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CONFORMALLY FLAT HYPERSURFACES IN EUCLIDEAN 4-SPACE II

Dedicated to Professor Shoshichi Kobayashi for his seventieth birthday

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Abstract

We study generic conformally flat hypersurfaces in the Euclidean 4-space. The conformal flatness condition for the Riemannian metric is given by a set of several differential equations of order three. In this paper, we first define a certain class Ξ of metrics for 3-manifolds which includes, as a large subset, all metrics of generic conformally flat hypersurfaces in the Euclidean 4-space. We obtain a kind of integrability condition on metrics of the class. Restricting our consideration to metrics of conformally flat hypersurfaces, we define a conformal invariant for generic conformally flat hypersurfaces and obtain a differential equation of order three from the integrability condition. The equation is equal to the simplest one in equations of conformal flatness condition. Next, we study some particular solutions of the equation. We will determine all generic conformally flat hyersurfaces corresponding to these particular solutions under an assumption on the first fundamental form, and characterize these hypersurfaces geometrically. The result includes all known examples of generic conformally flat hypersurfaces in the Euclidean 4-space. All known examples are the following: The hypersurfaces given by Lafontaine ([6]) which are made from constant Gaussian curvature surface in the three dimentional space forms, the hypersurfaces given by Suyama ([7]), and the flat metrics obtained by Hertrich-Jeromin ([3]). Furthermore, we explicitly construct a series of examples of generic conformally flat hypersurface, which have a geometrical property different from all known examples. Then, we have the following case: There exists a pair of hypersurfaces with the same conformal invariant, each of which is constructed from a surface with constant Gaussian curvature in either the Euclidean 3-space or the standard 3-sphere but does not belong to the known examples. Furthermore, no conformal transformation maps diffeomorphically one hypersurface of the pair to the other hypersurface.

1. Introduction

As a continuation of the paper [7], we study generic conformally flat hypersurfaces in the Euclidean 4-space $\mathbf{R}^4$ in this paper. A hypersurface is said to be generic if

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all principal curvatures are distinct (from each other) everywhere on the hypersurface. According to Cartan's theorem ([1], [6], [7]) and Hertrich-Jeromin ([3]) on generic conformally flat hypersurfaces in $\mathbf{R}^4$, there exists a special orthogonal curvature-line coordinate system, called a Guichard net, at each point of the hypersurface, namely a coordinate system $\{x^1, x^2, x^3\}$ which would represent the first fundamental form g as follows:

$$g = e^{2P(x)}\left\{\cos^2\varphi(x)(dx^1)^2 + \sin^2\varphi(x)(dx^2)^2 + (dx^3)^2\right\}, \tag{1.1}$$

or

$$g = e^{2P(x)}\left\{\cosh^2\varphi(x)(dx^1)^2 + \sinh^2\varphi(x)(dx^2)^2 + (dx^3)^2\right\}, \tag{1.2}$$

where $P(x) = P(x^1, x^2, x^3)$, $\varphi(x) = \varphi(x^1, x^2, x^3)$. Corresponding to the metric (1.1) or (1.2) respectively, the second fundamental form s is represented as

$$s = e^{2P(x)}\left\{\lambda(x)\cos^2\varphi(x)(dx^1)^2 + \mu(x)\sin^2\varphi(x)(dx^2)^2 + \nu(x)(dx^3)^2\right\}, \tag{1.3}$$

or

$$s = e^{2P(x)}\left\{\lambda(x)\cosh^2\varphi(x)(dx^1)^2 + \mu(x)\sinh^2\varphi(x)(dx^2)^2 + \nu(x)(dx^3)^2\right\}, \tag{1.4}$$

where $\lambda(x)$, $\mu(x)$ and $\nu(x)$ are principal curvatures in the direction of x^1-curve, x^2-curve and x^3-curve, respectively. Although the metric (1.2) is obtained from the metric (1.1) by interchanging variables, at the end of this section we will explain the reason to consider the two metrics. Furthermore, we note that our definition of the Guichard net is slightly different from that of the canonical Guichard net in [3] (cf. Remark at the end of §3.1).

We define a class Ξ of metrics for 3-manifolds which includes, as a large subset, all metrics for generic conformally flat hypersurfaces in $\mathbf{R}^4$: We say that a metric g for 3-manifolds (or open sets of the Euclidean 3-space $\mathbf{R}^3$) belongs to the class Ξ if in a suitable coordinate system $\{x^1, x^2, x^3\}$ g has the following properties (1) and (2):
(1) $\{x^1, x^2, x^3\}$ is an orthogonal coordinate system.
(2) The Riemannian curvature of g is diagonalizable, that is, the components R_{1323}, R_{1232} and R_{2131} of the curvature R identically vanish.

Then, a metric g of Ξ is represented in the following form with respect to the coordinate system $\{x^1, x^2, x^3\}$

$$g = e^{2P(x)}\left\{e^{2f(x)}(dx^1)^2 + e^{2h(x)}(dx^2)^2 + (dx^3)^2\right\} \tag{1.5}$$

by the property (1). This metric is an extension of (1.1) (resp. (1.2)), because we have (1.1) if take $e^{2f} = \cos^2\varphi$ and $e^{2h} = \sin^2\varphi$ in (1.5). Moreover, the Remannian curvature of the metric for a generic conformally flat hypersurface is diagonalized with respect to the Guichard net by (1.3) (resp. (1.4)) and Gauss's equation.

In §2.1, we obtain a kind of integrability condition on metrics of the class Ξ. Furthermore, when we restrict our consideration to metrics for conformally flat hypersurfaces, we will define a conformal invariant for generic conformally flat hypersurfaces and obtain a differential equation (see (2.1.8) (resp. (2.1.10))) for φ in (1.1) (resp. (1.2)) of order three from the integrability condition. If a function φ satisfies either at least one of the equations $\partial\varphi/\partial x^i = 0$ ($i = 1, 2, 3$) or $\partial^2\varphi/\partial x^1\partial x^3 = \partial^2\varphi/\partial x^2\partial x^3 = 0$, then it is a particular solution of the equation.

In §2.2, we reconsider all known conformally flat hypersurfaces (the hypersurfaces given by Lafontaine ([6]) and Suyama ([7])). For all known examples of generic conformally flat hypersurface, the function $\varphi(x)$ satisfies at least one of the equations $\partial\varphi/\partial x^i = 0$ ($i = 1, 2, 3$). Conversely, we can show that any generic conformally flat hypersurface such that the first fundamental form (1.1) (resp. (1.2)) satisfies at least one of the equations $\partial\varphi/\partial x^i = 0$ ($i = 1, 2, 3$) is conformally equivalent to the known examples. Here, we say that two hypersurfaces are conformally equivalent if there exists a conformal transformation Φ of $\mathbf{R}^4$ (resp. S^4) such that Φ maps diffeomorphically one hypersurface to the other hypersurface.

In §3, we explicitly construct new generic conformally flat hypersurfaces such that the φ in the first fundamental form (1.1) (resp. (1.2)) satisfies the equations $\partial^2\varphi/\partial x^1\partial x^3 = \partial^2\varphi/\partial x^2\partial x^3 = 0$ and $\partial\varphi/\partial x^i \neq 0$ ($i = 1, 2, 3$). When we add these hypersurfaces to those in Theorem 2 of the paper [7], we can classify all generic conformally flat hypersurfaces in $\mathbf{R}^4$ such that the function $P(x)$ in (1.1) (resp. (1.2)) depends only on the variable x^3. The condition for $P(x)$ is not conformally invariant but there exist many new generic conformally flat hypersurfaces under the condition. In particular, we note that, if the function $P(x)$ depends only on the variable x^3, then the equations $\partial^2\varphi/\partial x^1\partial x^3 = \partial^2\varphi/\partial x^2\partial x^3 = 0$ naturally hold. (see the claim before §3.1.)

All known hypersurfaces are conformally equivalent to the hypersurfaces constructed from surfaces with constant Gaussian curvature in 3-dimensional space forms in some way (see §2.2). The new hypersurfaces mentioned above are also constructed from surfaces in the Euclidean 3-space $\mathbf{R}^3$ and the standard 3-sphere S^3. Furthermore, there exist new hypersurfaces constructed from surfaces with constant Gaussian curvature in $\mathbf{R}^3$ and S^3. Then, we have the case where there exists a pair of conformally inequivalent hypersurfaces constructed from surfaces with constant Gaussian curvature in both $\mathbf{R}^3$ and S^3 such that these first fundamental forms (1.1) (resp. (1.2)) have the same function $\varphi(x)$. Moreover, even if the function $\varphi(x)$ varies continuously, there exists the case where the hypersurfaces corresponding to these functions can not deform continuously.

Now, we state the reason to consider the two metrics (1.1) and (1.2): When we define a function $\bar{\varphi}$ by $\cosh\bar{\varphi} = 1/\sin\varphi$ and $\sinh\bar{\varphi} = \cos\varphi/\sin\varphi$ for φ of (1.1), we can change the metric (1.1) into (1.2). However, the function $\bar{\varphi}$ no more satisfies the equations $\partial^2\bar{\varphi}/\partial x^1\partial x^3 = \partial^2\bar{\varphi}/\partial x^2\partial x^3 = 0$ even if φ satisfies the equations. Therefore, when we consider functions φ satisfying the equations, we need to study hypersurfaces with the metric (1.1) or (1.2), respectively.

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2. Several results for conformally flat hypersurfaces in Euclidean 4-space

2.1. Integrability condition on metrics of the class Ξ. Let Ξ be the class of metrics for 3-manifolds defined in the introduction. We denote by f_i the partial derivative of a function f with respect to x^i, and by f_{ij} the second derivative $\partial^2 f/\partial x^i \partial x^j$. Since the Riemannian curvature of any metric g in Ξ is diagonalized with the coordinate system $\{x^1, x^2, x^3\}$, we have the following equations:

$$(P+f)_2(P+h)_1 - P_{12} = f_2 h_1, \tag{2.1.1}$$

$$P_2(P+h)_3 - P_{23} = f_{23} + f_2(f-h)_3, \tag{2.1.2}$$

$$P_1(P+f)_3 - P_{13} = h_{13} - h_1(f-h)_3. \tag{2.1.3}$$

Theorem 2.1.1. *There exists a function $\psi = \psi(x^1, x^2, x^3)$ satisfying the following conditions* (1), (2) *and* (3) *for any metric* (1.5) *of the class Ξ*:

(1) $\psi_{12} = (P+f)_2(P+h)_1 - P_{12}$, (2) $\psi_{13} = (P+f)_3 P_1 - P_{13}$, (3) $\psi_{23} = (P+h)_3 P_2 - P_{23}$.

Moreover, the function ψ satisfying equations (1), (2) *and* (3) *is uniquely determined in the following sense*: *When another function $\bar{\psi}$ satisfies* (1), (2) *and* (3), *$\bar{\psi}$ is represented as $\bar{\psi}(x^1, x^2, x^3) = \psi(x^1, x^2, x^3) + A(x^1) + B(x^2) + C(x^3)$.*

Proof. By a direct calculation from the equations (2.1.1), (2.1.2) and (2.1.3), we derive the following equations:

$$\{(P+f)_3 P_1\}_2 = \{(P+h)_3 P_2\}_1, \tag{2.1.4}$$

$$\{P_2(P+h)_3\}_1 = \{(P+f)_2(P+h)_1\}_3, \tag{2.1.5}$$

$$\{(P+h)_1(P+f)_2\}_3 = \{P_1(P+f)_3\}_2. \tag{2.1.6}$$

Here, we only prove (2.1.4). We have

$$\begin{aligned}\{(P+f)_3 P_1\}_2 - \{(P+h)_3 P_2\}_1 &= (P+f)_{23} P_1 - (P+h)_{13} P_2 + (f-h)_3 P_{12} \\ &= P_1 P_2 (h-f)_3 - (f-h)_3 (f_2 P_1 + h_1 P_2 - P_{12}) \\ &= (h-f)_3 (P_1 P_2 + f_2 P_1 + h_1 P_2 - P_{12}) = 0.\end{aligned}$$

The second equality above follows from (2.1.2) and (2.1.3). The last equality follows from (2.1.1).

Next, we show that there exists a function $L = L(x^1, x^2, x^3)$ such that $L_{12} = (P + f)_2(P + h)_1, L_{13} = (P + f)_3 P_1$ and $L_{23} = (P + h)_3 P_2$. In fact: By (2.1.4), (2.1.5) and (2.1.6), there exist functions $K = K(x^1, x^2, x^3)$, $\overline{K} = \overline{K}(x^1, x^2, x^3)$ and $\hat{K} = \hat{K}(x^1, x^2, x^3)$ such that

$$\begin{aligned} &K_1 = (P + f)_3 P_1, &&K_2 = (P + h)_3 P_2, &&\overline{K}_1 = (P + f)_2(P + h)_1,\\ &\overline{K}_3 = P_2(P + h)_3, &&\hat{K}_3 = P_1(P + f)_3, &&\hat{K}_2 = (P + h)_1(P + f)_2. \end{aligned}$$

Then, by $K_1 = \hat{K}_3, K_2 = \overline{K}_3, \overline{K}_1 = \hat{K}_2$ there exist functions $L = L(x^1, x^2, x^3), \overline{L} = \overline{L}(x^1, x^2, x^3)$ and $\hat{L} = \hat{L}(x^1, x^2, x^3)$ such that

$$L_1 = \hat{K}, \quad L_3 = K, \quad \overline{L}_2 = \overline{K}, \quad \overline{L}_3 = K, \quad \hat{L}_1 = \hat{K}, \quad \hat{L}_2 = \overline{K}.$$

Therefore, we have $L_1 = \hat{L}_1 = \hat{K}$, $L_3 = \overline{L}_3 = K$ and $\overline{L}_2 = \hat{L}_2 = \overline{K}$. Furthermore, since $L - \hat{L} = U(x^2, x^3)$, $L - \overline{L} = V(x^1, x^2)$ and $\overline{L} - \hat{L} = W(x^1, x^3)$, we have

$$W(x^1, x^3) = \overline{L} - \hat{L} = (L - \hat{L}) - (L - \overline{L}) = U(x^2, x^3) - V(x^1, x^2).$$

Therefore, each variables of functions U, V and W have to separate to each other: $U(x^2, x^3) = X(x^2) + Y(x^3)$, $V(x^1, x^2) = Z(x^1) + X(x^2)$, $W(x^1, x^3) = Y(x^3) - Z(x^1)$. Hence, all functions $L, \overline{L}$ and $\hat{L}$ are equal to each other up to functions of one variable, and it satifies the statement.

From the argument above, we can define by $\psi = L - P$ the function ψ satisfying the conditions of Theorem. $\square$

From Theorem 2.1.1 and the equations (2.1.1), (2.1.2) and (2.1.3), we have the following Corollary.

Corollary 2.1.1. *Let g be a metric of the class Ξ and $\psi(x^1, x^2, x^3)$ be a function given in* Theorem 2.1.1. *Then we have the following equations*:

$$(1)\ \psi_{12} = f_2 h_1, \quad (2)\ \psi_{13} = h_{13} - h_1(f - h)_3, \quad (3)\ \psi_{23} = f_{23} + f_2(f - h)_3.$$

We restrict the statement of Corollary 2.1.1 to metrics for conformally flat hypersurfaces: In the case of the metric (1.1), we have

$$\psi_{12} = -\varphi_1\varphi_2, \quad \psi_{13} = \varphi_{13}\cot\varphi, \quad \psi_{23} = -\varphi_{23}\tan\varphi. \tag{2.1.7}$$

Then, the integrability condition for ψ is given by

$$\varphi_{123} = -\varphi_1\varphi_{23}\tan\varphi + \varphi_2\varphi_{13}\cot\varphi. \tag{2.1.8}$$

In the case of the metric (1.2), we have

$$\psi_{12} = \varphi_1\varphi_2, \quad \psi_{13} = \varphi_{13}\coth\varphi, \quad \psi_{23} = \varphi_{23}\tanh\varphi, \tag{2.1.9}$$

where $\coth\varphi=(\tanh\varphi)^{-1}$. The integrability condition for ψ is given by

$$\varphi_{123} = \varphi_1\varphi_{23}\tanh\varphi + \varphi_2\varphi_{13}\coth\varphi. \tag{2.1.10}$$

When a conformal transformation of $\mathbf{R}^4$ (resp. S^4) is restricted to a hypersurface, the function $P(x)$ in the metric (1.1) (resp. (1.2)) changes into another function $\bar{P}(x)$. However, since the function φ does not change, the function ψ also does not change by (2.1.7) (resp. (2.1.9)). Therefore, when we fix the first fundamental form on either (1.1) or (1.2), the pair $\{\varphi, \psi\}$ of functions is a conformal invariant for conformally flat hypersurfaces.

Furthermore, the invariant $\{\varphi, \psi\}$ for hypersurfaces (or metrics) is extended to an invariant for flat Riemannian manifolds conformally equivalent to conformally flat hypersurfaces, because the Riemannian curvature of a flat metric is trivially diagonalized. Hertrich-Jeromin [3] gave the examples of the Guichard net of $\mathbf{R}^3$ such that the canonical flat metric for $\mathbf{R}^3$ is represented as (1.1) (resp. (1.2)).

The equation (2.1.8) (resp. (2.1.10)) is one of the equations in the conformal flatness condition of the metric (1.1) (resp. (1.2)). Other equations in the conformal flatness condition of the metric (1.1) (resp. (1.2)) are given by (3.1.2), (3.1.3) and (3.1.4) (resp. (3.2.2), (3.2.3) and (3.2.4)) in §3. The equation (2.1.8) (resp. (2.1.10)) is the simplest equation in these conformal flatness conditions.

If a function φ satisfies either at least one of the equations $\varphi_i = 0$ ($i = 1, 2, 3$) or $\varphi_{13} = \varphi_{23} = 0$, then it is a particular solution of (2.1.8) (resp. (2.1.10)). In the following Subsection 2.2, we study the hypersurfaces with the first fundamental form satisfying at least one of the equations $\varphi_i = 0$ ($i = 1, 2, 3$).

In §3, we will use Theorem 2.1.1 and the equations (2.1.7), (2.1.9). We note: Although the function ψ of the conformal invariant is derived from the function φ when φ is given, we expect that other solutions φ of (2.1.8) (resp. (2.1.10)) could be found from the equation (2.1.7) (resp. (2.1.9)). Furthermore, it is interesting that there exists the class Ξ of the Riemannnian metrics including all metrics for conformally flat hypersurfaces.

2.2. Known examples of conformally flat hypersurfaces in Euclidean 4-space and in 4-sphere. The generic conformally flat hypersurfaces given by Lafontaine ([6]) are made from constant curvature surfaces in the 3-dimentional space forms. However, when we study a classification of conformally flat hypersurfaces, we need to characterize all images of these hypersurfaces under the action of conformal transformations of $\mathbf{R}^4$ (or S^4). In this subsection, we regard these hypersurfaces in $\mathbf{R}^4$ as ones in the standard 4-sphere S^4. Then we will find a common structure on S^4 for all such hypersurfaces.

In the Euclidean 5-space $\mathbf{R}^5$, we consider a rotation of a hyperplane P_0 preserving an affine 3-space U of P_0. We denote by t the rotation parameter and by P_t a hyperplane obtained by the rotation of P_0. When the intersection $P_t \cap S^4$ is a 3-sphere, we denote it by S^3_t. The 1-parameter family $\{S^3_t\}$ of 3-spheres in S^4 is determined by the hyperplane P_0 and the rotation. We say that two 1-parameter families of 3-spheres are equivalent if there exists a conformal transformation Φ of S^4 such that Φ maps each 3-sphere of one family to that of the other family. We donote by V the affine 2-space such that V is orthogonal to the space U and $V \cap S^4$ is a great circle of S^4. Then, the rotation above is determined by a rotation of a line l in V preserving a point p of l. According to each of the three cases where the point p is outside of $V \cap S^4$, included in $V \cap S^4$ and inside of $V \cap S^4$, these 1-parameter families $\{S^3_t\}$ are divided into three classes.

Let M be a generic conformally flat hypersurface in S^4 and let it be conformally equivalent to one of the hypersurfaces given by Lafontaine. Then, we can take a rotation of a hyperplane P_0 such that, for some t_0, M is made from the surface $S = M \cap S^3_{t_0}$ and the rotation. For the Guichard net of M, the coordinate functions x^1 and x^2 are given by a principal curvature-line coordinate system of S in $S^3_{t_0}$ and the coordinate function x^3 is obtained by a parameter change of t. When we represent the first fundamental form (1.1) or (1.2) for M in the form $e^{2P}(e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2 + (dx^3)^2)$ with respect to the Guichard net, the metric $e^{2f}(dx^1)^2 + e^{2h}(dx^2)^2$ for S has a constant Gaussian curvature.

For a hypersurface M given by Lafontaine, we say that M belongs to the hyperbolic class (resp. the parabolic class, the elliptic class) if the point p is outside of S^4 (resp. included in $V \cap S^4$, inside of S^4).

We recognize the above conformally flat hypesurfaces M in S^4 as hypersurfaces immersed in $\mathbf{R}^4$ through the stereographic projection. Then, we have three classes of the hypersurfaces in $\mathbf{R}^4$: The normal form of a hypersurface in $\mathbf{R}^4$ which belongs to the hyperbolic class is a cone hypersurface made from a constant Gaussian curvature surface in S^3. The normal form of a hypersurface which belongs to the parabolic class is made by the direct product of a constant Gaussian curvature surface in $\mathbf{R}^3$ and $\mathbf{R}$. The normal form of a hypersurface which belongs to the elliptic class is made from the revolution of a constant Gaussian curvature surface in the hyperbolic 3-space H^3. Here, H^3 is the upper half-space in $\mathbf{R}^3$ (of $\mathbf{R}^4$) with the Poincaré metric. These normal forms are the hypersurfaces given in [6].

When we fix any x^3 for the Guichard net above, each level surface is totally umbilic in the hypersurface. Furthermore, the function φ in the metric satisfies the equation $\varphi_3 = 0$.

Next, we reconsider the result of the paper [7] under the above consideration. In the paper [7], we gave an explicit representation of metrics for conformally flat hypersurfaces in $\mathbf{R}^4$ belonging to (T.1)-type and (T.2)-type. We note that all metrics obtained there satisfy at least one of the equations $\varphi_1 = 0$, $\varphi_2 = 0$ and $\varphi_3 = 0$.

In particular, when we regard hypersurfaces in $\mathbf{R}^4$ as ones in S^4, we can recognize that all hypersurfaces in Theorem 1 in [7] belong to the hyperbolic class, and further their normal forms are made from the Clifford tori in S^3. The hypersurfaces in Theorem 2-(3b) in [7] were constructed by two kind of revolutions of plane curves in $\mathbf{R}^4$: When the curve is in the $\{y^1, y^3\}$-plane of $\mathbf{R}^4$ with the canonical coordinate system $\{y^1, y^2, y^3, y^4\}$, one revolution preserves the $\{y^3, y^4\}$-plane and the other preserves the $\{y^1, y^2\}$-plane. We verify that the surfaces in $\mathbf{R}^3$ made by each revolution of the plane curves are constant curvarure surfaces when we regard them as the surfaces in the half-space H^3 with the Poincaré metric.

We can prove the converse of the above statements.

Theorem 2.2.1. *Let M be a generic conformally flat hypersurface in the Euclidean* 4*-space with the first fundamental form g of* (1.1) (*resp.* (1.2)). *Then the following three statements* (1), (2) *and* (3) *are equivalent*:
(1) *The metric satisfies at least one of the three equations $\varphi_1 = 0$, $\varphi_2 = 0$ and $\varphi_3 = 0$.*
(2) *Any level surface determined by x^i = constant with some coordinate function x^i is totally umbilic in M.*
(3) *M is a hypersurface belonging to one of the class of hyperbolic, parabolic and elliptic.*

It is interesting that all known examples of generic conformally flat hypersurface are characterized by only the metric condition (1). The fact that the statement (1) implies (3) is proved by constructing generic conformally flat hypersurfaces in $\mathbf{R}^4$ from the metric condition (1).

For the reconsideration of paper [7] and Theorem 2.2.1 mentioned above, see the paper [9].

3. New examples of conformally flat hypersurface

In the classification of generic conformally flat fypersurfaces of (T.2)-type (Theorem 2 in [7]), we missed the case where the metric (1.1) (resp. (1.2)) has the properties $P_1 = P_2 = \varphi_{13} = \varphi_{23} = 0$, $\varphi_3 \neq 0$ and further satisfies at least one of inequalities $\varphi_1 \neq 0$ or $\varphi_2 \neq 0$.

We first explain how this case occurs in the classification of (T.2)-type hypersurfaces. Next, we explicitely construct all generic conformally flat hypersurfaces satisfying these conditions. In particular, when the metrics satisfy both inequalities $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$, the hypersurfaces obtained here are new examples which are conformally inequivalent to the known examples.

In the equation (4.12) of the paper [7], we represented hypersurfaces M of (T.2)-type as follows: We took a curve $P(t)$ in a plane H_0 of $\mathbf{R}^4$, a mapping $A(x^1, x^2)$ into the group $SO(4)$ of orthogonal matrices and a mapping $\mathbf{a}(x^1, x^2)$ into $\mathbf{R}^4$ for any

(x^1, x^2). If $P(t)$ is not a line, then M was given by

$$M: \Phi(x^1, x^2, x^3) = A(x^1, x^2)P(x^3) + \mathbf{a}(x^1, x^2). \tag{4.12}$$

Then, we claimed in Proposition 4.3 that $\mathbf{a}(x^1, x^2)$ does not depend on the variable x^1 in the domain with $A_1(x^1, x^2)|_{H_0} \neq 0$. However, in addition to it we have one more case to consider. Namely, the mapping $\mathbf{a}(x^1, x^2)$ defines a surface in $\mathbf{R}^4$ and further the surface $\mathbf{a}(x^1, x^2)$ has an orthonormal frame field $\{\mathbf{n}(x^1, x^2), \bar{\mathbf{n}}(x^1, x^2)\}$ of the normal bundle satisfying the conditions

$$\mathbf{n}_1(x^1, x^2) = \kappa_1(x^1, x^2)\mathbf{a}_1(x^1, x^2), \quad \bar{\mathbf{n}}_1(x^1, x^2) = \bar{\kappa}_1(x^1, x^2)\mathbf{a}_1(x^1, x^2),$$

where κ_1 and $\bar{\kappa}_1$ are the principal curvatures. Indeed, this case follows from Lemmas 4.2 and 4.4 (in [7]). In the same way, in the domain with $A_2(x^1, x^2)|_{H_0} \neq 0$, we also have the case where the frame field of the surface $\mathbf{a}(x^1, x^2)$ satisfies the following conditions:

$$\mathbf{n}_2(x^1, x^2) = \kappa_2(x^1, x^2)\mathbf{a}_2(x^1, x^2), \quad \bar{\mathbf{n}}_2(x^1, x^2) = \bar{\kappa}_2(x^1, x^2)\mathbf{a}_2(x^1, x^2).$$

Therefore, we have to add, in the statement of Theorem 2 in the paper [7], the case where the representation (4.12) is written in the following form $(4.12)'$: Let a surface $\mathbf{a}(x^1, x^2)$ in $\mathbf{R}^4$ have an orthonormal frame field $\{\mathbf{n}(x^1, x^2), \bar{\mathbf{n}}(x^1, x^2)\}$ of the normal bundle satisfying above four conditions. Given functions $l(x^3)$, $m(x^3)$ and such a surface $\mathbf{a}(x^1, x^2)$ in $\mathbf{R}^4$, M is defined by

$$M : \Phi(x^1, x^2, x^3) = \mathbf{a}(x^1, x^2) + l(x^3)\mathbf{n}(x^1, x^2) + m(x^3)\bar{\mathbf{n}}(x^1, x^2) \tag{4.12$'$}$$

under the condition that the curve $(l(x^3), m(x^3))$ in $\mathbf{R}^2$ is not a line.

We will replace the above geometrical conditions by a metric condition. Since we need to change the parameter x^3 in the representation of the metric (T.2) of the paper [7], the metric condition for a (T.2)-type hypersurface is equivalent to the condition that the function $P(x)$ in (1.1) and (1.2) depends only on one variable x^3. Then, we have $\psi_{13} = \psi_{23} = 0$ by Theorem 2.1.1 and $P_1 = P_2 = 0$. The condition that the curve $(l(x^3), m(x^3))$ is not a line is equivalent to the inequality $\varphi_3 \neq 0$ (cf. Theorem 2-(2) in [7]). Furthermore, we must have at least one of the inequalities $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$ in this case (Compare this condition with Theorem 2-(3) of [7]). Therefore, if hypersurfaces $(4.12)'$ exist, then the first fundamental forms have to satisfy the conditions $P_1 = P_2 = \varphi_{13} = \varphi_{23} = 0$, $\varphi_3 \neq 0$ and further satisfy at least one of the inequalities $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$. Here, the equalities $\varphi_{13} = \varphi_{23} = 0$ follow from (2.1.7) and (2.1.9).

We study the existence problem of such hypersurfaces in the case of the metric (1.1) (resp. (1.2)) in the following Subsection 3.1 (resp. 3.2).

3.1. In this subsection, we study generic conformally flat hyperurfaces with the first fundamental form

$$g = e^{2P(x^3)}\left\{\cos^2\varphi(x)(dx^1)^2 + \sin^2\varphi(x)(dx^2)^2 + (dx^3)^2\right\}, \tag{3.1.1}$$

under the assumption $\varphi_3 \neq 0$ and further assumption that φ satisfies at least one of inequalities $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$. Since the metric $\overline{g} = \cos^2\varphi(x)(dx^1)^2 + \sin^2\varphi(x)(dx^2)^2 + (dx^3)^2$ is conformally flat, we have

$$\begin{aligned}
&2\cos 2\varphi\varphi_2(\varphi_{22} - \varphi_{11}) + \sin 2\varphi(\varphi_{112} - \varphi_{222}) - \sin 2\varphi\varphi_{233} \\
&\qquad + 2\cos 2\varphi\varphi_3\varphi_{23} = 2\varphi_3\varphi_{23} - 2\varphi_2\varphi_{33}, \qquad (3.1.2)\\
&2\cos 2\varphi\varphi_1(\varphi_{22} - \varphi_{11}) + \sin 2\varphi(\varphi_{111} - \varphi_{122}) + \sin 2\varphi\varphi_{133} \\
&\qquad - 2\cos 2\varphi\varphi_3\varphi_{13} = 2\varphi_3\varphi_{13} - 2\varphi_1\varphi_{33}, \qquad (3.1.3)\\
&\sin 2\varphi(\varphi_{113} + \varphi_{223} + \varphi_{333}) - 2\cos 2\varphi(\varphi_3\varphi_{33} + \varphi_1\varphi_{13} + \varphi_2\varphi_{23}) \\
&\qquad = 2\varphi_1\varphi_{13} - 2\varphi_2\varphi_{23} - 2\varphi_3(\varphi_{11} - \varphi_{22}), \qquad (3.1.4)
\end{aligned}$$

by (4.6), (4.7) and (4.8) in the paper [7].

Since the function P in (3.1.1) depends only on one variable x^3, we have the equations $\varphi_{13} = \varphi_{23} = 0$. The condition $\varphi_{13} = \varphi_{23} = 0$ is equivalent to the condition that the function φ is represented as $\varphi(x^1, x^2, x^3) = A(x^1, x^2) + B(x^3)$.

Proposition 3.1.1. *Let a metric* (3.1.1) *be conformally flat. We assume that the function φ is represented as*

$$\varphi(x^1, x^2, x^3) = A(x^1, x^2) + B(x^3), \tag{3.1.5}$$

where functions $A(x^1, x^2)$ and $B(x^3)$ satisfy $B_3 \neq 0$ and at least one of the two ineqalities $A_1 \neq 0$ and $A_2 \neq 0$.

Then, we have the following facts (1) *and* (2): *We denote by E and G positive constants.*

(1) *The function $A(x^1, x^2)$ satisfies the Sine-Gordon equation*:

$$A_{11} - A_{22} = \frac{E^2}{2}\sin(2A).$$

(2) *The function $B(x^3)$ is a Jacobi's amplitude function* (*an elliptic function*):

$$B_3(x^3) = \sqrt{G^2 - E^2\sin^2 B(x^3)}.$$

Proof. We first prove the statements (A) and (B) below.
(A) The function $A(x^1, x^2)$ satisfies the Sine-Gordon equation:

$$A_{11} - A_{22} = \frac{E^2}{2}\sin(2A - 2F), \tag{3.1.6}$$

where E and F are constant.

Proof of (A). By (3.1.5) and the conformal flatness conditions (3.1.2) and (3.1.3), we have

$$2\cos 2\varphi A_2(A_{22} - A_{11}) + \sin 2\varphi(A_{112} - A_{222}) = -2A_2B_{33}, \tag{3.1.7}$$

$$2\cos 2\varphi A_1(A_{22} - A_{11}) + \sin 2\varphi(A_{111} - A_{122}) = -2A_1B_{33}. \tag{3.1.8}$$

Therefore, the following function $C(x^1, x^2)$ is independent of $i = 1$ and 2:

$$2C(x^1, x^2) = -\frac{(A_{11} - A_{22})_i}{A_i} \quad (i = 1,\ 2).$$

Here, we assumed both inequalities $A_1 \neq 0$ and $A_2 \neq 0$ for the sake of simplicity. We define functions $D(x^1, x^2)$ and $\zeta(x^1, x^2)$ by

$$D(x^1, x^2) = A_{11} - A_{22}, \quad \sqrt{C^2 + D^2}\,(x^1, x^2)\cos\zeta(x^1, x^2) = C(x^1, x^2)$$
$$\text{and} \quad \sqrt{C^2 + D^2}\,(x^1, x^2)\sin\zeta(x^1, x^2) = D(x^1, x^2).$$

Then, we have

$$\begin{aligned} B_{33} &= C\sin 2\varphi + D\cos 2\varphi = \sqrt{C^2 + D^2}\,\sin(2\varphi + \zeta) \\ &= \left(\sqrt{C^2 + D^2}\sin(2A + \zeta)\right)\cos 2B + \left(\sqrt{C^2 + D^2}\,\cos(2A + \zeta)\right)\sin 2B \end{aligned} \tag{3.1.9}$$

by (3.1.7) and (3.1.8). Further, since the function B depends only on x^3, we have that

$$\bar{C} = \left(\sqrt{C^2 + D^2}\,\sin(2A + \zeta)\right)(x^1, x^2) \quad \text{and} \quad \bar{D} = \left(\sqrt{C^2 + D^2}\,\cos(2A + \zeta)\right)(x^1, x^2)$$

are constant functions. Therefore, $(C^2 + D^2)(x^1, x^2)$ and $(2A + \zeta)(x^1, x^2)$ are also constants. Then, we have

$$B_{33} = \bar{C}\cos 2B + \bar{D}\sin 2B. \tag{3.1.10}$$

On the other hand, by the conformal flatness condition (3.1.4) we have

$$\sin 2\varphi B_{333} = 2B_3(\cos 2\varphi B_{33} - A_{11} + A_{22}). \tag{3.1.11}$$

When we insert (3.1.10) into (3.1.11), we have the Sine-Gordon equation

$$A_{11} - A_{22} = \bar{C}\cos 2A - \bar{D}\sin 2A.$$

We take a constant F such that $\sqrt{C^2+D^2}\sin 2F = -\bar{C}$ and $\sqrt{C^2+D^2}\cos 2F = -\bar{D}$, and define a constant E^2 by $E^2 = 2\sqrt{C^2+D^2}$. Then we have $A_{11} - A_{22} = (E^2/2)\sin(2A-2F)$. Thus, this competes the proof of (A).

(B) The function $B(x^3)$ is given by the following equation:

$$B_3(x^3) = \sqrt{G^2 - E^2(\sin(B(x^3)+F))^2}, \tag{3.1.12}$$

where E, F are the same constants as in (1) and G is another constant.

Proof of (B). Since the function $B(x^3)$ satisfies the equation (3.1.10), we have $\{(B_3)^2\}_3 = -2B_3\{\sqrt{C^2+D^2}\sin(2B+2F)\}$. Since $E^2 = 2\sqrt{C^2+D^2}$, we have

$$(B_3)^2 = c + \frac{E^2}{2}\cos(2B+2F) \quad \text{with a constant } c.$$

Therefore, putting $G^2 = c + \sqrt{C^2+D^2}$, we have the statement (B).

Consequently, the statements (1) and (2) in Theorem follow from $\varphi = A + B = (A-F) + (B+F)$, when we replace $A - F$ and $B + F$ in the statements (A) and (B) by A and B respectively. □

On the right hand side of the Sine-Gordon equation in Proposition 3.1.1-(1), we gave a positive constant $(E^2/2)$ as the coefficient of $\sin(2A)$. In general, we can also give any negative constant as the coefficient (when we take $2F = \pi$ in (A) of the proof above). However, when we interchange x^1 and x^2 in the case of negative constant, we may consider it as a positive constant.

If we take $E^2 = G^2$ in Proposition 3.1.1-(2), then $B_3 = E\cos B(x^3)$. Hertrich-Jeromin ([3]) gave this case as a new example of the Guichard net in $\mathbf{R}^3$. After this work has been completed, Dr. Udo Hertrich-Jeromin informed the author that he also obtained a result similar to Proposition 3.1.1 ([5]).

Next, we construct a typical example of our generic conformally flat hypersurface in $\mathbf{R}^4$ whose metric is conformal to the metric in Proposition 3.1.1. This example is made from Dini's helix in $\mathbf{R}^3$ and a function $B(x^3)$ satisfying $G^2 < 1$ $(= E^2)$. The parametrization of Dini's helix in $\mathbf{R}^3$ used here was given by Hertrich-Jeromin ([3]).

EXAMPLE. For a constant $\sigma \in (-\pi/2, \pi/2)$ and a function $u(x^1, x^2) = (x^2 - x^1\sin\sigma)/\cos\sigma$, a Dini's helix with curvature -1 $(E^2 = 1)$ in $\mathbf{R}^3$ is defined by the mapping

$$\mathbf{f}(x^1, x^2) = (\cosh u)^{-1}(-\cos\sigma\sin x^1, \cos\sigma\cos x^1, x^2\cosh u - \cos\sigma\sinh u).$$

The unit normal vector field $\mathbf{n}(x^1, x^2)$ is given by

$$\begin{aligned}\mathbf{n} = (\cosh u)^{-1}(&-\cos\sigma\sin x^1\sinh u + \sin\sigma\cos x^1\cosh u,\\ &\cos\sigma\cos x^1\sinh u + \sin\sigma\sin x^1\cosh u, \cos\sigma).\end{aligned}$$

We define functions $l(x^3)$ and $m(x^3)$ by

$$l(x^3) = -\tan B(x^3), \quad m_3(x^3) = \frac{\sqrt{1-G^2}}{(\cos B(x^3))^2}$$

with a function $B(x^3)$ in Proposition 3.1.1 satisfying $G^2 < 1$ $(= E^2)$ and $B(0) = 0$.

Then, the mapping $\mathbf{p}(x^1, x^2, x^3)$ into $\mathbf{R}^4$ given by

$$\mathbf{p}(x^1, x^2, x^3) = \big(\mathbf{f}(x^1, x^2), 0\big) + \big(l(x^3)\mathbf{n}(x^1, x^2), m(x^3)\big) \tag{3.1.13}$$

defines a generic conformally flat hypersurface.

Proof of Example. The first derivatives $\mathbf{f}_1$ and $\mathbf{f}_2$ are given by the following:

$$\begin{aligned}\mathbf{f}_1 &= (\cosh u)^{-2}(-\cos\sigma\cos x^1\cosh u - \sin\sigma\sin x^1\sinh u,\\ &\qquad -\cos\sigma\sin x^1\cosh u + \sin\sigma\cos x^1\sinh u, \sin\sigma),\\ \mathbf{f}_2 &= \sinh u(\cosh u)^{-2}(\sin x^1, -\cos x^1, \sinh u).\end{aligned}$$

Thus, we have $\|\mathbf{f}_1\|^2 = 1/(\cosh u)^2$ and $\|\mathbf{f}_2\|^2 = (\sinh u/\cosh u)^2$. (We assume $u > 0$.) Therefore, as for the function $A(x^1, x^2)$ which defines Dini's helix, we have $\|\mathbf{f}_1\| = \cos A = 1/\cosh u$ and $\|\mathbf{f}_2\| = \sin A = \sinh u/\cosh u$.

Since $\mathbf{n}_1 = \sinh u\,\mathbf{f}_1$ and $\mathbf{n}_2 = -(\sinh u)^{-1}\mathbf{f}_2$, we have the following equations for the mapping $\mathbf{p}(x^1, x^2, x^3)$ of (3.1.13):

$$\mathbf{p}_1 = (1 + l\sinh u)(\mathbf{f}_1, 0), \quad \mathbf{p}_2 = (1 - l(\sinh u)^{-1})(\mathbf{f}_2, 0), \quad \mathbf{p}_3 = (l_3\mathbf{n}, m_3),$$

$$\|\mathbf{p}_1\|^2 = \frac{(1 + l\sinh u)^2}{(\cosh u)^2}, \quad \|\mathbf{p}_2\|^2 = \frac{(\sinh u - l)^2}{(\cosh u)^2}, \quad \|\mathbf{p}_3\|^2 = (l_3)^2 + (m_3)^2.$$

(If we take $l(x^3)$ with $l(0) = 0$, then the functions $(1 + l\sinh u)$ and $(\sinh u - l)$ are positive around $x^3 = 0$.)

Now, we have only to find functions $(l(x^3), m(x^3))$ such that

$$\cos(A + B) = \frac{\|\mathbf{p}_1\|}{\sqrt{l_3^2 + m_3^2}}, \quad \sin(A + B) = \frac{\|\mathbf{p}_2\|}{\sqrt{l_3^2 + m_3^2}}. \tag{3.1.14}$$

Indeed, if there exist such functions, then the metric g induced from $\mathbf{p}$ becomes

$$g = (1 + l^2)\big\{(\cos(A + B))^2(dx^1)^2 + (\sin(A + B))^2(dx^2)^2 + (dx^3)^2\big\}. \tag{3.1.15}$$

Then, the function $(1 + l^2)$ depends only on x^3 and the metric g is conformally flat.

We assume that the functions $l(x^3)$ and $m(x^3)$ satisfy the equations (3.1.14). Then, we have

$$\cos B = \frac{1}{\sqrt{l_3^2 + m_3^2}} \quad \text{and} \quad \sin B = -\frac{l}{\sqrt{l_3^2 + m_3^2}}.$$

Therefore, we first have $l = -\tan B$.

Next, by differentiating $l = -\tan B$, we have $l_3 = -B_3/(\cos B)^2$. Thus, we have $(l_3)^2 = (G^2 - (\sin B)^2)/(\cos B)^4$ by $(B_3)^2 = G^2 - (\sin B)^2$. Furthermore, from $(1 + l^2) = (l_3^2 + m_3^2)$ we can derive

$$(m_3)^2 = 1 + l^2 - (l_3)^2 = \frac{1 - G^2}{(\cos B)^4}.$$

Therefore, we have $G^2 < 1$ and $m_3 = \sqrt{1 - G^2}/(\cos B)^2$. (By a suitable parallel translation of x^4-axis, we can assume $m(0) = 0$.) Conversely, when we take $l = -\tan B$ and $m_3 = \sqrt{1 - G^2}/(\cos B)^2$, the equation (3.1.14) holds.

Finally, we prove that each x^i-curve of $\mathbf{p}$ is a principal curvature line. The unit normal vecter field $N(x^1, x^2, x^3)$ of $\mathbf{p}$ is $N = 1/\sqrt{1 + l^2}\,(m_3\mathbf{n}, -l_3)$. Therefore, we have

$$N_1 = \frac{\sqrt{1 - G^2}\ \sinh u}{\cos B - \sin B \sinh u}\mathbf{p}_1, \quad N_2 = \frac{-\sqrt{1 - G^2}}{\sin B + \cos B \sinh u}\mathbf{p}_2,$$
$$N_3 = -\sin B\sqrt{1 - G^2}\,\mathbf{p}_3.$$

The proof of the example is now completed.

We add some remarks: (1) The hypersurfaces obtained here deform continuously with the parameter G^2 $(< 1 = E^2)$. Then, at $G^2 = 1$ $(= E^2)$ the hypersurface degenerates to a domain in a hyperplane $\mathbf{R}^3$. (2) A Dini's helix for $\sigma = 0$ is a pseudosphere, which is a surface of revolution. Therefore, the obtained hypersurface $\mathbf{p}(x^1, x^2, x^3)$ belongs to the elliptic class. (We have $A_1 = 0$ in this case.) Then, the surface $\mathbf{p}(0, x^2, x^3)$ in the hyperbolic 3-space H^3 has a constant curvature $-G^2$. □

We extend the above result for Dini's helix to all constant negative curvature surfaces in $\mathbf{R}^3$ and S^3. In particular, the following Theorem 3.1.1 implies that, if there exist new conformally flat hypersurfaces with the metric (3.1.1), then they must be constructed from surfaces in $\mathbf{R}^3$ or S^3 with the metric $\cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2$.

Theorem 3.1.1. *Let $S\colon (x^1, x^2) \mapsto \mathbf{f}(x^1, x^2) \in \boldsymbol{R}^4$ be an immersed surface in $\boldsymbol{R}^4$. We assume that there exists an orthonormal frame field $\{\mathbf{n}, \bar{\mathbf{n}}\}$ of the normal bundle of S satisfying the following conditions*:

$$\text{(3.1.16)} \qquad \begin{aligned} \mathbf{n}_1 &= \kappa_1(x^1, x^2)\mathbf{f}_1, & \mathbf{n}_2 &= \kappa_2(x^1, x^2)\mathbf{f}_2,\\ \bar{\mathbf{n}}_1 &= \bar{\kappa}_1(x^1, x^2)\mathbf{f}_1, & \bar{\mathbf{n}}_2 &= \bar{\kappa}_2(x^1, x^2)\mathbf{f}_2, \end{aligned}$$

where $\kappa_i(x^1, x^2)$ and $\bar{\kappa}_i(x^1, x^2)$ denote the principal curvatures of S. Furthermore, we assume that, from the surface S and two functions $l(x^3)$ and $m(x^3)$ with $l(0) = m(0) = 0$, a hypersurface $\mathbf{p}\colon (x^1, x^2, x^3) \mapsto \mathbf{p}(x^1, x^2, x^3) \in \boldsymbol{R}^4$ is defined in the following form:

$$\text{(3.1.17)} \qquad \mathbf{p}(x^1, x^2, x^3) = \mathbf{f}(x^1, x^2) + l(x^3)\mathbf{n}(x^1, x^2) + m(x^3)\bar{\mathbf{n}}(x^1, x^2).$$

Then, we have the following fact: *If* $(l(x^3), m(x^3))$ *is not a line in* $\boldsymbol{R}^2$ *and the first fundamental form for* $\mathbf{p}$ *is represented in terms of the functions* $A(x^1, x^2)$ *and* $B(x^3)$ $(B(0) = 0)$ *in* Proposition 3.1.1 *as follows*:

$$e^{2P(x^3)}\left\{\cos^2(A+B)(dx^1)^2 + \sin^2(A+B)(dx^2)^2 + (dx^3)^2\right\}, \tag{3.1.18}$$

then the surface S *mentioned as above has to be included in either a hyperplane* $\boldsymbol{R}^3$ *or a standard* 3*-sphere* S^3 *in* $\boldsymbol{R}^4$.

Proof. First, we consider the map $\mathbf{p}(x^1, x^2, 0)$. Then, since $\mathbf{p}_1 = \mathbf{f}_1$ and $\mathbf{p}_2 = \mathbf{f}_2$, the first fundamental form for the map $\mathbf{f}$ is $e^{2P(0)}\{\cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2\}$. Replacing $e^{P(0)}\mathbf{f}(x^1, x^2)$ by $\mathbf{f}(x^1, x^2)$, we can assume $e^{P(0)} = 1$ and $(l_3^2 + m_3^2)(0) = 1$.

Next, by the definition of $\mathbf{p}(x^1, x^2, x^3)$, we have

$$\begin{aligned} &\|\mathbf{p}_1\|^2 = (1 + l\kappa_1 + m\bar{\kappa}_1)^2 \cos^2 A, \\ &\|\mathbf{p}_2\|^2 = (1 + l\kappa_2 + m\bar{\kappa}_2)^2 \sin^2 A, \qquad \|\mathbf{p}_3\|^2 = l_3^2 + m_3^2. \end{aligned} \tag{3.1.19}$$

For the sake of simplicity, we assume $\cos A \geq 0$, $\sin A \geq 0$, $(1 + l\kappa_1 + m\bar{\kappa}_1) \geq 0$ and $(1 + l\kappa_2 + m\bar{\kappa}_2) \geq 0$. By (3.1.18) and (3.1.19), we have

$$\cos(A+B) = \frac{1 + l\kappa_1 + m\bar{\kappa}_1}{\sqrt{l_3^2 + m_3^2}} \cos A, \quad \sin(A+B) = \frac{1 + l\kappa_2 + m\bar{\kappa}_2}{\sqrt{l_3^2 + m_3^2}} \sin A. \tag{3.1.20}$$

By (3.1.20), we also have

$$\begin{aligned} \cos B - \frac{1}{\sqrt{l_3^2 + m_3^2}} &= \frac{l}{\sqrt{l_3^2 + m_3^2}}(\kappa_1 \cos^2 A + \kappa_2 \sin^2 A) \\ &\quad + \frac{m}{\sqrt{l_3^2 + m_3^2}}(\bar{\kappa}_1 \cos^2 A + \bar{\kappa}_2 \sin^2 A) \end{aligned} \tag{3.1.21}$$

and

$$\sin B = \sin A \cos A \left\{ \frac{l}{\sqrt{l_3^2 + m_3^2}}(\kappa_2 - \kappa_1) + \frac{m}{\sqrt{l_3^2 + m_3^2}}(\bar{\kappa}_2 - \bar{\kappa}_1) \right\}. \tag{3.1.22}$$

Since, in the equations (3.1.21) and (3.1.22), the functions of the left hand side and $l\Big/\sqrt{l_3^2 + m_3^2}, m\Big/\sqrt{l_3^2 + m_3^2}$ depend only on x^3 and further $(l(x^3), m(x^3))$ is not a line, there exist constants C_1, C_2, C_3 and C_4 such that

$$(\kappa_1 \cos^2 A + \kappa_2 \sin^2 A) = C_1, \qquad (\bar{\kappa}_1 \cos^2 A + \bar{\kappa}_2 \sin^2 A) = C_2, \tag{3.1.23}$$

$$\sin A \cos A(\kappa_2 - \kappa_1) = C_3, \qquad \sin A \cos A(\bar{\kappa}_2 - \bar{\kappa}_1) = C_4. \tag{3.1.24}$$

By (3.1.23) and (3.1.24), we have more explicit forms for the conditions (3.1.16);

$$\mathbf{n}_1 = (C_1 - C_3 \tan A)\mathbf{f}_1, \qquad \mathbf{n}_2 = (C_1 + C_3 \cot A)\mathbf{f}_2, \tag{3.1.25}$$

$$\bar{\mathbf{n}}_1 = (C_2 - C_4 \tan A)\mathbf{f}_1, \qquad \bar{\mathbf{n}}_2 = (C_2 + C_4 \cot A)\mathbf{f}_2. \tag{3.1.26}$$

We assume $C_3 \neq 0$ and $C_4 \neq 0$ for the sake of simplicity.

By (3.1.25) and (3.1.26), we have, for $i = 1, 2$,

$$(C_4\mathbf{n} - C_3\bar{\mathbf{n}})_i = (C_1C_4 - C_2C_3)\mathbf{f}_i. \tag{3.1.27}$$

In the equation (3.1.27), if $(C_1C_4 - C_2C_3) = 0$, then the surface S is included in a hyperplane $\mathbf{R}^3$ orthogonal to the constant vector $(C_4\mathbf{n} - C_3\bar{\mathbf{n}})$.

Finally, we study the case $(C_1C_4 - C_2C_3) \neq 0$. For the sake of simplicity, we assume $(C_1C_4 - C_2C_3) = 1$. By (3.1.27), there exists a constant vector $\mathbf{a}$ such that $C_4\mathbf{n} - C_3\bar{\mathbf{n}} = \mathbf{a} + \mathbf{f}$. When we differenciate both sides of $\langle C_4\mathbf{n} - C_3\bar{\mathbf{n}}, \mathbf{f}\rangle = \langle \mathbf{a}, \mathbf{f}\rangle + \langle \mathbf{f}, \mathbf{f}\rangle$ with respect to x^i $(i = 1, 2)$, we have $\langle \mathbf{f}_i, \mathbf{f}\rangle + \langle C_4\mathbf{n} - C_3\bar{\mathbf{n}}, \mathbf{f}_i\rangle = \langle \mathbf{a}, \mathbf{f}_i\rangle + 2\langle \mathbf{f}_i, \mathbf{f}\rangle$. Since $\mathbf{n}$ and $\bar{\mathbf{n}}$ are normal vectors of S, we have $\langle \mathbf{a} + \mathbf{f}, \mathbf{f}_i\rangle = 0$; Namely, $\langle \mathbf{a} + \mathbf{f}, \mathbf{a} + \mathbf{f}\rangle = \text{constant}$. Therefore, the surface S is included in a sphere S^3 with center $(-\mathbf{a})$. □

Next, we study the existence problem of surfaces in $\mathbf{R}^3$ and S^3 (of $\mathbf{R}^4$) satisfying the following conditions (1) and (2): (1) The surface has a metric $\hat{g} = \cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2$. (2) There exists an orthonormal frame field of the normal bundle satisfying the conditions (3.1.25) and (3.1.26). However, when a surface in $\mathbf{R}^4$ is included in $\mathbf{R}^3$ (resp. S^3), we can take, as one of the normal vecter fields $\mathbf{n}$ and $\bar{\mathbf{n}}$, a constant vector orthogonal to the $\mathbf{R}^3$ (resp. the position vector of each point in the surface). Let us denote by $S^3(1)$ the standard sphere with radius 1.

Lemma 3.1.1. *For any function $A(x^1, x^2)$ in* Proposition 3.1.1-(1), *there exist surfaces S in both $\boldsymbol{R}^3$ and $S^3(1)$ such that the surface S has the first fundamental form $\hat{g} = \cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2$ and that S satisfies the following conditions* (1) *and* (2): (1) *Each x^i-curve* ($i = 1, 2$) *is a principal curvature line of S.* (2) *A unit vector field* $\mathbf{n}$ *normal to S satisfies the condition* (3.1.25).
Then, the second fundamental forms $\hat{s}$ for the surfaces S are respectively given as follows: *Let E be the positive constant at the definition of $A(x^1, x^2)$ in* Proposition 3.1.1-(1).
(1) *The case of the surface in $\boldsymbol{R}^3$*: $\hat{s} = E \sin A \cos A\{(dx^1)^2 - (dx^2)^2\}$.
(2) *The case of the surface in S^3*: $\hat{s} = \sqrt{E^2 + 1} \sin A \cos A\{(dx^1)^2 - (dx^2)^2\}$.

Proof. The Gaussian curvature K of the metric $\hat{g}$ is given by

$$K = -\frac{A_{11} - A_{22}}{\sin A \cos A} = -E^2. \tag{3.1.28}$$

We have only to find surfaces in $\mathbf{R}^3$ and $S^3(1)$ such that they satisfy the condition (2) and that these second fundamental forms are represented as $\hat{s} = s_{11}(dx^1)^2 + s_{22}(dx^2)^2$. We define functions $a(x^1, x^2)$ and $b(x^1, x^2)$ by

$$s_{11} = Ea\cos^2 A, \qquad s_{22} = Eb\sin^2 A. \tag{3.1.29}$$

By (3.1.28), we have respectively the following equations depending on whether the surface S is in $\mathbf{R}^3$ or $S^3(1)$:

$$ab = -1, \qquad \text{or} \qquad ab = -1 - E^{-2}. \tag{3.1.30}$$

From Codazzi's equation $\partial(s_{11})/\partial x^2 = \Gamma^1_{12}s_{11} - \Gamma^2_{11}s_{22}$ and $\partial(s_{22})/\partial x^1 = -\Gamma^1_{22}s_{11} + \Gamma^2_{12}s_{22}$, we have

$$a_2\cos A = (a-b)A_2\sin A, \qquad b_1\sin A = (a-b)A_1\cos A. \tag{3.1.31}$$

Then, functions a and b depend only on $A(x^1, x^2)$ by (3.1.25). By abuse of notation, we write $a(x^1, x^2) = a(A(x^1, x^2))$ and $b(x^1, x^2) = b(A(x^1, x^2))$. Thus, we can denote $a' = da/dA$, $b' = db/dA$. Then, we have $(a-b)' = (a-b)(\sin A/\cos A - \cos A/\sin A)$ by (3.1.31). Therefore, we have the following equation:

$$(a-b) = \frac{C}{\sin A\cos A} \quad \text{with a constant } C. \tag{3.1.32}$$

We only consider the case where S is a surface in $S^3(1)$. By (3.1.30) and (3.1.32), a and $-b$ are the solutions of the following quadratic equation for t:

$$t^2 - \frac{C}{\sin A\cos A}t + (1 + E^{-2}) = 0.$$

The discriminant D of the quadratic equation is given by

$$D = \frac{C^2 - (1+E^{-2})\sin^2(2A)}{\sin^2 A\cos^2 A}.$$

Since a and $-b$ are rational functions in $\sin A$ and $\cos A$ by (3.1.25), we have $C^2 = (1 + E^{-2})$. (In the case where S is a surface in $\mathbf{R}^3$, we have $C^2 = 1$.) Therefore, $a = (\sqrt{E^2+1}/E)\tan A$ and $-b = (\sqrt{E^2+1}/E)\cot A$ by (3.1.31). By the above argument, the second fundamental form $\hat{s}$ is given by $\hat{s} = \sqrt{E^2+1}\sin A\cos A\{(dx^1)^2 - (dx^2)^2\}$. $\square$

Theorem 3.1.2. *Let $A(x^1, x^2)$ and $B(x^3)$ be the functions in* Proposition 3.1.1. *Let S be a negative constant curvature surface in $\mathbf{R}^3$ determined from the function $A(x^1, x^2)$. The first fundamental form $\hat{g}$ and the second fundamental form $\hat{s}$ for S are* *respectively given as follows*:

$$\hat{g} = \cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2, \qquad \hat{s} = E \sin A \cos A\big\{(dx^1)^2 - (dx^2)^2\big\}.$$

Then, the map $\mathbf{p}(x^1, x^2, x^3)$ *into* $\boldsymbol{R}^4$, *made from the surface* S *and function* $B(x^3)$ *in the same way as in* (3.1.17), *defines a generic conformally flat hypersurface if and only if the constant* G^2 *in the definition of* $B(x^3)$ *satisfies the inequality* $E^2 > G^2$.

Furthermore, each x^i*-curve of the hypersurface* $\mathbf{p}$ *is a principal curvature line.*

Proof. The surface S is in $\mathbf{R}^3$ by the assumption. Let $S: (x^1, x^2) \to \mathbf{f}(x^1, x^2) \in \mathbf{R}^3$ denote the immersion and $\mathbf{n}(x^1, x^2)$ a unit vector field normal to S in $\mathbf{R}^3$. Then, the map $\mathbf{p}(x^1, x^2, x^3)$ of (3.1.17) is given by $\mathbf{p}(x^1, x^2, x^3) = (\mathbf{f}(x^1, x^2) + l(x^3)\mathbf{n}(x^1, x^2), m(x^3))$.

From the second fundamental form, it follows that $\mathbf{n}_1 = -E \tan A \mathbf{f}_1$, and $\mathbf{n}_2 = E \cot A \mathbf{f}_2$. The rest of the proof is same as in the proof of Example. Here, the functions l and m are respectively determined by

$$l(x^3) = \frac{\tan B(x^3)}{E}, \qquad m_3 = \frac{\sqrt{E^2 - G^2}}{E \cos^2 B(x^3)}. \tag{3.1.33}$$

Finally, we can directly show that each x^i-curve of the hypersurface $\mathbf{p}$ is a principal curvature line. □

Theorem 3.1.3. *Let* $A(x^1, x^2)$ *and* $B(x^3)$ *be the functions in* Proposition 3.1.1. *Let* S *be a negative constant curvature surface in* $S^3(1)$ *determined from* $A(x^1, x^2)$. *The first fundamental form* $\hat{g}$ *and the second fundamental form* $\hat{s}$ *for* S *are respectively given as follows*:

$$\hat{g} = \cos^2 A(dx^1)^2 + \sin^2 A(dx^2)^2, \qquad \hat{s} = \sqrt{E^2 + 1}\, \sin A \cos A\big\{(dx^1)^2 - (dx^2)^2\big\}.$$

Then, the map $\mathbf{p}(x^1, x^2, x^3)$ *into* $\boldsymbol{R}^4$, *made from* S *and* $B(x^3)$ *in the same way as in* (3.1.17), *defines a generic conformally flat hypersurface if and only if the constant* G^2 *in the definition of* $B(x^3)$ *satisfies the inequality* $E^2 + 1 > G^2$.

Furthermore, each x^i*-curve of the hypersurface* $\mathbf{p}$ *is a principal curvature line.*

Proof. Since the surface S is in $S^3(1)$, we denote the immersion by $S: (x^1, x^2) \mapsto \mathbf{f}(x^1, x^2) \in S^3(1)$. Let $\mathbf{n}(x^1, x^2)$ be a unit vector field normal to S in $S^3(1)$. Since $\mathbf{f}(x^1, x^2)$ is another unit vector field normal to S in $\mathbf{R}^4$, the map $\mathbf{p}(x^1, x^2, x^3)$ of (3.1.17) is given by

$$\mathbf{p}(x^1, x^2, x^3) = \big(1 + l(x^3)\big)\mathbf{f}(x^1, x^2) + m(x^3)\mathbf{n}(x^1, x^2).$$

Furthermore, from the second fundamental form it follows that $\mathbf{n}_1 = -\sqrt{E^2 + 1} \tan A \mathbf{f}_1$

and $\mathbf{n}_2 = \sqrt{E^2+1}\cot A\mathbf{f}_2$. Then, the equations (3.1.20) become as follows:

$$\cos(A+B) = \frac{(1+l)\cos A - \sqrt{E^2+1}\,m\sin A}{\sqrt{l_3^2+m_3^2}},$$
$$\sin(A+B) = \frac{(1+l)\sin A + \sqrt{E^2+1}\,m\cos A}{\sqrt{l_3^2+m_3^2}}.$$

Therefore, we have

$$\cos B = \frac{1+l}{\sqrt{l_3^2+m_3^2}}, \qquad \sin B = \frac{\sqrt{E^2+1}\,m}{\sqrt{l_3^2+m_3^2}}. \tag{3.1.34}$$

Conversely, if there exist functions $l(x^3)$ and $m(x^3)$ satisfying the equations (3.1.34), then the map $\mathbf{p}$ defines a conformally flat hypersurface.

For the sake of simplicity, we denote $\bar{l} = (1+l)$, $C(x^3) = \sqrt{(l_3^2+m_3^2)(x^3)}$. Then, we have

$$m = \frac{\bar{l}\tan B}{\sqrt{E^2+1}}, \qquad \bar{l}^2 + (E^2+1)m^2 = C^2. \tag{3.1.35}$$

When we substitute $m = \bar{l}\tan B/\sqrt{E^2+1}$ and

$$m_3 = \frac{\bar{l}_3\sin B\cos B + \bar{l}B_3}{\sqrt{E^2+1}\,\cos^2 B}$$

into the second equation of (3.1.35), we have

$$\cos^2 B(E^2\cos^2 B+1)\left(\frac{\bar{l}_3}{\bar{l}}\right)^2 + 2B_3\sin B\cos B\frac{\bar{l}_3}{\bar{l}} + \{B_3^2 - (E^2+1)\cos^2 B\} = 0.$$

Since $(B_3)^2 = G^2 - E^2\sin^2 B$ and the discriminant D of the quadratic equation for $(\bar{l}_3/\bar{l})$ is $D/4 = (E^2+1)(E^2+1-G^2)\cos^4 B$, $(\bar{l}_3/\bar{l})$ is a real function if and only if $E^2+1 \geq G^2$. Thus, in the case $E^2+1 > G^2$ we have two functions $l(x^3)$ under an initial condition $l(0) = m(0) = 0$. (Here, we assumed $B(0) = 0$.) This fact corresponds to the existence of an inversion which fixes the surface S. Furthermore, if $l(x^3)$ is determined, then $m(x^3)$ is also determined.

If $E^2+1 = G^2$, then all principal curvatures are the same constant; $\lambda = \mu = \nu = -1$. Furthermore, the hypersurface is a domain of the 3-sphere $S^3(1)$.

Finally, we can directly show that each x^i-curve of the hypersurface $\mathbf{p}$ is a principal curvature line. $\square$

By Theorems 3.1.2 and 3.1.3, when the functions $A(x^1, x^2)$ and $B(x^3)$ are defined from the constants E^2 and G^2 satisfying $E^2 > G^2$, we have two generic comformally flat hypersurfaces corresponding to these functions. We note that these hypersurfaces have the same conformal invariant $\{\varphi, \psi\}$. Furthermore, each x^i-curve in both hypersurfaces is a principal curvature line. In Corollary 3.1.1 below, we show that these two hypersurfaces are not conformally equivalent to each other. First, we prepare the following Proposition for Corollary 3.1.1.

Proposition 3.1.2. *Let Φ be an injective conformal map from $(\boldsymbol{R}^3, g_E)$ to $(S^3(1), g_S)$, where g_E and g_S are respectively the canonical metrics for $\boldsymbol{R}^3$ and $S^3(1)$. Then, the subset in $\boldsymbol{R}^3$ where Φ is isometric* (*or homothetic*) *is included in a* 2*-sphere S^2.*

Proof. We consider the metric $\Phi^* g_S$ for $\mathbf{R}^3$ induced by the map Φ. Let $\Phi_1 \colon \mathbf{R}^3 \to S^3(1)$ be a stereographic projection, and $\Phi_2 \colon S^3(1) \to S^3(1)$ a suitable conformal transformation of $S^3(1)$. Then, we have $\Phi = \Phi_2 \circ \Phi_1$. When we consider that the map Φ_1 gives a coordinate system on $S^3(1)$, the metric for $\mathbf{R}^3$ (as the metric for $S^3(1)$) is given by $\Phi_1^* g_S$:

$$\Phi_1^* g_S = \frac{4}{(1 + \|y\|^2)^2} \left\{(dy^1)^2 + (dy^2)^2 + (dy^3)^2\right\} = \frac{4}{(1 + \|y\|^2)^2} g_E.$$

Since we consider a transformation of $\mathbf{R}^3$, the group of conformal transformations consists of Euclidean motions, homotheties and inversions acting on $\mathbf{R}^3$. In both cases of Euclidean motions and homotheties, the metric $\Phi^* g_S$ is determined homothetically by $\Phi_1^* g_S$. Thus, as a conformal transformation Φ_2 of $S^3(1)$ we consider an inversion $(\Phi_1^{-1} \circ \Phi_2 \circ \Phi_1)(y) = R(y - y_0)/\|y - y_0\|^2$ ($(0 <)\ R \in \mathbf{R}$) acting on $\mathbf{R}^3$, where $y = \{y^1, y^2, y^3\}$ is the canonical coordinate system of $\mathbf{R}^3$ and $y_0 = \{y_0^1, y_0^2, y_0^3\}$ is a point. Then, we have

$$(\Phi_1^{-1} \circ \Phi_2 \circ \Phi_1)^* g_E = \frac{R^2}{\|y - y_0\|^4} \left\{(dy^1)^2 + (dy^2)^2 + (dy^3)^2\right\}.$$

Therefore, when we denote $\Phi^* g_S = 4R^2\{(1 + R^2/\|y - y_0\|^2)^2 \|y - y_0\|^4\}^{-1} g_E = e^{2P(y)} g_E$, the set $\{y \in \mathbf{R}^3 \mid P(y) = 0 \text{ (or } P(y) = c)\}$ is included in a 2-sphere S^2. □

Corollary 3.1.1. *Let $E^2 > G^2$. Then, two conformally flat hypersurfaces in* Theorems 3.1.2 *and* 3.1.3 *determined from a pair $\{A(x^1, x^2), B(x^3)\}$ are not conformally equivalent to each other.*

Proof. We denote by $\mathbf{p}(x^1, x^2, x^3)$ the hypersurface in Theorem 3.1.2, and by $\tilde{\mathbf{p}}(x^1, x^2, x^3)$ the hypersurface in Theorem 3.1.3. Then, we have

$$\mathbf{p} = \left(\mathbf{f}(x^1, x^2) + l(x^3)\mathbf{n}(x^1, x^2), m(x^3)\right), \qquad \tilde{\mathbf{p}} = \left(1 + \tilde{l}(x^3)\right)\tilde{\mathbf{f}}(x^1, x^2) + \tilde{m}(x^3)\tilde{\mathbf{n}}(x^1, x^2).$$

From the definitions of $\mathbf{n}$, l and m for the maps $\mathbf{p}$ and $\tilde{\mathbf{p}}$, it follows that

$$\mathbf{p}_1 = (1 - \tan A \tan B)\mathbf{f}_1, \qquad \mathbf{p}_2 = (1 + \cot A \tan B)\mathbf{f}_2,$$
$$\tilde{\mathbf{p}}_1 = \big(1 + \tilde{l}(x^3)\big)(1 - \tan A \tan B)\tilde{\mathbf{f}}_1, \qquad \tilde{\mathbf{p}}_2 = \big(1 + \tilde{l}(x^3)\big)(1 + \cot A \tan B)\tilde{\mathbf{f}}_2.$$

When we fix any x^3, the surface determined by the map $\mathbf{f}(x^1, x^2) + l(x^3)\mathbf{n}(x^1, x^2)$, which appears in the definition of $\mathbf{p}$, is included in a linear 3-space $\mathbf{R}^3_{x^3}$ and the surface determined by the map $\tilde{\mathbf{p}}$ is included in a 3-sphere $S^3_{x^3}$. Furthermore, in both cases, these surfaces are full in either $\mathbf{R}^3_{x^3}$ or $S^3_{x^3}$, respectively.

Now, we assume that there exists a conformal transformation Φ of $\mathbf{R}^4$ (or S^4) such that $\Phi(\mathbf{p}(x^1, x^2, x^3)) = \tilde{\mathbf{p}}(x^1, x^2, x^3)$. Then, Φ maps $\mathbf{R}^3_{x^3}$ to $S^3_{x^3}$ for a fixed x^3, and it satisfies $\Phi^*\hat{g}_{\tilde{\mathbf{p}}} = (1 + \tilde{l}(x^3))^2\hat{g}_{\mathbf{p}}$ on the surface in $\mathbf{R}^3_{x^3}$. This is a contradiction to Proposition 3.1.2. Therefore, there does not exist any conformal transformation satisfying the assumption. $\square$

REMARK. We assume that the first fundamental form g for a generic conformally flat hypersurface is represented as (1.1) with respect to a Guichard net $\{x^1, x^2, x^3\}$. Then, there exists a positive constant C such that

$$\frac{1}{(\mu - \nu)(\nu - \lambda)}(x^1, x^2, x^3) = Ce^{2P(x^1,x^2,x^3)} \tag{3.1.36}$$

by [3]. Two hypersurfaces are said to belong to the same associated family if the metrics (1.1) for two hypersurfaces are defined from the same function φ ([2], [4]). Thus, we have constructed two hypersurfaces in Theorems 3.1.2 and 3.1.3 belonging to the same associated family; Namely, we have $C = 1/(E^2 - G^2)$ or $C = 1/(E^2 + 1 - G^2)$ for each hypersurface, respectively. Furthermore, in the same way as the proof of Theorem 3.1.3 we can construct from surfaces in $S^3(r)$ a 1-parameter family of hypersurfaces belonging to the same associated family.

The definition of the canonical Guichard net on a conformally flat hypersurface in Hertrich-Jeromin ([3]) includes the condition $C = 1$ in the equation (3.1.36) besides our condition mentioned in the introduction. When we change the coordinate functions x^i to $\overline{x}^i = cx^i$ with a constant c, $\{\overline{x}^1, \overline{x}^2, \overline{x}^3\}$ is also a Guichard net and then the right hand side of (3.1.36) changes to $(C/c^2)e^{2P}$ with respect to the net. According to his definition, we can also consider that the metric for a hypersurface belonging to the same associated family is determined from a function $\overline{\varphi}$ obtained by such a parameter change of the same φ.

3.2. In this subsection, we list the results on generic conformally flat hypersurfaces in $\mathbf{R}^4$ with the first fundamental form

$$g = e^{2P(x^3)}\big\{\cosh^2\varphi(x)(dx^1)^2 + \sinh^2\varphi(x)(dx^1)^2 + (dx^3)^2\big\}, \tag{3.2.1}$$

under the assumption $\varphi_3 \neq 0$ and further assumption that φ satisfies at least one of inequalities $\varphi_1 \neq 0$ and $\varphi_2 \neq 0$. We note: In the metric (1.1), we replace $e^{2P(x)}\sin^2\varphi(x)$ by $e^{2P(x)}$. Then, we have the metric (1.2) if we exchange coordinates x^1, x^2 and x^3 for x^2, x^3 and x^1, respectively. We can prove each theorem (resp. equation) mentioned here in the same way as the proof of the theorem (resp. equation) in §3.1 corresponding to it.

The condition for the metric $\bar{g} = \cosh^2\varphi(x)(dx^1)^2 + \sinh^2\varphi(x)(dx^1)^2 + (dx^3)^2$ to be conformally flat is given by the following three equations together with (2.1.10):

(3.2.2)
$$\begin{aligned}&\varphi_2\varphi_{33} + 2\varphi_3\varphi_{23}\sinh^2\varphi - \varphi_{233}\sinh\varphi\cosh\varphi\\ &\quad = \varphi_2(\varphi_{11}+\varphi_{22})(\sinh^2\varphi + \cosh^2\varphi) - (\varphi_{11}+\varphi_{22})_2\sinh\varphi\cosh\varphi.\end{aligned}$$

(3.2.3)
$$\begin{aligned}&\varphi_1\varphi_{33} - 2\varphi_3\varphi_{13}\cosh^2\varphi + \varphi_{133}\sinh\varphi\cosh\varphi\\ &\quad = \varphi_1(\varphi_{11}+\varphi_{22})(\sinh^2\varphi + \cosh^2\varphi) - (\varphi_{11}+\varphi_{22})_1\sinh\varphi\cosh\varphi.\end{aligned}$$

(3.2.4)
$$\begin{aligned}&2\varphi_1\varphi_{13}\cosh^2\varphi - 2\varphi_2\varphi_{23}\sinh^2\varphi + \varphi_{223}\sinh\varphi\cosh\varphi - \varphi_3(\varphi_{11}+\varphi_{22})\\ &\quad = -\varphi_3\varphi_{33}(\sinh^2\varphi + \cosh^2\varphi) + (\varphi_{113} - \varphi_{223} + \varphi_{333})\sinh\varphi\cosh\varphi.\end{aligned}$$

Proposition 3.2.1. *Let a metric g of* (3.2.1) *be conformally flat. We assume that the function φ is represented as $\varphi(x^1,x^2,x^3) = A(x^1,x^2) + B(x^3)$, where the function $A(x^1,x^2)$ and $B(x^3)$ satisfy $B_3 \neq 0$ and at least one of the inequality $A_1 \neq 0$ and $A_2 \neq 0$. Then, the functions $A(x^1,x^2)$ and $B(x^3)$ are respectively determined by the equations in the following three cases* (a), (b) *and* (c):

We denote by E and G constant numbers.

(a) $$A_{11} + A_{22} = -\frac{1}{2}E\sinh(2A),\quad (B_3)^2 = \frac{1}{2}(E\cosh(2B) + G).$$

(b) $$A_{11} + A_{22} = E\cosh(2A),\quad (B_3)^2 = E\sinh(2B) + G.$$

(c) $$A_{11} + A_{22} = Ee^{-2A},\quad (B_3)^2 = Ee^{2B} + G.$$

We replace the metric (3.1.18) in Theorem 3.1.1 by the metric

(3.2.5) $$e^{2P(x^3)}\left\{\cosh^2(A+B)(dx^1)^2 + \sinh^2(A+B)(dx^2)^2 + (dx^3)^2\right\},$$

where $A(x^1,x^2)$ and $B = B(x^3)$ are functions given in Proposition 3.2.1. Then, the statement of Theorem 3.1.1 is also true in this case. The equations (3.1.25) and (3.1.26) for the orthonormal frame field $\{\mathbf{n},\bar{\mathbf{n}}\}$ in the proof of Theorem 3.1.1 are replaced by the following equations: We denote by C_1, C_2, C_3 and C_4 constant numbers.

(3.2.6) $$\mathbf{n}_1 = (C_1 + C_3\tanh A)\mathbf{f}_1,\qquad \mathbf{n}_2 = (C_1 + C_3\coth A)\mathbf{f}_2.$$

(3.2.7) $$\bar{\mathbf{n}}_1 = (C_2 + C_4\tanh A)\mathbf{f}_1,\qquad \bar{\mathbf{n}}_2 = (C_2 + C_4\coth A)\mathbf{f}_2.$$

Lemma 3.2.1. *Let a function* $A(x^1, x^2)$ *be given in* Proposition 3.2.1. *We assume that there exists a surface* S *in either* $\boldsymbol{R}^3$ *or* $S^3(1)$ *such that* S *has the first fundamental form* $\hat{g} = \cosh^2 A(dx^1)^2 + \sinh^2 A(dx^2)^2$ *and that* S *satisfies the following conditions* (1) *and* (2): (1) *Each* x^i*-curve* ($i = 1,\ 2$) *is a principal curvature line of* S. (2) *A unit vector field* $\mathbf{n}$ *normal to* S *satisfies the condition* (3.2.6).

Then, we have only the following two cases (1) *and* (2):

(1) *The case of* Proposition 3.2.1-(a). *There exists such a surface in* $\boldsymbol{R}^3$ *if and only if* $E > 0$. *Its second fundamental form is given by* $\hat{s} = -\sqrt{E}\sinh A\cosh A\{(dx^1)^2 + (dx^2)^2\}$.

There also exists such a surface in $S^3(1)$ *if and only if* $E > 1$. *Its second fundamental form is given by* $\hat{s} = -\sqrt{E-1}\sinh A\cosh A\{(dx^1)^2 + (dx^2)^2\}$.

(2) *The case of* Proposition 3.2.1-(c). *There exists such a surface in* $\boldsymbol{R}^3$ *if and only if* $E > 0$. *Its second fundamental form is given by* $\hat{s} = \sqrt{E}e^{-A}\{\cosh A(dx^1)^2 - \sinh A(dx^2)^2\}$.

We will write down the important equations used in the proof of Lemma 3.2.1: For the sake of simplicity, we assume that the constant E in the definition of the function $A(x^1, x^2)$ is a positive number $E = \bar{E}^2$. The Gaussian curvature K of $\hat{g}$ is given by $K = -(A_{11} + A_{22})/(\sinh A\cosh A)$. We have only to find surfaces in $\mathbf{R}^3$ and $S^3(1)$ such that they satisfy the condition (2) and that these second fundamental forms are represented as $\hat{s} = s_{11}(dx^1)^2 + s_{22}(dx^2)^2$. We define functions $a(x^1, x^2)$ and $b(x^1, x^2)$ by $s_{11} = \bar{E}a\cosh^2 A, s_{22} = -\bar{E}b\sinh^2 A$. If S is a surface in $\mathbf{R}^3$, then we have $\bar{E}^2 ab = (A_{11} + A_{22})/(\sinh A\cosh A)$. If S is a surface in $S^3(1)$, then we have $\bar{E}^2 ab - 1 = (A_{11} + A_{22})/(\sinh A\cosh A)$.

From Codazzi's equation, we can derive the following equations:

$$a_2\cosh A = -(a + b)A_2\sinh A, \qquad b_1\sinh A = -(a + b)A_1\cosh A.$$

We can assume $a(x^1, x^2) = a(A(x^1, x^2))$ and $b(x^1, x^2) = b(A(x^1, x^2))$ from (3.2.6). Therefore, when we denote $a' = da/dA$ and $b' = db/dA$, we have $(a + b)' = -(a + b)(\cosh^2 A + \sinh^2 A)/(\sinh A\cosh A)$. Thus, we have the following equation:

$$(a + b) = \frac{C}{\sinh A\cosh A} \qquad \text{with a constant } C.$$

Here, by using the equations above, we will only show the non-existence of surfaces in $\mathbf{R}^3$ corresponding to functions A in Proposition 3.2.1-(b). The functions a and b are the solutions of the following quadratic equation for t:

$$t^2 - \frac{C}{\sinh A\cosh A}t + \frac{\cosh(2A)}{\sinh A\cosh A} = 0.$$

The discriminant D of the equation is

$$D = e^{4A}\frac{(e^{-4A}+C^2)^2 - 1 - C^4}{2\sinh^2 A\cosh^2 A}.$$

Furthermore, a and b are rational functions in $\sinh A$ and $\cosh A$ by (3.2.6). Therefore, the solutions of the equation can not be the functions a and b for any C.

Theorem 3.2.1. *Let $A(x^1, x^2)$ and $B(x^3)$ be the functions in* Proposition 3.2.1-(a), *and $E > 0$ in the definitions of $A(x^1, x^2)$ and $B(x^3)$. Let S be a positive constant curvature surface in $\boldsymbol{R}^3$ determined from A. The first fundamental form $\hat{g}$ and the second fundamental form $\hat{s}$ for S are respectively given as follows*:

$$\hat{g} = \cosh^2 A(dx^1)^2 + \sinh^2 A(dx^2)^2, \qquad \hat{s} = -\sqrt{E}\sinh A\cosh A\left\{(dx^1)^2 + (dx^2)^2\right\}.$$

Then, the map $\mathbf{p}(x^1, x^2, x^3)$ into $\boldsymbol{R}^4$, made from S and $B(x^3)$ in the same way as in (3.1.17), *defines a generic conformally flat hypersurface if and only if the constant G in the definition of $B(x^3)$ satisfies the inequality $E > G$.*

Furthermore, each x^i-curve of the hypersurface $\mathbf{p}$ is a principal curvature line.

In Theorem 3.2.1, the functions $l(x^3)$ and $m(x^3)$ in the definition of the map $\mathbf{p}(x^1, x^2, x^3)$ are respectively detemined as follows: $l = \tanh B/\sqrt{E}$, $m_3^2 = (E-G)/(2E\cosh^4 B)$.

Theorem 3.2.2. *Let $A(x^1, x^2)$ and $B(x^3)$ be the functions in* Proposition 3.2.1-(a), *and $E > 1$ in the definitions of $A(x^1, x^2)$ and $B(x^3)$. Let S be a positive constant curvature surface in $S^3(1)$ determined from A. The first fundamental form $\hat{g}$ and the second fundamental form $\hat{s}$ for S are respectively given as follows*:

$$\hat{g} = \cosh^2 A(dx^1)^2 + \sinh^2 A(dx^2)^2, \qquad \hat{s} = -\sqrt{E-1}\sinh A\cosh A\left\{(dx^1)^2 + (dx^2)^2\right\}.$$

Then, the map $\mathbf{p}(x^1, x^2, x^3)$ into $\boldsymbol{R}^4$, made from S and $B(x^3)$ in the same way as in (3.1.17), *defines a generic conformally flat hypersurface if and only if the constant G in the definition of $B(x^3)$ satisfies the inequality $E > G + 2$.*

Furthermore, each x^i-curve of the hypersurface $\mathbf{p}$ is a principal curvature line.

In Theorem 3.2.2, the function $m(x^3)$ is given by $m = (1+l)\tanh B/\sqrt{E-1}$. The function $l(x^3)$ is determined from the solutions of the following quadratic equation: We denote $\bar{l}(x^3) = 1 + l(x^3)$.

$$\cosh^2 B(E\cosh^2 B - 1)\left(\frac{\bar{l}_3}{l}\right)^2 + 2B_3\sinh B\cosh B\frac{\bar{l}_3}{l} + \left\{B_3^2 - (E-1)\cosh^2 B\right\} = 0.$$

Then, the inequality $E \geq G + 2$ is the condition under which the equation has a real solution.

Corollary 3.2.1. *Let $E > 1$ and $E > G + 2$. Then, two conformally flat hypersurfaces in* Theorems 3.2.1 *and* 3.2.2 *determined from a pair* $\{A(x^1, x^2), B(x^3)\}$ *are not conformally equivalent to each other.*

We note that the hypersurfaces given in the following Theorem are not hypersurfaces obtained from constant curvature surfaces in $\mathbf{R}^3$. We can also prove this theorem in the same way as the proof of Theorem 3.1.2.

Theorem 3.2.3. *Let $A(x^1, x^2)$ and $B(x^3)$ be the functions in* Proposition 3.2.1-(c), *and $E > 0$ in the definitions of $A(x^1, x^2)$ and $B(x^3)$. Let S be a surface in $\boldsymbol{R}^3$ determined from $A(x^1, x^2)$. The first fundamental form $\hat{g}$ and the second fundamental form $\hat{s}$ for S are respectively given as follows*:

$$\hat{g} = \cosh^2 A(dx^1)^2 + \sinh^2 A(dx^2)^2, \qquad \hat{s} = \sqrt{E}e^{-A}\left\{\cosh A(dx^1)^2 - \sinh A(dx^2)^2\right\}.$$

Then, the map $\mathbf{p}(x^1, x^2, x^3)$ into $\boldsymbol{R}^4$, made from S and $B(x^3)$ in the same way as in (3.1.17), *defines a generic conformally flat hypersurface if and only if the constant G in the definition of $B(x^3)$ satisfies the inequalities $G < 0$ and $Ee^{2B} + G > 0$.*

Furthermore, each x^i-curve of the hypersurface $\mathbf{p}$ is a principal curvature line.

In Theorem 3.2.3, the functions $l(x^3)$ and $m(x^3)$ in the definition of the map $\mathbf{p}(x^1, x^2, x^3)$ are respectively detemined as follows: $2\sqrt{E}\,l = 1 - e^{-2B}$, $Em_3^2 = -Ge^{-4B}$. Furthermore, we have $B_3^2 = Ee^{2B} + G$.

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