\vartheta}^A \,\in\, \widetilde{\mathcal{M}}_2, \tag{10}$$

where $e = \{e_a = e_a{}^{\tilde{C}}\, e_{\tilde{C}}\}$ is the frame field and $\vartheta = \{\vartheta^a = e^a{}_{\tilde{C}}\, \vartheta^{\tilde{C}}\}$ is the coframe field defined on $\widetilde{\mathcal{M}}_2$, such that $e_a \rfloor \vartheta^b = \delta^b_a$. The deformation tensors $\widetilde{\Omega}_{\tilde{A}}^{\ \tilde{B}} = \pi_{\tilde{A}}{}^{\tilde{C}}\, \pi_{\tilde{C}}{}^{\tilde{B}}$, and $\Omega_A{}^B$ imply

$$\widetilde{\Omega}_{\tilde{A}}^{\ \tilde{B}} = \breve{D}^C_{\tilde{A}}\, \Omega_C{}^D\, \breve{Y}^{\tilde{B}}_D, \quad \Omega_A{}^B = Y^{\tilde{C}}_A\, D^B_{\tilde{C}}\,, \tag{11}$$

provided

$$D^A_{\tilde{C}} = \pi_{\tilde{C}}{}^{\tilde{B}}\, \breve{D}^A_{\tilde{B}}, \quad Y^{\tilde{C}}_B = \pi^{\tilde{C}}{}_{\tilde{A}}\, \breve{Y}^{\tilde{A}}_B, \tag{12}$$

such that

$$e_{\tilde{C}} = \pi_{\tilde{C}}{}^{\tilde{B}}\, \breve{e}_{\tilde{B}} \equiv \widetilde{\partial}_{\tilde{C}}, \quad \vartheta^{\tilde{C}} = \pi^{\tilde{C}}{}_{\tilde{A}}\, \breve{\vartheta}^{\tilde{V}} \equiv d\,\widetilde{\eta}^{\tilde{C}}, \quad \widetilde{\eta}^{\tilde{C}} \,\in \mathcal{U} \in \widetilde{\mathcal{M}}_2. \tag{13}$$

Hence the anholonomic deformation tensor, $\Omega_b{}^a = \pi_c{}^a\, \pi_b{}^c = \widetilde{\Omega}_{\tilde{A}}^{\ \tilde{B}}\, \breve{e}^a{}_{\tilde{B}}\, \breve{e}_b{}^{\tilde{A}}$, yields local tetrad deformations

$$e_c = \pi_c{}^a\, \breve{e}_a, \quad \vartheta^c = \pi^c{}_b\, \breve{\vartheta}^b, \quad e\,\vartheta = e_a \otimes \vartheta^a = \Omega^a{}_b\, \breve{e}_a \otimes \breve{\vartheta}^b. \tag{14}$$

The matrices, $\pi(\widetilde{\eta}) :\ \ = (\pi_b{}^a)(\widetilde{\eta})$, are referred to as the *first deformation matrices*, and the matrices $\gamma_{cd}(\widetilde{\eta}) = {}^{*}o_{ab}\, \pi_c{}^a(\widetilde{\eta})\, \pi_d{}^b(\widetilde{\eta})$, - *second deformation matrices*. The matrices, $\pi^a{}_c(\widetilde{\eta}) \,\in\, GL(2,R)\, \forall\, \widetilde{\eta}$, in general, give rise to right cosets of the Lorentz group, i.e. they are the elements of the quotient group $GL(2,R)/SO(1,1)$, because the Lorentz matrices, Λ^r_s, $(r,s = 1,0)$ leave the Minkowski metric invariant. A right-multiplication of $\pi(\widetilde{\eta})$ by a Lorentz matrix gives an other deformation matrix. So, all the fundamental geometrical structures on deformed/distorted MS in fact - the metric as much as the coframes and connections - acquire a *deformation/distortion* induced theoretical interpretation. If we deform the tetrad according to (66), in general, we have two choices to recast metric as follows: either writing the deformation of the metric in the space of tetrads or deforming the tetrad field:

$$g = {}^{*}o_{ab}\, \pi^a{}_c \pi^b{}_d \breve{\vartheta}^c \otimes \breve{\vartheta}^d = \gamma_{cd}\, \breve{\vartheta}^c \otimes \breve{\vartheta}^d = {}^{*}o_{ab}\, \vartheta^a \otimes \vartheta^b. \tag{15}$$

In the first case, the contribution of the Christoffel symbols, constructed by the metric γ_{ab}, reads

$$\Gamma^a{}_{bc} = \tfrac{1}{2}\left(\breve{C}^a{}_{bc} - \gamma^{aa'}\, \gamma_{bb'}\, \breve{C}^{b'}{}_{a'c} - \gamma^{aa'}\, \gamma_{cc'}\, \breve{C}^{c'}{}_{a'b}\right) + \tfrac{1}{2}\, \gamma^{aa'}\, \left(\breve{e}_c \rfloor d\,\gamma_{ba'} - \breve{e}_b \rfloor d\,\gamma_{ca'} - \breve{e}_{a'} \rfloor d\,\gamma_{bc}\right). \tag{16}$$

The deformed metric can be split as [51]:

$$g_{\tilde{A}\tilde{B}}(\pi) = \Upsilon^2(\pi)\, \breve{g}_{\tilde{A}\tilde{B}} + \gamma_{\tilde{A}\tilde{B}}(\pi), \tag{17}$$

where $\Upsilon(\pi) = \pi^a_a$, and

$$\gamma_{\tilde{A}\tilde{B}}(\pi) = [\gamma_{ab} - \Upsilon^2(\pi)\, {}^{*}o_{ab}]\, \breve{e}^a{}_{\tilde{A}}\, \breve{e}^b{}_{\tilde{B}}. \tag{18}$$

In the second case, we may write the commutation table for the anholonomic frame, $\{e_a\}$,

$$[e_a,\, e_b] = -\tfrac{1}{2}\, C^c{}_{ab}\, e_c, \tag{19}$$

and define the anholonomy objects

$$C^a{}_{bc} = \pi^a{}_e\, \pi^{-1d}{}_b\, \pi^{-1f}{}_c\, \breve{C}^e{}_{df} + 2\, \pi^a{}_f\, \breve{e}_g{}^{\tilde{A}} \left(\pi^{-1g}{}_{[b}\partial_{\tilde{A}}\, \pi^{-1f}{}_{c]}\right). \tag{20}$$

Taking into account (10), the metric (15) can be alternatively written in a general form of the spacetime or frame objects:

$$g = g_{\tilde{A}\tilde{B}}\, \vartheta^{\tilde{A}} \otimes \vartheta^{\tilde{B}} = \left(\widetilde{\Omega}_{\tilde{A}}{}^{\tilde{B}}\, \widetilde{\Omega}_{\tilde{C}}{}^{\tilde{D}}\right) \breve{g}_{\tilde{B}\tilde{D}}\, \breve{\vartheta}^{\tilde{A}} \otimes \breve{\vartheta}^{\tilde{C}} = {}^{*}o_{ab}\, \vartheta^a \otimes \vartheta^b = (\Omega_a{}^c\, \Omega_b{}^d)^{*} o_{cd}\, \breve{\vartheta}^a \otimes\, \breve{\vartheta}^b = \gamma_{cd}\, \breve{\vartheta}^c \otimes\, \breve{\vartheta}^d = \left(\Omega_A{}^C\, \Omega_B{}^D\right) {}^{*}o_{CD}\, \bar{\vartheta}^A \otimes \bar{\vartheta}^B. \tag{21}$$

A significantly more rigorous formulation of the spacetime deformation technique as we have presented it may be found in [51].

IV. MODEL BUILDING IN THE 4D BACKGROUND MINKOWSKI SPACETIME

In this section, we construct the RTI in particular case when the relativistic test particle accelerated in the Minkowski 4D background flat space, M_4, under an unbalanced net force other than gravitational. Here and henceforth we simplify DC for our use by imposing the constraints

$$D^A_{\tilde{C}} = \breve{D}^A_{\tilde{B}}, \quad \breve{Y}^{\tilde{A}}_B = \breve{D}^{\tilde{A}}_B, \tag{22}$$

and, therefore,

$$DC\, \sim\, (D,\, \Omega) \rightarrow \widetilde{\Omega}.$$

The (11), by virtue of (10) and (22), gives

$$\widetilde{\Omega}_{\tilde{A}}{}^{\tilde{B}} = \breve{D}^C_{\tilde{A}}\, \Omega_C{}^D\, \breve{D}^{\tilde{B}}_D = \pi_{\tilde{A}}{}^{\tilde{B}}, \quad Y^{\tilde{C}}_B = \widetilde{\Omega}_{\tilde{A}}{}^{\tilde{C}}\, \breve{D}^{\tilde{A}}_B, \tag{23}$$

where the deformation tensor, $\widetilde{\Omega}_{\tilde{A}}{}^{\tilde{B}}$, yields the partial holonomic frame transformations

$$e_{\tilde{C}} = \breve{e}_{\tilde{C}}, \quad \vartheta^{\tilde{C}} = \widetilde{\Omega}^{\tilde{C}}_{\tilde{A}}\, \breve{\vartheta}^{\tilde{V}}, \tag{24}$$

or, respectively, the $\Omega_b{}^a$ yields the partial local tetrad deformations

$$e_c = \breve{e}_c, \quad \vartheta^c = \Omega^c{}_b\, \breve{\vartheta}^b, \quad e\,\vartheta = e_a \otimes \vartheta^a = \Omega^a{}_b\, \breve{e}_a \otimes \breve{\vartheta}^b. \tag{25}$$

Hence, (10) defines a diffeomorphism $\widetilde{\eta}^{\tilde{A}}(\eta):\ \ M_2 \rightarrow \widetilde{\mathcal{M}}_2$:

$$e_{\tilde{A}}\, Y^{\tilde{A}}_B = \Omega_B{}^A\, \overline{e}_A\,, \tag{26}$$

where $Y^{\tilde{A}}_B = \partial\widetilde{\eta}^{\tilde{A}}/\partial\eta^B$. The conditions of integrability, $\partial_A\, Y^{\tilde{C}}_B = \partial_B\, Y^{\tilde{C}}_A$, and non-degeneracy, $det|Y^{\tilde{A}}_B| \neq 0$, immediately define a general form of the *flat-deformation tensor* $\Omega_B{}^A :\ \ = D^A_{\tilde{C}}\, \partial_B \Theta^{\tilde{C}}$, where $\Theta^{\tilde{C}}$ is an arbitrary holonomic function. To make the remainder of our discussion a bit more concrete, it proves necessary to provide, further, a constitutive ansatz of simple, yet tentative, linear *distortion transformations*, which, according to RLI-Conjecture, can be written in terms of *local rate* $\varrho(\eta, m, f)$ of instantaneously change of the measure v^A of massive (m) test particle under the unbalanced net force (f):

$$e_{(\tilde{+})}(\varrho) = D_{(\tilde{+})}{}^B(\varrho)\, \overline{e}_B = \overline{e}_{(+)} - \varrho(\eta, m, f)\, v^{(-)}\, \overline{e}_{(-)}, \quad e_{(\tilde{-})}(\varrho) = D_{(\tilde{-})}{}^B(\varrho)\, \overline{e}_B = \overline{e}_{(-)} + \varrho(\eta, m, f)\, v^{(+)}\, \overline{e}_{(+)}. \tag{27}$$

Clearly, these transformations imply a violation of the relation (3) $(e^2_{\tilde{A}}(\varrho) \neq 0)$ for the null vectors $\overline{e}_A$. The (10), for dual vectors of differential forms $\vartheta = \begin{pmatrix} \vartheta^{(\tilde{+})} \\ \vartheta^{(\tilde{-})} \end{pmatrix}$ and $\overline{\vartheta} = \begin{pmatrix} \overline{\vartheta}^{(+)} \\ \overline{\vartheta}^{(-)} \end{pmatrix}$, gives

$$\vartheta = \begin{pmatrix} \Omega^{\ C}_{(+)} < e^{(\tilde{+})},\, \overline{e}_C > & \Omega^{\ C}_{(-)} < e^{(\tilde{+})},\, \overline{e}_C > \\ \Omega^{\ C}_{(+)} < e^{(\tilde{-})},\, \overline{e}_C > & \Omega^{\ C}_{(-)} < e^{(\tilde{-})},\, \overline{e}_C > \end{pmatrix} \overline{\vartheta}. \tag{28}$$

We may parameterize the tensor $\Omega_B{}^A$ in terms of the parameters τ_1 and τ_2 as

$$\Omega_{(+)}{}^{(+)} = \Omega_{(-)}{}^{(-)} = \tau_1(1+\tau_2\,\overline{\varrho}^2), \quad \Omega_{(+)}{}^{(-)} = -\tau_1(1-\tau_2)\varrho v^{(-)}, \quad \Omega_{(-)}{}^{(+)} = \tau_1(1-\tau_2)\varrho v^{(+)}, \tag{29}$$

where $\overline{\varrho}^2 = v^2\varrho^2$, $v^2 = v^{(+)}v^{(-)} = 1/2\gamma_q^2$ and $\gamma_q = (1-v_q^2)^{-1/2}$. Then, the relation (28) can be recast in an alternative form

$$\vartheta = \tau_1 \begin{pmatrix} 1 & -\tau_2\varrho\, v^{(+)} \\ \tau_2\varrho\, v^{(-)} & 1 \end{pmatrix} \overline{\vartheta}. \tag{30}$$

Suppose a second observer, who makes measurements using a frame of reference $\widetilde{S}_{(2)}$ which is held stationary in deformed/distorted space $\widetilde{\mathcal{M}}_2$, uses for the test particle the corresponding spacetime coordinates $\widetilde{q}^{\tilde{r}}\,\left((\widetilde{q}^{\tilde{0}},\, \widetilde{q}^{\tilde{1}}) \equiv (\widetilde{t},\, \widetilde{q})\right)$. The (10) can be rewritten in terms of spacetime variables as

$$id = e\,\vartheta \equiv d\widetilde{\hat{q}} = \widetilde{e}_0 \otimes d\widetilde{t} + \widetilde{e}_q \otimes d\widetilde{q}, \tag{31}$$

where $\widetilde{e}_0$ and $\widetilde{e}_q$ are, respectively, the temporal and spatial basis vectors:

$$\widetilde{e}_0(\varrho) = \tfrac{1}{\sqrt{2}}\left[e_{(\tilde{+})}(\varrho) + e_{(\tilde{-})}(\varrho)\right], \quad \widetilde{e}_q(\varrho) = \tfrac{1}{\sqrt{2}}\left[e_{(\tilde{+})}(\varrho) - e_{(\tilde{-})}(\varrho)\right]. \tag{32}$$

The transformation equation for the coordinates, according to (30), becomes

$$\vartheta^{(\tilde{\pm})} = \tau_1\,(\overline{\vartheta}^{(\pm)} \mp \tau_2\,\varrho v^{(\pm)}\overline{\vartheta}^{(\mp)}) = \tau_1\,(v^{(\pm)} \mp \tau_2\,\varrho v^2)dt, \tag{33}$$

which gives the general transformation equations for spatial and temporal coordinates as follows ($\vec{e}_q \equiv e_1$, $q \equiv q^1$):

$$d\widetilde{t} = \tau_1\,dt, \quad d\widetilde{q} = \tau_1\left[dq(1+\tfrac{\tau_2\varrho v_q}{\sqrt{2}}) - \tfrac{\tau_2\varrho}{\sqrt{2}}dt\right] = \tau_1\,(dq - \tfrac{\tau_2\varrho}{\sqrt{2}\gamma_q^2}dt). \tag{34}$$

Hence, the general metric (21) in $\widetilde{\mathcal{M}}_2$ reads

$$\begin{aligned} g \equiv d\widetilde{s}_q^2 = g_{\tilde{r}\tilde{s}}\,d\widetilde{q}^{\tilde{r}} \otimes d\widetilde{q}^{\tilde{s}} = &\left[(\Omega_{(+)}{}^{(+)})^2 + \Omega_{(-)}{}^{(+)}\Omega_{(+)}{}^{(-)}\right] ds_q^2 + \Omega_{(+)}{}^{(+)}\left(\Omega_{(-)}{}^{(+)} + \Omega_{(+)}{}^{(-)}\right)(dt\otimes dt + dq\otimes dq) - \\ &2\Omega_{(+)}{}^{(+)}\left(\Omega_{(-)}{}^{(+)} - \Omega_{(+)}{}^{(-)}\right) dt\otimes dq, \end{aligned} \tag{35}$$

provided

$$g_{\tilde{0}\tilde{0}} = (1+\tfrac{\varrho v_q}{\sqrt{2}})^2 - \tfrac{\varrho^2}{2}, \quad g_{\tilde{1}\tilde{1}} = -(1-\tfrac{\varrho v_q}{\sqrt{2}})^2 + \tfrac{\varrho^2}{2}, \quad g_{\tilde{1}\tilde{0}} = g_{\tilde{0}\tilde{1}} = -\sqrt{2}\varrho. \tag{36}$$

The difference of the vector, $d\hat{q} \in M_2$ (5), and the vector, $d\widetilde{\hat{q}} \in \widetilde{\mathcal{M}}_2$ (31), can be interpreted by the second observer as being due to the deformation/distortion of flat space M_2. However, this difference with equal justice can be interpreted by him as a definite criterion for the *absolute* character of his own state of acceleration in M_2, rather than to any absolute quality of a deformation/distortion of M_2. To prove this assertion, note that the transformation equations (34) give a reasonable change at low velocities $v_q \simeq 0$, as

$$d\widetilde{t} = \tau_1\,dt, \quad d\widetilde{q} \simeq \tau_1\,(dq - \tfrac{\tau_2\varrho}{\sqrt{2}}dt), \tag{37}$$

thereby

$$\Omega_{(+)}{}^{(+)} = \Omega_{(-)}{}^{(-)} = \tau_1(1+\tau_2\overline{\varrho}^2), \quad \Omega_{(-)}{}^{(+)} = -\Omega_{(+)}{}^{(-)} = \tau_1(1-\tau_2)\overline{\varrho}. \tag{38}$$

The (37) becomes conventional transformation equations to accelerated ($a \neq 0$) axes if we assume $d(\tau_2\varrho)/\sqrt{2}dt = a$ and $\tau_1(v_q \simeq 0) = 1$. In high velocity limit $v_q \simeq 1$, $\overline{\varrho} \simeq 0$, ($d\eta^{(-)} = v^{(-)}dt \simeq 0$, $v^{(+)} \simeq v \simeq \sqrt{2}$), we have

$$\Omega_{(+)}{}^{(+)} = \Omega_{(-)}{}^{(-)} = \tau_1, \quad \Omega_{(+)}{}^{(-)} = 0, \quad \Omega_{(-)}{}^{(+)} = \tau_1(1-\tau_2)\sqrt{2}\varrho, \tag{39}$$

so (34) and (35), respectively, give

$$d\widetilde{t} = \tau_1\,dt \simeq \tau_1\,dq \simeq d\widetilde{q}, \tag{40}$$

and

$$d\widetilde{s}_q^2 \simeq \left[(1+\frac{\varrho}{\sqrt{2}})^2-\frac{\varrho^2}{2}\right] d\widetilde{t}\otimes d\widetilde{t}+\left[-(1-\frac{\varrho}{\sqrt{2}})^2+\frac{\varrho^2}{2}\right] d\widetilde{q}\otimes d\widetilde{q}-2\sqrt{2}\varrho\, d\widetilde{t}\otimes d\widetilde{q}\simeq \tau_1^2\, ds_q^2=0. \tag{41}$$

To this end, *the inertial effects become zero.* Let $\vec{a}_{net}$ be a local net 3-acceleration of an arbitrary observer with proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ measured in the rest frame:

$$\vec{a}_{net}=\frac{d\vec{u}}{ds}=\vec{a}\wedge\vec{u}+\vec{\omega}\times\vec{u}, \tag{42}$$

where $\mathbf{u}$ is the 4-velocity. A magnitude of $\vec{a}_{net}$ can be computed as the simple invariant of the absolute value $|\frac{d\mathbf{u}}{ds}|$ as measured in rest frame:

$$|\mathbf{a}|=|\frac{d\mathbf{u}}{ds}|=\left(\frac{du^l}{ds},\, \frac{du_l}{ds}\right)^{1/2}. \tag{43}$$

Following [46, 52], let us define an orthonormal frame $e_{\hat{a}}$, carried by an accelerated observer, who moves with proper linear 3-acceleration and $\vec{a}(s)$ and proper 3-rotation $\vec{\omega}(s)$. Let the zeroth leg of the frame $e_{\hat{0}}$ be 4-velocity $\mathbf{u}$ of the observer that is tangent to the worldline at a given event $x^l(s)$ and we parameterize the remaining spatial triad frame vectors $e_{\hat{i}}$, orthogonal to $e_{\hat{0}}$, also by (s). The spatial triad $e_{\hat{i}}$ rotates with proper 3-rotation $\vec{\omega}(s)$. The 4-velocity vector naturally undergoes Fermi-Walker transport along the curve C, which guarantees that $e_{\hat{0}}(s)$ will always be tangent to C determined by $x^l=x^l(s)$:

$$\frac{de_{\hat{a}}}{ds}=-\Omega\, e_{\hat{a}} \tag{44}$$

where the antisymmetric rotation tensor Ω splits into a Fermi-Walker transport part Ω_{FW} and a spatial rotation part Ω_{SR}:

$$\Omega_{FW}^{lk}=a^l u^k-a^k u^l, \quad \Omega_{SR}^{lk}=u_m\omega_n\,\varepsilon^{mnlk}. \tag{45}$$

The 4-vector of rotation ω^l is orthogonal to 4-velocity u^l, therefore, in the rest frame it becomes $\omega^l(0,\,\vec{\omega})$, and ε^{mnlk} is the Levi-Civita tensor with $\varepsilon^{0123}=-1$. The (37) immediately indicates that we may introduce the very concept of the local *absolute acceleration* (in Newton's terminology) brought about via the Fermi-Walker transported frames as

$$\vec{a}_{abs}\equiv\vec{e}_q\frac{d(\tau_2\varrho)}{\sqrt{2}ds_q}=\vec{e}_q\,|\frac{de_{\hat{0}}}{ds}|=\vec{e}_q\,|\mathbf{a}|, \tag{46}$$

where we choose the system $S_{(2)}$ in such a way as the axis $\vec{e}_q$ lies along the net 3-acceleration $(\vec{e}_q\,||\,\vec{e}_a)$, $(\vec{e}_a=\vec{a}_{net}/|\vec{a}_{net}|)$. Combining (33) and (46), we obtain the key relation between a so-called *inertial* acceleration

$$\vec{a}_{in}=\vec{e}_a\,\frac{d^2\widetilde{q}}{ds_q^2}=\vec{e}_a\frac{1}{\sqrt{2}}(\frac{d^2\widetilde{\eta}^{(+)}}{ds_q^2}-\frac{d^2\widetilde{\eta}^{(-)}}{ds_q^2}), \tag{47}$$

and a local *absolute acceleration* as follows:

$$\gamma_q\,\vec{a}_{in}=-\vec{a}_{abs}. \tag{48}$$

The (48) provides a quantitative means for the *inertial force* $\vec{f}_{(in)}$:

$$\vec{f}_{(in)}=m\vec{a}_{in}=-\frac{m\vec{a}_{abs}}{\gamma_q}, \tag{49}$$

In case of absence of rotation, we may write the local *absolute acceleration* (46) in terms of the relativistic force f^l acting on a particle with coordinates $x^l(s)$ ([53]):

$$f^l(f^0,\,\vec{f})=m\frac{d^2x^l}{ds^2}=\Lambda^l_k(\vec{v})F^k. \tag{50}$$

Here $F^k(0,\,\vec{F})$ is the force defined in the rest frame of the test particle, $\Lambda^l_k(\vec{v})$ is the Lorentz transformation matrix $(i,j=1,2,3)$:

$$\Lambda^i_j=\delta_{ij}-(\gamma-1)\frac{v_iv_j}{|\vec{v}|^2}, \quad \Lambda^0_i=\gamma v_i, \tag{51}$$

where $\gamma=(1-\vec{v}^2)^{-1/2}$. So the local rate $\varrho(m,\, f^l)$ of change of the measure of difference from the *absolute* state for massive (m) test particle under the unbalanced net force $f^l(x^0,\, x^i)(f^0,\, \vec{f})$ other than gravitational at the instant x^0 when the acceleration begins, can be determined as

$$\frac{1}{\sqrt{2}}\frac{d(\tau_2\varrho)}{ds_q}=|\mathbf{a}|=\frac{1}{m}|f^l|=\frac{1}{m}(f^lf_l)^{1/2}=\frac{1}{m\gamma}|\vec{f}|. \tag{52}$$

The (46), (52) and (45) give

$$\vec{f}_{(in)}=-\frac{1}{\gamma_q\gamma}[\vec{F}+(\gamma-1)\frac{\vec{v}(\vec{v}\cdot\vec{F})}{|\vec{v}|^2}]. \tag{53}$$

At low velocities $v_q\simeq|\vec{v}|\simeq 0$, the (53) reduces to the conventional non-relativistic law of inertia

$$\vec{f}_{(in)}=-m\vec{a}_{abs}=-\vec{F}. \tag{54}$$

At high velocities $v_q\simeq|\vec{v}|\simeq 1$, if $(\vec{v}\cdot\vec{F})\neq 0$, the inertial force (53) becomes

$$\vec{f}_{(in)}\simeq-\frac{1}{\gamma}\vec{e}_v(\vec{e}_v\cdot\vec{F}), \tag{55}$$

and, in agreement with (41), it vanishes in the limit of the photon ($|\vec{v}|=1,\, m=0$). Thus, it takes force to disturb an inertia state, i.e. to make the *absolute acceleration* ($\vec{a}_{abs}\neq 0$). The *absolute acceleration* is due to the real deformation/distortion of the space M_2. The *relative* ($d(\tau_2\varrho)/ds_q=0$) acceleration (in Newton's terminology) (both magnitude and direction), to the contrary, has nothing to do with the deformation/distortion of the space M_2 and, thus, it cannot produce an inertia effects.

V. THE ACCELERATING AND ROTATING OBSERVER IN MINKOWSKI SPACETIME

The standard geometrical structures, related to the noninertial coordinate frame of accelerating and rotating observer in Minkowski spacetime, were computed on the base of the hypothesis of locality [23]-[29]), which in effect replaces an accelerated observer at each instant with a momentarily comoving inertial observer along its wordline. This assumption represents strict restrictions, because in other words, it approximately replaces a noninertial frame of reference $\widetilde{S}_{(2)}$, which is held stationary in the deformed/distorted space $\widetilde{\mathcal{M}}_2\equiv V_2^{(\varrho)}$ ($\varrho\neq 0$), with a continuous infinity set of the inertial frames $\{\widetilde{S}_{(2)},\, \widetilde{S}'_{(2)},\, \widetilde{S}''_{(2)},...\}$ given in the flat M_2 ($\varrho=0$). Therefore, it appears natural to go beyond the hypothesis of locality with an emphasis on distortion of MS, which we might expect will essentially improve the standard results. The notation will be slightly different from the previous section. We denote the orthonormal frame $e_{\hat{a}}$ (44), carried by an accelerated observer, with the over '*breve*' such that

$$\breve{e}_{\hat{a}}=\overline{e}^{\,\mu}_{\ \hat{a}}\,\overline{e}_\mu=\breve{e}^{\,\mu}_{\ \hat{a}}\,\breve{e}_\mu, \quad \breve{\vartheta}^{\hat{b}}=\overline{e}_\mu^{\ \hat{b}}\,\overline{\vartheta}^\mu=\breve{e}_\mu^{\ \hat{b}}\,\breve{\vartheta}^\mu, \tag{56}$$

with $\overline{e}_\mu=\partial_\mu=\partial/\partial x^\mu$, $\breve{e}_\mu=\breve{\partial}_\mu=\partial/\partial\breve{x}^\mu$, $\overline{\vartheta}^\mu=dx^\mu$, $\breve{\vartheta}^\mu=d\breve{x}$. Here, following [28, 52], we introduced a geodesic coordinate system $\breve{x}^\mu$- "coordinates relative to the accelerated observer" (laboratory coordinates), in the neighborhood of the accelerated path. The coframe members $\{\breve{\vartheta}^{\hat{b}}\}$ are the objects of dual counterpart: $\breve{e}_{\hat{a}}\rfloor\breve{\vartheta}^{\hat{b}}=\delta^b_a$. We choose the zeroth leg of the frame, $\breve{e}_{\hat{0}}$, as before, to be the unit vector $\mathbf{u}$ that is tangent to the worldline at a given event $x^\mu(s)$, where (s) is a proper time measured along the accelerated path by the standard (static inertial) observers in the underlying global inertial frame. The condition of orthonormality for the frame field $\overline{e}^{\,\mu}_{\ \hat{a}}$ reads $\eta_{\mu\nu}\,\overline{e}^{\,\mu}_{\ \hat{a}}\,\overline{e}^{\,\nu}_{\ \hat{b}}=o_{\hat{a}\hat{b}}=diag(+---)$. The antisymmetric acceleration tensor Φ_{ab} [28]-[31] is given by

$$\Phi_a^{\ b}:=\overline{e}_\mu^{\ \hat{b}}\,\frac{d\overline{e}^{\,\mu}_{\ \hat{a}}}{ds}, \tag{57}$$

where according to (44) and (45), and in analogy with the Faraday tensor, one can identify $\Phi_{ab}\to(-\mathbf{a},\,\omega)$, with $\mathbf{a}(s)$ as the translational acceleration $\Phi_{0i}=-a_i$, and $\omega(s)$ as the frequency of rotation of the local spatial frame with respect to a nonrotating (Fermi- Walker transported) frame $\Phi_{ij}=-\varepsilon_{ijk}\,\omega^k$. The hypothesis of locality holds for huge proper acceleration lengths $|I|^{-1/2}\gg 1$ and $|I^*|^{-1/2}\gg 1$, where the scalar invariants are given by $I=(1/2)\,\Phi_{ab}\,\Phi^{ab}=-\vec{a}^2+\vec{\omega}^2$ and $I^*=(1/4)\,\Phi^*_{ab}\,\Phi^{ab}=-\vec{a}\cdot\vec{\omega}$ ($\Phi^*_{ab}=\varepsilon_{abcd}\,\Phi^{cd}$) [28, 29]. Suppose the displacement vector $z^\mu(s)$ represents the position of the accelerated observer. According to the hypothesis of locality, at any time (s) along the accelerated worldline the hypersurface orthogonal to the worldline is Euclidean space and we usually describe some event on this hypersurface ("local coordinate system") at x^μ to be at $\breve{x}^\mu$, where x^μ and $\breve{x}^\mu$ are connected via $\breve{x}^{\,0}=s$ and

$$x^\mu=z^\mu(s)+\breve{x}^{\,i}\,\overline{e}^{\,\mu}_{\ \hat{i}}(s). \tag{58}$$

Let $\breve{q}^{\,r}(\breve{q}^{\,0},\,\breve{q}^{\,1})$ be "coordinates relative to the accelerated observer" in the neighborhood of the accelerated path in MS, with spacetime components implying

$$d\breve{q}^{\,0} = d\breve{x}^{\,0}, \quad d\breve{q}^{\,1} = |d\vec{\breve{x}}|, \quad \vec{\breve{e}} = \frac{d\vec{\breve{x}}}{d\breve{q}^{\,1}} = \frac{d\vec{\breve{x}}}{|d\vec{\breve{x}}|}, \quad \vec{\breve{e}}\cdot\vec{\breve{e}} = 1. \tag{59}$$

As long as a locality assumption holds, we may describe, with equal justice, the event at x^{μ} (58) to be at point $\breve{q}^{\,r}$, such that x^{μ} and $\breve{q}^{\,r}$, in full generality, are connected via $\breve{q}^{\,0} = s$ and

$$x^{\mu} = z_{q}^{\mu}(s) + \breve{q}^{\,1}\,\overline{\beta}^{\mu}{}_{\hat{1}}(s), \tag{60}$$

where the displacement vector from the origin reads $dz_{q}^{\mu}(s) = \overline{\beta}^{\,\mu}{}_{\hat{0}}\,d\breve{q}^{\,0}$, and the components $\overline{\beta}^{\,\mu}{}_{\hat{r}}$ can be written in terms of $\overline{e}^{\,\mu}{}_{\hat{a}}$. Actually, from (58) and (60) we may obtain

$$\begin{gathered} dx^{\mu} = dz_{q}^{\mu}(s) + d\breve{q}^{\,1}\,\overline{\beta}^{\,\mu}{}_{\hat{1}}(s) + \breve{q}^{\,1}\,d\overline{\beta}^{\,\mu}{}_{\hat{1}}(s) = \left[\overline{\beta}^{\,\mu}{}_{\hat{0}}(1+\breve{q}^{\,1}\breve{\varphi}_0) + \overline{\beta}^{\,\mu}{}_{\hat{1}}\,\breve{q}^{\,1}\breve{\varphi}_1\right] d\breve{q}^{\,0} + \overline{\beta}^{\,\mu}{}_{\hat{1}}\,d\breve{q}^{\,1} \equiv \\ dz^{\mu}(s) + d\breve{x}^{\,i}\,\overline{e}^{\,\mu}{}_{\hat{i}}(s) + \breve{x}^{\,i}\,d\overline{e}^{\,\mu}{}_{\hat{i}}(s) = \left[\overline{e}^{\,\mu}{}_{\hat{0}}(1+\breve{x}^{\,i}\Phi_i^0) + \overline{e}^{\,\mu}{}_{\hat{j}}\,\breve{x}^{\,i}\Phi_i^j)\right] d\breve{x}^{\,0} + \overline{e}^{\,\mu}{}_{\hat{i}}\,d\breve{x}^{\,i}, \end{gathered} \tag{61}$$

where $d\overline{\beta}^{\,\mu}{}_{\hat{1}}(s)$ is written in the basis $\overline{\beta}^{\,\mu}{}_{\hat{a}}$ as $d\overline{\beta}^{\,\mu}{}_{\hat{1}} = (\breve{\varphi}_0\overline{\beta}^{\,\mu}{}_{\hat{0}} + \breve{\varphi}_1\overline{\beta}^{\,\mu}{}_{\hat{1}})d\breve{q}^{\,0}$. The equation (61) holds if one identifies

$$\overline{\beta}^{\,\mu}{}_{\hat{0}}\left(1+\breve{q}^{\,1}\breve{\varphi}_0\right) \equiv \overline{e}^{\,\mu}{}_{\hat{0}}\left(1+\breve{x}^{\,i}\Phi_i^0\right), \quad \overline{\beta}^{\,\mu}{}_{\hat{1}}\,\breve{q}^{\,1}\breve{\varphi}_1 \equiv \overline{e}^{\,\mu}{}_{\hat{j}}\,\breve{x}^{\,i}\Phi_i^j, \quad \overline{\beta}^{\,\mu}{}_{\hat{1}}\,d\breve{q}^{\,1} \equiv \overline{e}^{\,\mu}{}_{\hat{i}}\,d\breve{x}^{\,i}. \tag{62}$$

Choosing $\overline{\beta}^{\,\mu}{}_{\hat{0}} \equiv \overline{e}^{\,\mu}{}_{\hat{0}}$, we have then

$$\breve{q}^{\,1}\breve{\varphi}_0 = \breve{x}^{\,i}\,\Phi_i^0, \quad \overline{\beta}^{\,\mu}{}_{\hat{1}} = \overline{e}^{\,\mu}{}_{\hat{i}}\,\breve{e}^{\,i}, \quad \breve{q}^{\,1}\breve{\varphi}_1 = \breve{x}^{\,i}\,\Phi_i^j\,\breve{e}_i^{\,-1}, \tag{63}$$

with $\breve{e}^{j}\,\breve{e}_i^{\,-1} = \delta_i^j$. Consequently, (61) yields the standard metric of semi-Riemannian 4D background space $V_4^{(0)}$, in noninertial system of the accelerating and rotating observer, computed on the base of hypothesis of locality:

$$\breve{g} = \eta_{\mu\nu}\,dx^{\mu}\otimes dx^{\nu} = \left[(1+\vec{a}\cdot\vec{\breve{x}})^2 + (\vec{\omega}\cdot\vec{\breve{x}})^2 - (\vec{\omega}\cdot\vec{\omega})(\vec{\breve{x}}\cdot\vec{\breve{x}})\right] d\breve{x}^0\otimes d\breve{x}^0 - 2\,(\vec{\omega}\wedge\vec{\breve{x}})\cdot d\vec{\breve{x}}\otimes d\breve{x}^0 - d\vec{\breve{x}}\otimes d\vec{\breve{x}}, \tag{64}$$

This metric was derived by [23] and [24], in agreement with [25] -[27] (see also [28, 29]). We see that the hypothesis of locality leads to the 2D semi-Riemannian MS space : $V_2^{(0)}$ with the incomplete metric $\breve{g} \quad (\varrho = 0)$:

$$\breve{g} = \left[(1+\breve{q}^{\,1}\breve{\varphi}_0)^2 - (\breve{q}^{\,1}\breve{\varphi}_1)^2\right] d\breve{q}^{\,0}\otimes d\breve{q}^{\,0} - 2\,(\breve{q}^{\,1}\breve{\varphi}_1)d\breve{q}^{\,1}\otimes d\breve{q}^{\,0} - d\breve{q}^{\,1}\otimes d\breve{q}^{\,1}, \tag{65}$$

Therefore, our strategy now is to deform the metric (65) by an additional deformation of semi-Riemannian 4D background space $V_4^{(0)} \to \widetilde{\mathcal{M}}_4 \equiv V_4^{(\varrho)}$, which, as a corollary, will recover the complete metric $g \quad (\varrho \neq 0)$ (35) of the distorted MS - $V_2^{(\varrho)}$. Following [51], this means that we should find the first deformation matrices, $\pi(\varrho) : \quad = (\pi_{\hat{a}}{}^{\hat{b}})(\varrho)$, which yield the local tetrad deformations

$$e_{\hat{c}} = \pi_{\hat{c}}{}^{\hat{a}}\,\breve{e}_{\hat{a}}, \quad \vartheta^{\hat{c}} = \pi^{\hat{c}}{}_{\hat{b}}\,\breve{\vartheta}^{\hat{b}}, \quad e\,\vartheta = e_{\hat{a}}\otimes\vartheta^{\hat{a}} = \Omega^{a}{}_{\hat{b}}\,\breve{e}_{\hat{a}}\otimes\breve{\vartheta}^{\hat{b}}, \tag{66}$$

where $\Omega^{\hat{a}}{}_{\hat{b}}(\varrho) = \pi_{\hat{c}}{}^{\hat{a}}(\varrho)\,\pi_{\hat{b}}{}^{\hat{c}}(\varrho)$ is referred to as the anholonomic *deformation tensor*, and that the resulting deformed metric of the space $V_4^{(\varrho)}$ can be split as

$$g_{\mu\nu}(\varrho) = \Upsilon^2(\varrho)\,\breve{g}_{\mu\nu} + \gamma_{\mu\nu}(\varrho), \tag{67}$$

provided

$$\gamma_{\mu\nu}(\varrho) = \left[\gamma_{\hat{a}\hat{b}} - \Upsilon^2(\varrho)\,o_{\hat{a}\hat{b}}\right]\breve{e}^{\,\hat{a}}{}_{\mu}\,\breve{e}^{\,\hat{b}}{}_{\nu}, \quad \gamma_{\hat{c}\hat{d}} = o_{\hat{a}\hat{b}}\,\pi_{\hat{c}}{}^{\hat{a}}\,\pi_{\hat{d}}{}^{\hat{b}}, \tag{68}$$

where $\Upsilon(\varrho) = \pi_{\hat{a}}^{\hat{a}}(\varrho)$ and $\gamma_{\hat{a}\hat{b}}(\breve{x})$ are the second deformation matrices. Let the Latin letters $\hat{r}, \hat{s}, ... = 0, 1$ be the anholonomic indices related to the anholonomic frame $e_{\hat{r}} = e^{s}{}_{\hat{r}}\,\partial_{\tilde{s}}$, defined on the $V_2^{(\varrho)}$, with $\partial_{\tilde{s}} = \partial/\partial\,\widetilde{q}^{\tilde{s}}$ as the vectors tangent to the coordinate lines. So, a smooth differential 2D-manifold $V_2^{(\varrho)}$ has at each point $\widetilde{q}^{s}$ a tangent space $\widetilde{T}_{\widetilde{q}}V_2^{(\varrho)}$, spanned by the frame, $\{e_{\hat{r}}\}$, and the coframe members $\vartheta^{\hat{r}} = e_{s}{}^{\hat{r}}\,d\widetilde{q}^{\tilde{s}}$, which constitute a basis of the covector space $\widetilde{T}^{\star}_{\widetilde{q}}V_2^{(\varrho)}$. All this nomenclature can be given for $V_2^{(0)}$ too. Then, we may calculate corresponding vierbein fields $\breve{e}_r{}^{\hat{s}}$ and $e_r{}^{\hat{s}}$ from the equations

$$\breve{g}_{rs} = \breve{e}_r{}^{\hat{r}'}\,\breve{e}_s{}^{\hat{s}'}\,o_{\hat{r}'\hat{s}'}, \quad g_{\tilde{r}\tilde{s}} = e_r{}^{\hat{r}'}\,e_s{}^{\hat{s}'}\,o_{\hat{r}'\hat{s}'}, \tag{69}$$

with $\breve{g}_{rs}$ and $g_{\tilde{r}\tilde{s}}$ given by (65) and (36), respectively. Hence

$$\begin{array}{l} \breve{e}_0{}^{\hat{0}} = 1 + \vec{a}\cdot\vec{\breve{x}}, \quad \breve{e}_0{}^{\hat{1}} = \vec{\omega}\cdot\vec{\breve{x}}, \quad \breve{e}_1{}^{\hat{0}} = 0, \quad \breve{e}_1{}^{\hat{1}} = 1, \\ e_0{}^{\hat{0}} = 1 + \frac{\varrho v_q}{\sqrt{2}}, \quad e_0{}^{\hat{1}} = \frac{\varrho}{\sqrt{2}}, \quad e_1{}^{\hat{0}} = -\frac{\varrho}{\sqrt{2}}, \quad e_1{}^{\hat{1}} = 1 - \frac{\varrho v_q}{\sqrt{2}}. \end{array} \tag{70}$$

Since a distortion of MS may affect only the MS-part of the components $\overline{\beta}^{\mu}{}_{\hat{r}}$, without relation to the background spacetime part, therefore, a deformation $V_4^{(0)} \to V_4^{(\varrho)}$ is equivalent to a straightforward generalization $\overline{\beta}^{\mu}{}_{\hat{r}} \to \beta^{\mu}{}_{\hat{r}}$, where

$$\beta^{\mu}{}_{\hat{r}} = E_{\hat{r}}{}^{\hat{s}}\,\overline{\beta}^{\mu}{}_{\hat{s}}, \quad E_{\hat{r}}{}^{\hat{s}} : \quad = e^{r'}{}_{\hat{r}}\,\breve{e}_{r'}{}^{\hat{s}}. \tag{71}$$

Consequently, the (71) gives a generalization of (58) as

$$x^{\mu} \;\to\; x^{\mu}_{(\varrho)} = z^{\mu}_{(\varrho)}(s) + \breve{x}^{\,i}\,e^{\mu}{}_{\hat{i}}(s), \tag{72}$$

provided, as before, $\breve{x}^{\mu}$ denotes the coordinates relative to the accelerated observer in 4D background space $V_4^{(\varrho)}$, and according to (62), we have

$$e^{\mu}{}_{\hat{0}} = \beta^{\mu}{}_{\hat{0}}, \quad e^{\mu}{}_{\hat{i}} = \beta^{\mu}{}_{\hat{i}}\,\breve{e}_i^{-1}. \tag{73}$$

A displacement vector from the origin is then $dz^{\mu}_{\varrho}(s) = e^{\mu}{}_{\hat{0}}\,d\breve{x}^0$, Combining (71) and (73), and inverting $e_r{}^{\hat{s}}$ (70), we obtain $e^{\mu}{}_{\hat{a}} = \pi_{\hat{a}}{}^{\hat{b}}(\varrho)\,\overline{e}^{\mu}{}_{\hat{b}}$, where

$$\begin{array}{l} \pi^{\hat{0}}_{\hat{0}}(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma_q^2})^{-1}(1 - \frac{\varrho v_q}{\sqrt{2}})\,(1 + \vec{a}\cdot\vec{\breve{x}}), \quad \pi^{\hat{i}}_{\hat{0}}(\varrho) \equiv -(1 + \frac{\varrho^2}{2\gamma_q^2})^{-1}\frac{\varrho}{\sqrt{2}}\,\breve{e}^i\,(1 + \vec{a}\cdot\vec{\breve{x}}), \\ \pi^{\hat{0}}_{\hat{i}}(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma_q^2})^{-1}\left[(\vec{\omega}\cdot\vec{\breve{x}})(1 - \frac{\varrho v_q}{\sqrt{2}}) - \frac{\varrho}{\sqrt{2}}\right]\breve{e}_i^{-1}, \quad \pi^{\hat{j}}_{\hat{i}}(\varrho) = \delta^j_i\,\pi(\varrho), \\ \pi(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma_q^2})^{-1}\,\left[(\vec{\omega}\cdot\vec{\breve{x}})\,\frac{\varrho}{\sqrt{2}} + 1 + \frac{\varrho v_q}{\sqrt{2}}\right]. \end{array} \tag{74}$$

Thus,

$$dx^{\mu}_{\varrho} = dz^{\mu}_{\varrho}(s) + d\breve{x}^{\,i}\,e^{\,\mu}{}_{\hat{i}} + \breve{x}^{\,i}\,de^{\,\mu}{}_{\hat{i}}(s) = (\tau^{\hat{b}}\,d\breve{x}^0 + \pi^{\hat{b}}_{\hat{i}}\,d\breve{x}^{\,i})\,\overline{e}^{\,\mu}{}_{\hat{b}}, \tag{75}$$

where

$$\tau^{\hat{b}} \equiv \pi^{\hat{b}}_{\hat{0}} + \breve{x}^{\,i}\,\left(\pi^{\hat{a}}_{\hat{i}}\Phi^b_a + \frac{d\pi^{\hat{b}}_{\hat{i}}}{ds}\right). \tag{76}$$

Hence, in general, the metric in noninertial frame of arbitrary accelerating and rotating observer in Minkowski spacetime is

$$g(\varrho) = \eta_{\mu\nu}\,dx^{\mu}_{\varrho}\otimes dx^{\nu}_{\varrho} = W_{\mu\nu}(\varrho)\,d\breve{x}^{\mu}\otimes d\breve{x}^{\nu}, \tag{77}$$

which can be conveniently decomposed according to

$$\begin{array}{l} W_{00}(\varrho) = \pi^2\left[(1 + \vec{a}\cdot\vec{\breve{x}})^2 + (\vec{\omega}\cdot\vec{\breve{x}})^2 - (\vec{\omega}\cdot\vec{\omega})(\vec{\breve{x}}\cdot\vec{\breve{x}})\right] + \gamma_{00}(\varrho), \\ W_{0i}(\varrho) = -\pi^2\,(\vec{\omega}\wedge\vec{\breve{x}})^i + \gamma_{0i}(\varrho), \quad W_{ij}(\varrho) = -\pi^2\,\delta_{ij} + \gamma_{ij}(\varrho), \end{array} \tag{78}$$

and that

$$\begin{array}{l} \gamma_{00}(\varrho) = \pi\,\left[(1 + \vec{a}\cdot\vec{\breve{x}})\zeta^0 - (\vec{\omega}\wedge\vec{\breve{x}})\cdot\vec{\zeta}\right] + (\zeta^0)^2 - (\vec{\zeta})^2, \quad \gamma_{0i}(\varrho) = -\pi\,\zeta^i + \tau^{\hat{0}}\,\pi^{\hat{0}}_{\hat{i}}, \\ \gamma_{ij}(\varrho) = \pi^{\hat{0}}_{\hat{i}}\,\pi^{\hat{0}}_{\hat{j}}, \quad \zeta^0 = \pi\,\left(\tau^{\hat{0}} - 1 - \vec{a}\cdot\vec{\breve{x}}\right), \quad \vec{\zeta} = \pi\,\left(\vec{\tau} - \vec{\omega}\wedge\vec{\breve{x}}\right). \end{array} \tag{79}$$

As we expected, according to (77)- (79), the matric $g(\varrho)$ is decomposed in the form of (67):

$$g(\varrho) = \pi^2(\varrho)\,\breve{g} + \gamma(\varrho), \tag{80}$$

where $\gamma(\varrho) = \gamma_{\mu\nu}(\varrho)\, d\breve{x}^\mu \otimes d\breve{x}^\nu$ and $\Upsilon(\varrho) = \pi^{\hat{a}}_{\hat{a}}(\varrho) = \pi(\varrho)$, provided (46) gives $(s = s_q)$

$$\tfrac{\tau_2 \varrho}{\sqrt{2}} = \int_0^s |\mathbf{a}| ds'. \tag{81}$$

In general, the geodesic coordinates are admissible as long as

$$\left(1 + \vec{a}\cdot\vec{\breve{x}} + \tfrac{\zeta^0}{\pi}\right)^2 > \left(\vec{\omega}\wedge\vec{\breve{x}} + \tfrac{\vec{\zeta}}{\pi}\right)^2. \tag{82}$$

The equations (64) and (77) say that the vierbein fields, with entries $\eta_{\mu\nu}\,\overline{e}^{\,\mu}_{\;\hat{a}}\,\overline{e}^{\,\nu}_{\;\hat{b}} = o_{\hat{a}\hat{b}}$, lead to the relations

$$\breve{g} = o_{\hat{a}\hat{b}}\,\breve{\vartheta}^{\hat{a}}\otimes\breve{\vartheta}^{\hat{b}}, \quad g = o_{\hat{a}\hat{b}}\,\vartheta^{\hat{a}}\otimes\vartheta^{\hat{b}}, \tag{83}$$

and that (61) and (75) readily give the coframe fields:

$$\begin{aligned} \breve{\vartheta}^{\hat{b}} &= \overline{e}_{\mu}^{\;\hat{b}}\, dx^\mu = \breve{e}^{\hat{b}}_{\;\mu}\, d\breve{x}^{\mu}, \quad \breve{e}^{\hat{b}}_{\;0} = N^b_0, \quad \breve{e}^{\hat{b}}_{\;i} = N^b_i, \\ \vartheta^{\hat{b}} &= \overline{e}_{\mu}^{\;\hat{b}}\, dx^\mu_\varrho = e^{\hat{b}}_{\;\mu}\, d\breve{x}^{\mu} = \pi^{\hat{b}}_{\;\hat{a}}\,\breve{\vartheta}^{\hat{a}}, \quad e^{\hat{b}}_{\;0} = \tau^{\hat{b}}, \quad e^{\hat{b}}_{\;i} = \pi^{\hat{b}}_{\;\hat{i}}. \end{aligned} \tag{84}$$

where $N^0_0 = N \equiv \left(1 + \vec{a}\cdot\vec{\breve{x}}\right)$, $\quad N^0_i = 0, \quad N^i_0 = N^i \equiv \left(\vec{\omega}\cdot\vec{\breve{x}}\right)^i$, $\quad N^j_i = \delta^j_i$. In the standard $(3+1)$-decomposition of spacetime, N and N^i are known as *lapse function* and *shift vector*, respectively [54]. Hence, we may easily recover the frame field $e_{\hat{a}} = e_{\hat{b}}^{\;\mu}\,\breve{e}_\mu = \pi_{\hat{a}}^{\;\hat{b}}\,\breve{e}_{\hat{b}}$ by inverting (84):

$$e_{\hat{0}} = \frac{\pi}{\pi\,\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\,\tau^{\hat{k}}}\,\breve{e}_0 - \frac{\tau^{\hat{i}}}{\pi\,\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\,\tau^{\hat{k}}}\,\breve{e}_i, \quad e_{\hat{i}} = -\frac{\pi^{\hat{0}}_{\hat{i}}}{\pi\,\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\,\tau^{\hat{k}}}\,\breve{e}_0 + \pi^{-1}\left[\delta^j_i + \frac{\tau^j\,\pi^{\hat{0}}_{\hat{i}}}{\pi\,\tau^{\hat{0}} - \pi^{\hat{0}}_{\hat{k}}\,\tau^{\hat{k}}}\right]\breve{e}_j. \tag{85}$$

VI. INVOLVING THE BACKGROUND SEMI-RIEMANN SPACE V_4; JUSTIFICATION FOR THE INTRODUCTION OF THE PE

We can always choose *natural coordinates* $X^\alpha(T, X, Y, Z) = (T, \vec{X})$ with respect to the axes of the local free-fall coordinate frame $S^{(l)}_4$ in an immediate neighbourhood of any spacetime point $(\breve{x}_p) \in V_4$ in question of the background semi- Riemann space, V_4, over a differential region taken small enough so that we can neglect the spatial and temporal variations of gravity for the range involved. The values of the metric tensor $\breve{g}_{\mu\nu}$ and the affine connection $\breve{\Gamma}^\lambda_{\mu\nu}$ at the point $(\breve{x}_p)$ are necessarily sufficient information for determination of the natural coordinates $X^\alpha(\breve{x}^\mu)$ in the small region of the neighbourhood of the selected point [53]. Then the whole scheme outlined in the section 4 will be held in the frame $S^{(l)}_4$. The relativistic gravitational force $\breve{f}^\mu_g(\breve{x})$ exerted on the test particle of the mass (m) is given by

$$\breve{f}^\mu_g(\breve{x}) = m\frac{d^2\breve{x}^\mu}{d\breve{s}^2} = -m\breve{\Gamma}^\mu_{\nu\lambda}(a)\frac{d\breve{x}^\nu}{d\breve{s}}\frac{d\breve{x}^\lambda}{d\breve{s}}, \tag{86}$$

such that the gravitational force in the free-fall coordinate frame $S^{(l)}_4$ will be

$$f^\alpha_{g(l)} = \frac{\partial X^\alpha}{\partial\breve{x}^\mu}\breve{f}^\mu_g, \tag{87}$$

As before, the two systems S_2 and $S^{(l)}_4$ can be chosen in such a way as the axis $\vec{e}_q$ of $S_{(2)}$ lies $(\vec{e}_q = \vec{e}_f)$ along the acting net force $\vec{f} = \vec{f}_{(l)} + \vec{f}_{g(l)}$, where $\vec{f}_{(l)}$ is the SR value of the unbalanced relativistic force other than gravitational in the frame $S^{(l)}_4$, while the time coordinates in the two systems are taken the same, $q^0 = t = X^0 = T$. The (52) now can be replaced by

$$\frac{1}{\sqrt{2}}\frac{d(\tau_2\varrho)}{ds_q} = \frac{1}{m}|f^\alpha_{(l)} + f^\alpha_{g(l)}|, \tag{88}$$

and according to (49), the general *inertial force* reads

$$\breve{\vec{f}}_{(in)} = m\vec{a}_{in} = -\frac{m\vec{a}_{abs}}{\gamma_q} = -\frac{\vec{e}_f}{\gamma_q}|f^\alpha_{(l)} - m\frac{\partial X^\alpha}{\partial\breve{x}^\sigma}\breve{\Gamma}^\sigma_{\mu\nu}\frac{d\breve{x}^\mu}{dS}\frac{d\breve{x}^\nu}{dS}|. \tag{89}$$

Despite of totally different and independent sources of gravitation and inertia, at $f^{\alpha}_{(l)} = 0$, the (89) establishes the independence of free-fall trajectories of the mass, internal composition and structure of bodies. This furnishes a justification for the introduction of the PE. A remarkable feature is that, although the inertial force has a nature different than the gravitational force, nevertheless both are due to a distortion of the local inertial properties of, respectively, 2D MS and 4D-background space. The non-vanishing inertial force acting on the photon of energy $h\nu$, and that of effective mass $\left(h\nu/c^2\right)$, after inserting units $(h,\, c)$ which so far was suppressed, can be obtained from the (89) $(f^{\alpha}_{(l)} = 0)$ as

$$\breve{\vec{f}}_{(in)} = -\left(\frac{h\nu}{c^2}\right)\vec{e}_f \left|\frac{\partial X^\alpha}{\partial \breve{x}^\sigma}\Gamma^\sigma_{\mu\nu}\frac{d\breve{x}^\mu}{dT}\frac{d\breve{x}^\nu}{dT}\right| = -\left(\frac{h\nu}{c^2}\right)\vec{e}_f \left|\left(\frac{d^2\tilde{t}}{dT^2}\right)\frac{dX^\alpha}{d\tilde{t}} + \left(\frac{d\tilde{t}}{dT}\right)^2\frac{\partial X^\alpha}{\partial \breve{x}^{\,i}}\frac{du_i}{d\tilde{t}}\right|, \tag{90}$$

provided $\vec{e}_f = (\vec{X}/|\vec{X}|)$, $v_q = (\vec{e}_f \cdot \breve{\vec{u}}) = |\breve{\vec{u}}|$, $(\gamma_q = \gamma)$ where $\breve{\vec{u}}$ is the velocity of a photon and $(d\breve{\vec{u}}/d\tilde{t})$ is the acceleration, and that, $\breve{g}_{\mu\nu}(d\breve{x}^\mu/dT) \otimes (d\breve{x}^\nu/dT) = 0$. To obtain some feeling for this, in the (PPN) approximation [55]-[58] we may calculate the inertial force exerted on the photon [59], in a gravitating system of particles that are bound together by their mutual gravitational attraction to order $\bar{v}^2 \sim G_N\bar{M}/\bar{r}$ of a small parameter, where $\bar{v}$, $\bar{M}$ and $\bar{r}$ are typically the average values of their velocities, masses and separations, respectively. To this aim, we may expand the metric tensor to the following order: $\breve{g}_{00} = 1 + \overset{2}{g}_{00} + \overset{4}{g}_{00} + \ldots$, $\breve{g}_{ij} = -\delta_{ij} + \overset{2}{g}_{ij} + \overset{4}{g}_{ij} + \ldots$, $\breve{g}_{i0} = \overset{3}{g}_{i0} + \overset{5}{g}_{i0} + \ldots$, where $\overset{N}{g}_{\mu\nu}$ denotes the term of order $\bar{v}^N$. Taking into account the standard expansions of the affine connection [53]: $\breve{\Gamma}^\sigma_{\mu\nu} = \overset{2}{\Gamma}{}^\sigma_{\mu\nu} + \overset{4}{\Gamma}{}^\sigma_{\mu\nu} + \ldots$ for the components $\breve{\Gamma}^i_{00}$, $\breve{\Gamma}^i_{jk}$, $\breve{\Gamma}^0_{0i}$, and that $\breve{\Gamma}^\sigma_{\mu\nu} = \overset{3}{\Gamma}{}^\sigma_{\mu\nu} + \overset{5}{\Gamma}{}^\sigma_{\mu\nu} + \ldots$ for the components $\breve{\Gamma}^i_{0j}$, $\breve{\Gamma}^0_{00}$, $\breve{\Gamma}^0_{ij}$, where $\overset{2}{\Gamma}{}^i_{00} = \overset{2}{\Gamma}{}^0_{0i} = -(1/2)(\partial\, \overset{2}{g}_{00}\,/\partial \breve{x}^{\,i})$ etc., hence to the required accuracy we obtain

$$\breve{\vec{f}}{}^{(2)}_{(in)} = -\left(\frac{h\nu}{c^2}\right)\vec{e}_f \left|\,\left(\overset{1}{\frac{\partial X^\alpha}{\partial \breve{x}^\sigma}}\right)\left(\overset{2}{\frac{d^2\breve{x}^\sigma}{dT^2}}\right)\,\right| = -\left(\frac{h\nu}{c^2}\right)\left(\overset{2}{\frac{d\breve{\vec{u}}}{dt}}\right) = -\left(\frac{h\nu}{\gamma c^2}\right)\left[-2\vec{\nabla}\phi + 4\breve{\vec{u}}(\breve{\vec{u}}\cdot\vec{\nabla}\phi) + O(\bar{v}^3)\right], \tag{91}$$

where ϕ is the Newton potential, such that $\overset{2}{g}_{00} = 2\phi$, $\overset{2}{g}_{ij} = 2\delta_{ij}\phi$, and $|\breve{\vec{u}}| = 1 + 2\phi + O(\bar{v}^3)$.

VII. RTI IN THE BACKGROUND POST RIEMANNIAN GEOMETRY

Recall that the general metric-affine space, $(\widetilde{\mathcal{M}}_4,\ g,\, \Gamma)$, is defined to have equipped with two independent geometrical structures: the pseudo-Riemannian metric, g and the linear affine connection Γ. The new geometrical property of the spacetime, are the *nonmetricity* 1-form N_{ab} and the affine *torsion* 2-form T^a representing a translational misfit (for a comprehensive discussion see [60]-[63]. These, together with the *curvature* 2-form $R_a{}^b$, symbolically can be presented as $\left(N_{ab},\, T^a,\, R_a{}^b\right) \;\sim\; \mathcal{D}\left(g_{ab},\, \vartheta^a,\, \Gamma_a{}^b\right)$, where for a tensor-valued $p-$form density of representation type $\rho(L^b{}_a)$, the $GL(4,R)$-covariant exterior derivative reads $\mathcal{D} : \;= d + \Gamma_a{}^b\rho(L^b{}_a)\wedge$. To avoid any possibility of confusion, here and throughout we use the first half of Latin alphabet $(a, b, c, ... = 0, 1, 2, 3$ rather than $(\pm))$ now to denote the anholonomic indices related to the tangent space, which is endowed with the Lorentzian metric $o_{ab} := diag(+ - - -)$. If the nonmetricity tensor $N_{\lambda\mu\nu} = -\mathcal{D}_\lambda\, g_{\mu\nu} \equiv -g_{\mu\nu\,;\lambda}$ does not vanish, the general formula for the affine connection written in the spacetime components is [63]

$$\Gamma^\rho{}_{\mu\,\nu} = \overset{\circ}{\Gamma}{}^{\,\rho}{}_{\mu\,\nu} + K^\rho{}_{\mu\nu} - N^\rho{}_{\mu\nu} + \frac{1}{2}N_{(\mu}{}^\rho{}_{\nu)}, \tag{92}$$

where the metric alone determines the torsion-free Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\,\rho}{}_{\mu\nu}$, $K^\rho{}_{\mu\nu} : \;= 2Q_{(\mu\nu)}{}^\rho + Q^\rho{}_{\mu\nu}$ is the non-Riemann part - the affine *contortion tensor*. The torsion, $Q^\rho{}_{\mu\nu} = \frac{1}{2}\,T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{[\mu\,\nu]}$ given with respect to a holonomic frame, $d\,\vartheta^\rho = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components.

A. The principle of equivalence in the RC space

The RC manifold, U_4, is a particular case of general metric-affine manifold $\widetilde{M}_4$, restricted by the metricity condition $N_{ab} = 0$, when a nonsymmetric linear connection, Γ, is said to be metric compatible. The space, U_4, also locally has the structure of M_4, as has been first pointed out by [64] and developed by [65]-[68]. In the case of the RC space there also exist orthonormal reference frames which realize an 'anholonomic' free-fall elevator. In Hartley's formulation [68], this reads: *For any single point $P \in U_4$, there exist coordinates $\{x^\mu\}$ and an orthonormal frame $\{e_a\}$ in a neighborhood of P such that*

$$\left\{ \begin{array}{rcl} e_a &=& \delta^\mu_a\, \partial_{x^\mu} \\ \Gamma_a{}^b &=& 0 \end{array} \right\} \quad \text{at } P$$

where $\Gamma_a{}^b$ are the connection 1-forms referred to the frame $\{e_a\}$. Therefore the existence of torsion does not violate the PE. Note that, since $\nabla\, \mathbf{g} = 0$ holds in U_4, the arguments showing that $\mathbf{g}$ can be transformed to o at any point P in U_4 are the same as in the case of V_4, while the treatment of the connection must be different: the antisymmetric part of ω can be eliminated only by a suitable choice for the relative orientation of neighbouring tetrads. Actually, let us choose new local coordinates at P, $\;d\,x^\mu \to d\,x^a = e^a_\mu\, d\,x^\mu$, related to an inertial frame. Then,

$$g'_{ab} = e_a{}^\mu\, e_b{}^\nu\, g_{\mu\nu} = o_{ab}, \quad \Gamma'^b{}_{ac} = e^b{}_\mu\, e_a{}^\nu\, e_c{}^\lambda\, (\Delta^\mu{}_{\nu\lambda} + K^\mu{}_{\nu\lambda}) \equiv e_c{}^\lambda\, \omega^b{}_{a\lambda}. \tag{93}$$

As it is argued in [69], the metricity condition ensures that this can be done consistently at every point in spacetime. Suppose that we have a tetrad $\{e_a(x)\}$ at the point P, and a tetrad $\{e_a(x+d\,x)\}$ at another point in a neighbourhood of P; then, we can apply a suitable Lorentz rotation to $e_a(x+d\,x)$, so that it becomes parallel to $e_a(x)$. Given a vector v at P, it follows that the components $v_c = v \cdot e_c$ do not change under parallel transport from x to $x + d\,x$, provided the metricity condition holds. Hence, the connection coefficients $\omega^{ab}{}_\mu(x)$ at P, defined with respect to this particular tetrad field, vanish: $\omega^{ab}{}_\mu(P) = 0$. This property is compatible with $g'_{ab} = o_{ab}$, since Lorentz rotation does not influence the value of the metric at a given point. In more general geometries, where the symmetry of the tangent space is higher than the Poincare group, the usual form of the PE is violated and local physics differs from SR.

B. The generalized inertial force exerted on the extended spinning body in the U_4

We now compute the relativistic inertial force for the motion of the matter, which is distributed over a small region in the U_4 space and consists of points with the coordinates x^μ, forming an extended body whose motion in the space, U_4, is represented by a world tube in spacetime. Suppose the motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $\tilde{X}^\mu(\tau)$, where τ is the proper time on γ. Define

$$\delta x^\mu = x^\mu - \tilde{X}^\mu, \;\; \delta x^0 = 0, \;\; u^\mu = \tfrac{d\,\tilde{X}^\mu}{d\,s}. \tag{94}$$

The *Papapetrou equation of motion for the modified momentum* ([70]-[72], [63]) is

$$\frac{\overset{\circ}{\mathcal{D}}\,\Theta^\nu}{\mathcal{D}\,s} = -\tfrac{1}{2}\, \overset{\circ}{R}{}^\nu{}_{\mu\sigma\rho}\, u^\mu\, J^{\sigma\rho} - \tfrac{1}{2}\, N_{\mu\rho\lambda}\, K^{\mu\rho\lambda:\,\nu}, \tag{95}$$

where $K^\mu_{\nu\lambda}$ is the contortion tensor,

$$\Theta^\nu = P^\nu + \tfrac{1}{u^0}\, \overset{\circ}{\Gamma}{}^\nu_{\mu\,\rho}\, (u^\mu\, J^{\rho 0} + N^{0\mu\rho}) - \tfrac{1}{2u^0}\, K_{\mu\rho}{}^\nu\, N^{\mu\rho 0} \tag{96}$$

is referred to as the *modified 4-momentum*, $P^\lambda = \int \tau^{\lambda 0}\, d\,\Omega$ is the ordinary 4-momentum, $d\,\Omega := d\,x^4$, and the following integrals are defined:

$$\begin{array}{l} M^{\mu\rho} = u^0 \int \tau^{\mu\rho}\, d\,\Omega, \quad M^{\mu\nu\rho} = -u^0 \int \delta x^\mu\, \tau^{\nu\rho}\, d\,\Omega, \quad N^{\mu\nu\rho} = u^0 \int s^{\mu\nu\rho}\, d\,\Omega, \\ J^{\mu\rho} = \int (\delta x^\mu\, \tau^{\rho 0} - \delta x^\rho\, \tau^{\mu 0} + s^{\mu\rho 0})\, d\,\Omega = \tfrac{1}{u^0}(-M^{\mu\rho 0} + M^{\rho\mu 0} + N^{\mu\rho 0}), \end{array} \tag{97}$$

where $\tau^{\mu\rho}$ is the energy-momentum tensor for particles, $s^{\mu\nu\rho}$ is the spin density. The quantity $J^{\mu\rho}$ is equal to $\int (\delta x^\mu\, \tau^{kl} - \delta x^\rho\, \tau^{\mu\lambda} + s^{\mu\rho\lambda})\, d\,S_\lambda$ taken for the volume hypersurface, so it is a tensor, which is called the *total spin tensor*. The quantity $N^{\mu\nu\rho}$ is also a tensor. The relation $\delta x^0 = 0$ gives $M^{0\nu\rho} = 0$. It was assumed that the dimensions of the body are small, so integrals with two or more factors δx^μ multiplying $\tau^{\nu\rho}$ and integrals with one or more factors δx^μ multiplying $s^{\nu\rho\lambda}$ can be neglected. The *Papapetrou equations of motion for the spin* ([70]-[72], [63]) is

$$\frac{\overset{\circ}{\mathcal{D}}}{\mathcal{D}s}\, J^{\lambda\nu} = u^\nu\, \Theta^\lambda - u^\lambda\, \Theta^\nu + K^\lambda_{\;\mu\rho}\, N^{\nu\mu\rho} + \tfrac{1}{2}\, K_{\mu\rho}{}^\lambda\, N^{\mu\nu\rho} - K^\nu{}_{\mu\rho}\, N^{\lambda\mu\rho} - \tfrac{1}{2}\, K_{\mu\rho}{}^\nu\, N^{\mu\rho\lambda}. \tag{98}$$

Calculating from (95) the particle 4-acceleration is

$$\begin{array}{l} \tfrac{1}{m}\, f^\mu_g(x) = \tfrac{d^2 x^\mu}{d s^2} = -\Gamma^\mu_{\nu\lambda}\, \left[u^\nu\, u^\lambda + \tfrac{1}{u^0}\, \overset{\circ}{\Gamma}{}^\mu_{\nu\,\rho}\, (u^\nu\, J^{\rho 0} + N^{0\nu\rho})\right] + \tfrac{1}{2u^0}\, K_{\nu\rho}{}^\mu\, N^{\nu\rho 0} - \\ \tfrac{1}{2}\, \overset{\circ}{R}{}^\mu{}_{\nu\sigma\rho}\, u^\nu\, J^{\sigma\rho} - \tfrac{1}{2}\, N_{\nu\rho\lambda}\, K^{\nu\rho\lambda:\,\mu}. \end{array} \tag{99}$$

Thus, the relativistic inertial force, exerted on the extended spinning body moving in the RC space U_4, can be found to be

$$\vec{f}_{(in)}(x) = m\vec{a}_{in}(x) = -\frac{m\vec{a}_{abs}(x)}{\gamma_q} = -m\,\frac{\vec{e}_f}{\gamma_q}\,\Big| \frac{1}{m}\, f^{\alpha}_{(l)} - \frac{\partial X^{\alpha}}{\partial\, x^{\mu}}\, \Big[\overset{\circ}{\Gamma}{}^{\mu}_{\ \nu\lambda}\, u^{\nu}\, u^{\lambda} + \frac{1}{u^0}\, \overset{\circ}{\Gamma}{}_{\nu\rho}^{\ \ \mu}\, (u^{\nu}\, J^{\rho 0} + N^{0\nu\rho}) - \frac{1}{2u^0}\, K_{\nu\rho}{}^{\mu}\, N^{\nu\rho 0} + \frac{1}{2}\, \overset{\circ}{R}{}^{\mu}{}_{\nu\sigma\rho}\, u^{\nu}\, J^{\sigma\rho} + \frac{1}{2}\, N_{\nu\rho\lambda}\, K^{\nu\rho\lambda:\,\mu} \Big] \Big|. \tag{100}$$

In particular, if the spin density vanishes, $s^{\mu\nu\rho} = 0$, from the conservation law we get then $\tau^{\mu\rho} = \tau^{\rho\mu}$, $M^{\mu\rho} = M^{\rho\mu}$, $M^{\mu\nu\rho} = M^{\mu\rho\rho}$, $N^{\mu\nu\rho} = 0$, and that

$$J^{\mu\rho} = L^{\mu\rho} = \int (\delta\, x^{\mu}\, \tau^{\rho 0} - \delta\, x^{\rho}\, \tau^{\mu 0})\, d\,\Omega = \frac{1}{u^0}(-M^{\mu\rho 0} + M^{\rho\mu 0}), \tag{101}$$

where $L^{\mu\rho}$ is the angular momentum tensor. The modified 4-momentum (96) reduces to

$$\Theta^{\nu} = P^{\nu} + \frac{\overset{\circ}{\mathcal{D}}}{\mathcal{D}s}\, L^{\nu\lambda}\, u_{\lambda}. \tag{102}$$

The Eq. (98) can be recast in the form

$$\frac{\overset{\circ}{\mathcal{D}}}{\mathcal{D}s}\, L^{\lambda\nu} = u^{\nu}\, \Theta^{\lambda} - u^{\lambda}\, \Theta^{\nu}, \tag{103}$$

while the Eq. (95) becomes

$$\frac{\overset{\circ}{\mathcal{D}}\,\Theta^{\nu}}{\mathcal{D}\,s} = -\frac{1}{2}\, \overset{\circ}{R}{}^{\nu}{}_{\mu\sigma\rho}\, u^{\mu}\, L^{\sigma\rho}, \tag{104}$$

which give the relativistic inertial force exerted on the spinless extended body moving in the RC space U_4 as follows:

$$\vec{f}_{(in)}(x) = -m\,\frac{\vec{e}_f}{\gamma_q}\,\Big| \frac{1}{m}\, f^{\alpha}_{(l)} - \frac{\partial X^{\alpha}}{\partial\, x^{\mu}}\, \Big[\overset{\circ}{\Gamma}{}^{\mu}_{\ \nu\lambda}\, u^{\nu}\, u^{\lambda} + \frac{1}{u^0}\, \overset{\circ}{\Gamma}{}_{\nu\rho}^{\ \ \mu}\, u^{\nu}\, L^{\rho 0} + \frac{1}{2}\, \overset{\circ}{R}{}^{\mu}{}_{\nu\sigma\rho}\, u^{\nu}\, L^{\sigma\rho} \Big] \Big|. \tag{105}$$

If the body is not spatially extended then it is referred to as a *particle*. The corresponding condition $\delta\, x^{\alpha} = 0$ gives $M^{\mu\nu\rho} = 0$, $L^{\mu\rho} = 0$. Therefore $\frac{u^{\lambda}}{u^0}\, N^{\mu\nu 0} - N^{\mu\nu\lambda} = 0$, which gives $N^{\mu\nu\rho} = u^{\mu}\, J^{\nu\rho}$, so $J^{\mu\nu} = S^{\mu\nu} = N^{\mu\nu\rho}\, u_{\rho}$, where $S^{\mu\nu}$ is the *intrinsic spin tensor*. If the body is spatially extended then the difference $R^{\mu\rho} = J^{\mu\rho} - S^{\mu\rho}$ is the *rotational spin tensor*. The relativistic inertial force is then

$$\vec{f}_{(in)}(x) = -m\,\frac{\vec{e}_f}{\gamma_q}\,\Big| \frac{1}{m}\, f^{\alpha}_{(l)} - \frac{\partial X^{\alpha}}{\partial\, x^{\mu}}\, \Big[\overset{\circ}{\Gamma}{}^{\mu}_{\ \nu\lambda}\, u^{\nu}\, u^{\lambda} + \frac{1}{u^0}\, \overset{\circ}{\Gamma}{}_{\nu\rho}^{\ \ \mu}\, (u^{\nu}\, S^{\rho 0} + \overline{u^0}\, S^{\nu\rho}) - \frac{1}{2u^0}\, K_{\nu\rho}{}^{\mu}\, u^{\nu}\, S^{\rho 0} + \frac{1}{2}\, \overset{\circ}{R}{}^{\mu}{}_{\nu\sigma\rho}\, u^{\nu}\, S^{\sigma\rho} + \frac{1}{2}\, u_{\nu}\, S_{\rho\lambda}\, K^{\nu\rho\lambda:\,\mu} \Big] \Big|. \tag{106}$$

In case of the Riemann space, V_4 $(\breve{Q} = 0)$, the relativistic inertial force (100) exerted on the extended spinning body can be written in terms of the Ricci coefficient of rotation only:

$$\breve{\vec{f}}_{(in)}(\breve{x}) = -m\,\frac{\vec{e}_f}{\gamma_q}\,\Big| \frac{1}{m}\, f^{\alpha}_{(l)} - \frac{\partial X^{\alpha}}{\partial\, \breve{x}^{\mu}}\, \Big[\breve{\Gamma}^{\mu}_{\ \nu\lambda}\, \breve{u}^{\nu}\, \breve{u}^{\lambda} + \frac{1}{\breve{u}^0}\, \breve{\Gamma}_{\nu\rho}^{\ \ \mu}\, (\breve{u}^{\nu}\, \breve{J}^{\rho 0} + \breve{N}^{0\nu\rho}) + \frac{1}{2}\, \breve{R}^{\mu}{}_{\nu\sigma\rho}\, \breve{u}^{\nu}\, \breve{J}^{\sigma\rho} \Big] \Big|. \tag{107}$$

In case of the Weitzenböck space, W_4 $(\overset{\bullet}{R} = 0)$, the (100) reduces to its teleparallel equivalent,

$$\overset{\bullet}{\vec{f}}_{(in)}\, (\overset{\bullet}{x}) = -m\,\frac{\vec{e}_f}{\gamma_q}\,\Big| \frac{1}{m}\, f^{\alpha}_{(l)} - \frac{\partial X^{\alpha}}{\partial\, \overset{\bullet}{x}^{\mu}}\, \Big[\overset{\circ}{\Gamma}{}^{\mu}_{\ \nu\lambda}\, \overset{\bullet}{u}{}^{\nu}\, \overset{\bullet}{u}{}^{\lambda} + \frac{1}{\overset{\bullet}{u}{}^0}\, \overset{\circ}{\Gamma}{}_{\nu\rho}^{\ \ \mu}\, (\overset{\bullet}{u}{}^{\nu}\, \overset{\bullet}{J}{}^{\rho 0} + \overset{\bullet}{N}{}^{0\nu\rho}) - \frac{1}{2\overset{\bullet}{u}{}^0}\, \overset{\bullet}{K}{}_{\nu\rho}{}^{\mu}\, \overset{\bullet}{N}{}^{\nu\rho 0} + \frac{1}{2}\, \overset{\bullet}{N}{}_{\nu\rho\lambda}\, \overset{\bullet}{K}{}^{\nu\rho\lambda:\,\mu} \Big] \Big|. \tag{108}$$

All magnitudes related with the teleparallel gravity is denoted with an over '•'. Finally, the non-vanishing inertial force, $\mathbf{f}^{(phot)}_{(in)}(x)$, acting on the photon of energy $h\nu$ in the U_4, can be obtained from the (106), at $\vec{f}_{(l)} = 0$, as

$$\vec{f}^{(phot)}_{(in)}(x) = -\left(\frac{h\nu}{c^2}\right)\, \vec{e}_f\, \Big| \frac{\partial X^{\alpha}}{\partial\, x^{\mu}}\, \Big[\overset{\circ}{\Gamma}{}^{\mu}_{\ \nu\lambda}\, \frac{d\,x^{\nu}}{dT}\, \frac{d\,x^{\lambda}}{dT} + \frac{d\,T}{d\,\overline{t}}\, \overset{\circ}{\Gamma}{}_{\nu\rho}^{\ \ \mu}\, \Big(\frac{d\,x^{\nu}}{dT}\, S^{\rho 0} + \frac{d\,\overline{t}}{dT}\, S^{\nu\rho}\Big) - \frac{d\,T}{2d\,\overline{t}}\, K_{\nu\rho}{}^{\mu}\, \frac{d\,x^{\nu}}{dT}\, S^{\rho 0} + \frac{1}{2}\, \overset{\circ}{R}{}^{\mu}{}_{\nu\sigma\rho}\, \frac{d\,x^{\nu}}{dT}\, S^{\sigma\rho} + \frac{1}{2}\, \frac{d\,x_{\nu}}{dT}\, S_{\rho\lambda}\, K^{\nu\rho\lambda:\,\mu} \Big] \Big|, \tag{109}$$

where $\vec{e}_f = (\vec{X}/|\vec{X}|)$, $v_q = (\vec{e}_f \cdot \overline{\vec{u}}) = |\overline{\vec{u}}|$, $(\gamma_q = \gamma)$, $\overline{\vec{u}}$ is the velocity of the photon in U_4, $(d\,\overline{\vec{u}}/d\,\overline{t})$ is the acceleration, $g_{\mu\nu}\, (d\,x^{\mu}/dT) \otimes (d\,x^{\nu}/dT) = 0$.

VIII. CONCLUDING REMARKS

We construct the RTI, which treats the inertia as a distortion of local internal properties of hypothetical 2D, so-called, *master-space* (MS). The MS is an indispensable companion of individual particle, without relation to the other matter, embedded in the background 4D-spacetime. The RTI allows to compute the *inertial force*, acting on an arbitrary point-like observer or particle due to its *absolute acceleration.* In this framework we essentially improve standard metric and other relevant geometrical structures related to noninertial frame for an arbitrary velocities and characteristic acceleration lengths. Despite the totally different and independent physical sources of gravitation and inertia, this approach furnishes justification for the introduction of the principle of equivalence. We relate the inertia effects to the more general post-Riemannian geometry. We derive a general expression of the relativistic inertial force exerted on the extended spinning body moving in the Rieman-Cartan (RC) space.

References

[1] Gillies G T 1997 *Rep. Prog. Phys.* **60** 151
[2] Will C M, 2006 *Living Rev. Relativity* **9** 3
http://www.livingreviews.org/lrr-2006-3
Will C M 1998 *The Confrontation between General Relativity and Experiment*: An Update (Lecture notes from the 1998 SLAC Summer Institute on Particle Physics), arXiv:9811036[gr-qc]
[3] Turyshev S G 2008 *Annu. Rev. Nucl. Part. Sci.* **58** 207 arXiv:0806.1731 [gr-qc]
[4] Gabriel M D and Haugan M P 1990 *Phys. Rev. D* **41** 2943
[5] Haugan M P and Kauffmann T F: 1995 *Phys. Rev. D* **52** 3168
[6] Haugan M P and Lämmerzahl C 2001 in *Gyrod, Clocks, Interferometers...: Testing Relativistic Gravity in Space*, Lämmerzahl C, Everitt C W F, Hehl F W (eds.): SpringerVelag, p. 195
[7] Lämmerzahl C, Ch.J. Bordé: 2001 in Lämmerzahl C L, Everitt C W F, Hehl F W (eds.): *Gyrod, Clocks, Interferometers...: Testing Relativistic Gravity in Space* SpringerVelag, p. 466
[8] Ni W-T 2005 *Chin. Phys. Lett.* **22** 33 arXiv:0407113[gr-qc]
[9] Ni W-T 2005 *Int. J. Mod. Phys. D* **14** 901 (arXiv:0504116[gr-qc]
[10] Ni W-T 2008 *Prog. Theor. Phys. Suppl.* **172** 49 arXiv:0712.4082[gr-qc]
[11] Ni W-T, 2010 *Rep. Prog. Phys.* **73** 056901
[12] de Sabbata V and Gasperini M -II 1981 *Nuovo Cimento A* **66** 479
[13] Gasperini M 1988 *Phys. Rev.* **D38** 2635; 1989 *Phys. Rev. D* **39** 3606
[14] Halprin A and Leung C N 1991 *Phys. Rev. Lett.* **67** 1833; 1992 *Nucl. Phys. B (Proc. Suppl.)* **28A** 139
[15] Pantaleone J, Halprin A, Leung C N 1993 *Phys. Rev. D* **47** R4199; Iida K, Minakata H, Yasuda O 1993 Mod. Phys. Lett. A **8** 1037; Bultler M N, Nozawa S, Malaney R, Boothroyd A I 1993 *Phys. Rev. D* **47** (1993) 2615.
[16] Piniz D, Roy M, Wudka J 1996 *Phys. Rev. D* **54** 1587; 1997 *Phys. Rev. D* **56** 2403
[17] Cardall C Y, Fuller G M 1997 *Phys. Rev. D* **55** 7960
[18] Bonse U, Wroblewski T 1983 *Phys. Rev. Lett.* **51** 1401
[19] Colella R, Overhausen A W, Werner S A 1975 *Phys. Rev. Lett.* **34** 1472
[20] Atwood D K, Horne M A, Shull C G, Arthur J 1984 *Phys. Rev. Lett.* **52** 1673
[21] Cai Y Q, Lloyd D G and Papini G 1993 *Phys. Lett. A* **178** 225
[22] Capozziello S and Lambiase G 2000 *Eur. Phys. J C* **12** 343; 2000 *ibidem* **16** 155
[23] Hehl F W, Ni W-T 1990 *Phys. Rev. D* **42** 2045
[24] Hehl F W, Lemke J and Mielke E W: 1991 *Two lectures on fermions and gravity.* In: *Geometry and Theoretical Physics*, Debrus J and Hirshfeld A C (eds.) (Springer, Berlin) p. 56
[25] Ni W-T 1977 *Chinese J. Phys.* **15** 51
[26] Li W.-Q. and Ni W-T 1978 *Chinese J. Phys.* **16** 214
[27] Ni W-T and Zimmermann M 1978 *Phys. Rev. D* **17** 1473
[28] Mashhoon B 2002 *Ann. der Physik* **514**532
[29] Mashhoon B 2011 *Ann. der Physik* **523** 226; 2008 *ibidem* **520** 705; 2005 arXiv:hep-th/0507157; 2003 **Ann. der Physik 12** 586 arXive:0309124[hep-th]; 1990 Phys. Lett. A **143** 176; 1990 ibidem **145** 147 ; 1988 *Phys. Rev. Lett.* **61** 2639; 1995 ibidem **198** 9
[30] Maluf J W and Faria F F and Ulhoa S C 2007 *Class. Quantum Grav.* **24** 2743, arXiv:0704.0986[gr-qc].
[31] Maluf J W and Faria F F 2008 *Ann. der Physik* **520** 326, arXiv:0804.2502[gr-qc]
[32] Marzlin K-P 1996 *Phys. Lett. A* **215** 1
[33] J. Audretsch J, Hehl F W, and Lämmerzahl C 1991 In: *Relativistic Gravity Research* Proc., Bad Honnef, Germany, Ehlers J and Schäfer G eds., Lecture Notes in Physics (Springer) **410** (1992) 368;

[34] Pan Y Y and Ren J 2011 *Int. J Theor. Phys.*, **50** 724
[35] Bakke K and Furtado C 2010 *Phys. Rev. D* **82** id. 084025; 2010 *Europ Phys. J. C* **69** 531
[36] Bakke K 2010 *Phys. Rev. A* **81** id. 052117
[37] Korunur M, Salti M and Acikgoz I 2010 *Comm. in Theor. Phys.* **53** 864
[38] Ryder L 2008 *Gen. Rel. and Gr.* **40** 1111; 1998 *J. Phys. A: Math. Gen.* **31** 2465
[39] Arminjon M and Reifler F 2008 *Brazilian J. of Phys.* **38** 248
[40] Silenko A J, Teryaev O V 2007 *Phys. Rev. D* **76** id. 061101
[41] Mashhoon B and Kaiser H 2006 *Physica B: Physics of Condensed Matter* **385** 1381
[42] Bini D, Cherubini C and Mashhoon B 2004 *Class. and Quant. Gr.* **21** 3893
[43] Vulcanov D N and Cotaescu I I 1997 *Int. J. of Mod. Phys. C* **8** 345
[44] Vetharaniam I and Stedman G E 1994 *Class. Quantum Grav.* **11** 1069
[45] Huang J C 1994 *Ann. der Physik* **506** 53
[46] Synge J L 1960 *Relativity: The General Theory* North-Holland, Amsterdam
[47] Damour T, 2001 *Missions spatiales en physique fondamentale* (Chatillon, France), edited by Bordé C and Touboul P, published in Comptes Rendus de l'Academie des Sciences (Serie IV) **2** 1249 arXiv:0109063[gr-qc]
[48] Cocconi G and Salpeter E E 1960 *Phys. Rev. Lett.* **4** 176
[49] Hughes V W et al. 1960 *Phys. Rev. Lett.* **4** 342
[50] Drever R W P 1961 *Phil. Mag.* **6** 683
[51] Ter-Kazarian G 2011 *Class. Quantum Grav.* 28 055003, arXiv:1102.2491[gr-qc]
[52] Misner C W, Thorne K S, and Wheeler J A 1973 *Gravitation* (Freeman, San Francisco)
[53] Weinberg S 1972 *Gravitation and Cosmology* J. W. and Sons, New York
[54] Gronwald F and Hehl F W 1996 *Proc. of the 14th Course of the School of Cosmology and Gravitation on Quantum Gravity*, held at Erice, Italy, Eds. Bergmann P G, de Sabbata V, and Treder H.-J., (World Scientific, Singapore 1996); arXiv:9602013[gr-qc]
[55] Will C M 1993 *Theory and experiment in gravitational physics*, revised edition (Cambridge University Press) Cambridge
[56] Will C M and Nordtvedt K J 1972 *Astrophys. J.* **177** 757
[57] Nordtvedt K J and Will C M 1972 *Astrophys. J.* 177, 775
[58] Will C M 1973 *Astrophys. J.* **185** 31
[59] Ter-Kazarian G 2010 Ap&SS, **327** 91, arXiv:1001.2990[gr-qc]
[60] Hehl F W, Lord E A, Smalley L L 1981 *Gen. Relat. Grav.* **13** 1037; Hehl F W et al. 1995 *Phys. Rep.* **258** 1
[61] de Sabbata V and Gasperini M 1985 *Introduction to gravitation*, (Worid Scientific Publishing Co Pte Ltd., Singapore); 1983 in *Unified field theories of more than four dimensions*, ed. by de Sabbata V and Schmutzer E (World Scientific, Singapore) p.152 and references therein. de Sabbata V and Gasperini M 1980 *Phys.Lett. A* **77** 300
[62] de Sabbata V and Sivaram C 1994 *Spin and Torsion in Gravitation* (World Scientific, Singapore)
[63] Poplawski N J 2009 *Spacetime and fields* [gr-qc/0911.0334]
[64] von der Heyde P 1975 *Lettre al Nuovo Cimento* **14** 250
[65] Hehl F W 1980 *Four lectures on Poincaré gauge theory.* In: Proc. of the *6th Course of the School of Cosmology and Gravitation on 'Spin, Torsion, Rotation, and Supergravity* held at Erice, Italy, May 1979, Bergmann P G, de Sabbata V, eds. (Plenum, New York.) 5
[66] Modanese G and Toller M 1990 *J. Math. Phys.* **31** 452
[67] Iliev B Z 1992 Preprints: E5-92-507, E5-92-508, E5-92-543 (Joint Institute for Nuclear Research, Dubna)
[68] Hartley D 1995 *Class. Quantum Grav.* **12** L103
[69] Blagojevic M, 2002 *Gravitation and gauge symmetries, series in high energy physics, Cosmology and gravitation* Series Eds Foster B, Grishchuk L, Kolb E W, MacCallum M A H, Perkins D H and Schutz B F
[70] Papapetrou A 1948 *Proc. Roy. Irish Acad. A* **52** 11; 1951 *Proc.R.Soc. A* **202** 248; 1974 *Lectures on general relativity* (Reidel D)(ISBN 9027705402)
[71] Bergmann P G and Thompson R 1953 *Phys. Rev.* **89** 400
[72] Møller C 1958 *Ann. Phys.* (NY) **4** 347