J. Hano
Nagoya Math. J.
Vol. 141 (1996), 79–105

CONFORMAL IMMERSIONS OF COMPACT RIEMANN SURFACES INTO THE $2n$-SPHERE ($n \geq 2$)

JUN-ICHI HANO

The purpose of this article is to prove the following theorem:

Let n be a positive integer larger than or equal to 2, and let S^{2n} be the unit sphere in the $2n+1$ dimensional Euclidean space. Given a compact Riemann surface, we can always find a conformal and minimal immersion of the surface into S^{2n} whose image is not lying in any $2n-1$ dimensional hyperplane.

This is a partial generalization of the result by R. L. Bryant. In this papers, he demonstrates the existence of a conformal and minimal immersion of a compact Riemann surface into S^{2n}, which is generically $1:1$, when $n = 2$ ([2]) and $n = 3$ ([1]).

We start with an idea formulated by Bryant in his paper [2], which is also fundamental for our proof. Let $\mathbf{V}$ be the set of all maximal isotropic subspaces in $\mathbf{C}^{2n+1}$ with respect to the complex symmetric bilinear form, the extension of the standard inner product on $\mathbf{R}^{2n+1}$. The set $\mathbf{V}$ is a connected compact complex manifold and has a natural projection π on the unit sphere S^{2n}, defining the twistor bundle $(\mathbf{V}, \pi, S^{2n})$, where the $SO(2n+1)$-actions on $\mathbf{V}$ and on S^{2n} are equivariant under the projection π. Beginning with E. Calabi's work ([5], [6]), the twister bundle plays an important role in the geometry of minimal surfaces, or more generally harmonic maps of surfaces, in S^{2n}. (For recent developments on twistor bundles over even dimensional Riemannian symmetric spaces and their applications, we refer to Bryant [3], Burstall-Rawnsley [4]).

There is a distribution $\mathbf{T}$ on $\mathbf{V}$ perpendicular to the fibre at each point with respect to any Riemannian metric invariant under the $SO(2n+1)$-action, which is not integrable, but is holomorphic [2]. An oriented surface immersed in S^{2n} has a complex structure canonically determined by the orientation and the first fundamental form. The basic idea of Bryant's proof [2] is that if a Riemann surface M admits an anti-holomorphic immersion φ into $\mathbf{V}$ whose image is tangent to the distribution $\mathbf{T}$ at each point on M, then $\pi{,}\varphi : M \to S^{2n}$ is a minimal and conformal

Received November 25, 1994.

immersion.

Furthermore, the complex manifold $\mathbf{V}$ admits a holomorphic imbedding into the complex projective space $\mathbf{P}^{2^n-1}$, introduced by É. Cartan [7] in connection with the spinor representation. For our purpose, it is crucial that the imbedding can be written in an explicit form in terms of the Cartan coordinates on a dense open subset in $\mathbf{V}$, so that $\mathbf{V}$ is realized as a projective submanifold of a simple form.

Our task is to combine the above two known results. In the section 1, we study the distribution $\mathbf{T}$ on the twistor space $\mathbf{V}$ and give its concrete description in terms of Cartan's local holomorphic coordinates (Lemma 1.1). In the section 2, making use of the Clifford algebra, we treat the projective imbedding of $\mathbf{V}$ (Lemma 2.1). In his lecture notes [7] (Chap. V, 92), É. Cartan suggests a quite different, more direct approach to the projective imbedding. We would like to explore his idea elsewhere.

The first half of the section 3 is a survey of differential geometry of a surface in S^{2n} which admits an anti-holomorphic section into the bundle space $\mathbf{V}$ whose image is tangent to the distribution $\mathbf{T}$. Corresponding to two different aspacts of the twistor space, we state two characterizations of such immersion (Lemma 3.2 and 3.4). In Lemma 3.6, we show that if the image of such section is in general position in $\mathbf{P}^{2^n-1}$ (not contained in any linear submanifold), then the surface in S^{2n} can not lie in any hyperplane of dimension $2n-1$. In the last section 4, using the Riemann-Roch theorem, we construct an anti-holomorphic immersion of a given compact Riemann surface into $\mathbf{P}^{2^n-1}$ whose image is contained in $\mathbf{V}$, tangent to the distribution $\mathbf{T}$ and in general position in $\mathbf{P}^{2^n-1}$. This yields immediately the main theorem.

1. The twistor space over S^{2n}

1.1. The real Cartesian space $\mathbf{R}^{2n+1}$ is contained in $\mathbf{C}^{2n+1}$ canonically and its standard inner product extends to a complex symmetric bilinear form on $\mathbf{C}^{2n+1}$, which will be denoted by B.

Using the standard basis $\{\varepsilon_\lambda\,;\,\lambda=0,1,\ldots,n,1',\ldots,n'\}$ of $\mathbf{R}^{2n+1}$ we put

$$e_0=\varepsilon_0,\ e_i=(1/\sqrt{2})(\varepsilon_i-\sqrt{-1}\,\varepsilon_{i'}),\ e_{i'}=(1/\sqrt{2})(\varepsilon_i+\sqrt{-1}\,\varepsilon_{i'}). \tag{1.1.1}$$

Then $\{e_\lambda\,;\,\lambda=0,1,\ldots,n,1',\ldots,n'\}$ is a basis of $\mathbf{C}^{2n+1}$, and

$$B(\textstyle\sum_\lambda a_\lambda e_\lambda,\ \sum_\lambda b_\lambda e_\lambda)=a_0b_0+\sum_i(a_ib_{i'}+a_{i'}b_i).$$

With this basis, the standard hermitian form is given by

$$H(\Sigma_\lambda a_\lambda e_\lambda, \Sigma_\lambda b_\lambda e_\lambda) = \Sigma_\lambda a_\lambda \bar{b}_\lambda.$$

We denote by G_c and by G respectively the matrix representations of the special complex orthogonal group $SO(2n+1, \mathbf{C})$ and the special orthogonal group $SO(2n+1)$ with respect to the basis $\{e_\lambda\}$. The group G_c consists of all complex matrices leaving the complex bilinear form B and the wedge product $\varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_{1'} \wedge \ldots \wedge \varepsilon_n \wedge \varepsilon_{n'}$ invariant, and the group G is the intersection of G_c and the unitary group $U(2n+1)$.

Let $\mathfrak{g}$ and $\mathfrak{g}_c$ be the Lie algebras of G and G_C respectively. A complex $(2n+1, 2n+1)$ matrix X belongs to $\mathfrak{g}_c$ if and only if its entries $X_{\lambda\mu}$ satisfy the following conditions:

$$X_{00} = 0,\ X_{0i'} = -X_{i0},\ X_{0i} = -X_{i'0},\ X_{i'j'} = -X_{ji},$$
$$X_{ij'} = -X_{ji'},\ X_{i'j} = -X_{j'i},\quad (i, j = 1, \ldots, n),$$

and X belongs to $\mathfrak{g}$ if and only if X is skew-hermitian and belongs to $\mathfrak{g}_c$.

1.2. A complex subspace V of the vector space $\mathbf{C}^{2n+1}$ is said to be isotropic if the restriction of B to V is identically zero. Every maximal isotropic subspace in $\mathbf{C}^{2n+1}$ is of the same dimension n, by Witt's Theorem. We denote by $\mathbf{V}$ the set of all maximal isotropic subspaces in $\mathbf{C}^{2n+1}$.

The subspace V_0 spanned by $e_1, \ldots, e_n$ in the basis (1.1.1) is a maximal isotropic subspace. Take an arbitrary maximal isotropic subspace V, and choose an orthonormal basis $\{f_1, \ldots, f_n\}$ of V with respect to H. Let $\bar{f}_i$ denote the complex conjugate of f_i with respect to $\mathbf{R}^{2n+1}$. Then, there exists one and only one unit vector f_0 such that f_0 is orthogonal to $f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n$ and that

$$\varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_{1'} \wedge \ldots \wedge \varepsilon_n \wedge \varepsilon_{n'} = (-\sqrt{-1})^n f_0 \wedge (f_1 \wedge \bar{f}_1) \wedge \ldots \wedge (f_n \wedge \bar{f}_n). \tag{1.2.1}$$

Arranging these $2n+1$ column vectors $f_0, f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n$, we obtain a matrix belonging to G, which maps e_i to f_i, $(i = 0, 1, \ldots, n)$, and $\bar{e}_i$ to $\bar{f}_i$, $(i = 1, \ldots, n)$, and hence V_0 to V. Thus, both G and G_c act transitively on $\mathbf{V}$.

Moreover, the correspondence $\pi : V \mapsto f_0$ is a G-equivariant map from $\mathbf{V}$ onto the unit sphere S^{2n}. As the bundle space of the fiber bundle $(\mathbf{V}, \pi, S^{2n})$, $\mathbf{V}$ is the twistor space of the sphere S^{2n} ([13]IV, 9).

The subset $(G_c)_o$ of all matrices in the complex Lie group G_c which leave the complex subspace V_0 invariant is a complex Lie subgroup of G_c. Thus, the quotient space $\mathbf{V}$ of the complex Lie group G_c modulo $(G_c)_o$ is a connected compact complex manifold. Its complex dimension is $n(n+1)/2$.

We denote by H the subgroup in G consisting of all matrices leaving the vector e_0 invariant, and by K the subgroup of all matrices leaving the subspace V_0 invariant or equivalently the complex conjugate of V_0 invariant. The subgroup H is isomorphic to $\mathrm{SO}(2n)$, and K is a subgroup of H and isomorphic to $\mathrm{U}(n)$. As quotient spaces of G, $\mathbf{V} = G/K$ and $S^{2n} = G/H$. We denote by Π the quotient map $G \to G/K = \mathbf{V}$. The composite $\pi . \Pi$ is the quotient map $G \to G/H = S^{2n}$.

The Lie subalgebras in $\mathfrak{g}$ corresponding to the Lie subgroups K and H are denoted by $\mathfrak{k}$ and $\mathfrak{h}$ respectively.

1.3. Given a point $p \in S^{2n}$, let us take an arbitrary maximal isotropic subspace V lying over p and its complex conjugate $\bar{V}$ with respect to the real vector space $\mathbf{R}^{2n+1}$. Clearly $V \cap \bar{V} = \{0\}$. The direct sum $V + \bar{V}$ is the complexification of the tangent space S_p to S^{2n} at p. There exists a unique complex structure J_p on S_p such that V is the subspace of all eigen-vectors belonging to the eigen-value $\sqrt{-1}$ of J_p (i.e., the (1.0)-component of the complexification of S_p. The endomorphism J_p is orthogonal with respect to the inner product on S_p.

Conversely, take an orthogonal complex structure J_p on the tangent space S_p, and a unitary basis $\{f_1, \ldots, f_n\}$ of the (1.0)-component V of the complexification of S_p. Obviously, V is a maximal isotropic subspace in $\mathbf{C}^{2n+1}$. As a point in $\mathbf{V}$, V is lying over p, this is $\pi(V) = p$ if and only if (1.2.1) is satisfied, with $f_0 = p$. If n is even, $\pi(V) = \pi(\bar{V}) = p$, but if n is odd, one and only one of $\pi(V)$ and $\pi(\bar{V})$ is p.

1.4. Consider $\mathbf{V}$ as the quotient space $G_c/(G_c)_o$, where $(G_c)_o$ is the isotropy subgroup at the point $o = V_0 \in \mathbf{V}$. We denote by L (resp. L_1) the subgroup of matrices in $(G_c)_o$ which induce the identity on the subspace V_0 (resp. leave not only V_0, but also its complex conjugate $\bar{V}_0$). The subgroup L is nilpotent, connected and simply connected. The subgroup L_1 is isomorphic to $\mathrm{GL}(n, \mathbf{C})$.

The isotropy subgroup $(G_c)_o$ is the semi-direct product of its normal subgroup L and the subgroup L_1 and hence connected. Later we need the fact that the normalizer of $(G_c)_o$ in G_c coincides with $(G_c)_o$. This follows easily from that a vector in $\mathbf{C}^{2n+1}$ kept fixed by the subgroup L belongs to V_0.

Let $(\mathfrak{g}_c)_o$ be the Lie subalgebra corresponding to the subgroup $(G_c)_o$. We regard the quotient space $\mathfrak{g}_c/(\mathfrak{g}_c)_o$ as the (1,0)-component of the complexification of the tangent space $T(\mathbf{V})_o$ to $\mathbf{V}$ at the point o. Then, the isomorphism: $\mathfrak{g}/\mathfrak{k} \to \mathfrak{g}_c/(\mathfrak{g}_c)_o$ induced by the inclusion $\mathfrak{g} \subset \mathfrak{g}_c$ maps a real vector in $T(\mathbf{V})_o$ to its (1,0) component with respect to the complex structure on $\mathfrak{g}_c/(\mathfrak{g}_c)_o$. (The same vector in $T(\mathbf{V})_o$ can be a real vector as an element of $\mathfrak{g}/\mathfrak{k}$ and its (1,0) component as an element of $\mathfrak{g}_c/(\mathfrak{g}_c)_o$.)

1.5. Let $\mathfrak{n}$ be the nilpotent subalgebra of $\mathfrak{g}_c$ consisting of matrices $\xi = (X_{\lambda\mu})$ with

$$X_{0i'} = X_{i0} = 0,\ X_{ij} = X_{i'j'} = 0,\ X_{ij'} = 0.$$

As a vector space, $\mathfrak{g}_c$ is the direct sum of subspaces $\mathfrak{n}$ and $(\mathfrak{g}_c)_o$.

If $\xi = (X_{\lambda\mu}) \in \mathfrak{n}$, we set $\xi_i = X_{0i} = -X_{i'0}$ and $\xi_{ij} = X_{i'j} = -X_{j'i}$. We have

$$(1.5.1)\qquad \exp \xi = \left(\begin{array}{c|c|c} 1 & {}^t(\xi_i) & 0 \\ \hline 0 & (\xi_{ij}) & 0 \\ \hline (-\xi_i) & (-(1/2)\xi_i\xi_i + \xi_{ij}) & (\delta_{ij}) \end{array}\right).$$

The connected Lie subgroup corresponding to the Lie algebra $\mathfrak{n}$ intersects with $(G_c)_o$ at the identity, and the correspondence

$$\xi \mapsto (\exp \xi)(V_0)$$

difines a 1 : 1 holomorphic map from the complex vector space $\mathfrak{n}$ onto an open subset $\mathbf{V}_0$ in $\mathbf{V}$. We regard (ξ_i, ξ_{jk}), $1 \leq i, j, k \leq n$, $\xi_{jk} + \xi_{kj} = 0$, the complex coordinates of the point $(\exp \xi)(V_0)$ on $\mathbf{V}_0$.

Let $x_0, x_1, \ldots, x_n, x_{1'}, \ldots, x_{n'}$ be the complex coordinates of $\mathbf{C}^{2n+1}$ with respect to the basis $\{e_\lambda\}$ in **1.1**. These coordinate functions form the dual basis of $\{e_\lambda\}$. If $(\exp \xi)(V_0) = V$, the restrictions of $x_1, \ldots, x_n$ form the dual basis of the basis $\{(\exp \xi)e_1, \ldots, (\exp \xi)e_n\}$ of V by (1.5.1), and V is the solutions subspace of the following $n + 1$ linear equations (Cartan [7] Chap. V, 92):

$$(1.5.2)\qquad \begin{cases} x_0 - \sum_{i=1}^n \xi_i x_i = 0, \\ x_{j'} + (1/2)\xi_j x_0 - \sum_{i=1}^n \xi_{ji} x_i = 0, \quad (1 \leq j \leq n). \end{cases}$$

Conversely, given (ξ_i, ξ_{jk}) satisfying $\xi_{jk} + \xi_{kj} = 0$ $(1 \leq i, j, k \leq n)$, the subspace V of solutions of the above $n + 1$ linear equations is a maximal isotropic subspace in $\mathbf{C}^{2n+1}$ and belongs to $\mathbf{V}_0$.

The image of the identity element e of G_c under Π is the point $o = V_0$, whose coordinates are all zero. Take $X = (X_{\lambda\mu}) \in \mathfrak{g}_c$, and denote by X' a matrix in $\mathfrak{n}$ determined by $X \equiv X' \pmod{(\mathfrak{g}_c)_o}$. Then $(\Pi_*)_e(X_e) = (\Pi_*)_e(X'e)$ and

$$(1.5.3)\qquad (\Pi_*)_e(X_e) = \textstyle\sum_i X_{0i}(\partial/\partial\xi_i)_o + \sum_{i'j} X_{i'j}(\partial/\partial\xi_{ij})_o.$$

1.6. Let $(\mathbf{V}, \pi, S^{2n})$ be the fibre bundle constructed in **1.2**. We show that the fibre $\mathbf{V}(p)$ over an arbitrary point $p \in S^{2n}$ is a connected complex submanifold.

Since $\mathbf{V}(e_0) = H/K$, it is connected and its real dimension is $n(n+1)$. Let $\mathfrak{f}$ be the ideal of $\mathfrak{n}$ consisting of matrices ξ such that $\xi_i = 0$ $(1 \leq i \leq n)$. If $\xi \in \mathfrak{f}$, $(\exp \xi)(e_0) = e_0$ and the matrix $\exp \xi$ leaves the wedge product $e_0 \wedge (e_1 \wedge e_{1'}) \wedge \ldots \wedge (e_n \wedge e_{n'})$ invariant by (1.5.1). Hence, the image of $(\exp \xi)(V_0)$ under π is e_0 by definition, and $(\exp \xi)(V_0)$ belongs to the fibre $\mathbf{V}(e_0)$. Comparing dimensions, we see that in open subset $\mathbf{V}_0$, the fibre $\mathbf{V}(e_0)$ is the complex submanifold defined by $\xi_1 = \cdots = \xi_n = 0$. By the homogeneity of the G-action on $\mathbf{V}$, we obtain the desired result.

1.7. Let $\mathfrak{t}$ be the subspace in the complex nilpotent subalgebra $\mathfrak{n}$ defined by $\xi_{jk} = 0$ $(1 \leq j, k \leq n)$. We have $\mathfrak{n} = \mathfrak{t} + \mathfrak{f}$ and $\mathfrak{t} \cap \mathfrak{f} = \{0\}$. We denote by $\mathbf{T}_o$ the complex subspace $\mathfrak{t} + (\mathfrak{g}_c)_o/(\mathfrak{g}_c)_o$ in the tangent space $T(\mathbf{V})_o$ at $o = V_0$ of $\mathbf{V}$, which is spanned by $(\partial/\partial\xi_1)_o, \ldots, (\partial/\partial\xi_n)_o$.

Since $[(\mathfrak{g}_c)_o, \mathfrak{t} + (\mathfrak{g}_c)_o] \subset \mathfrak{t} + (\mathfrak{g}_c)_o$ and since $(G_c)_o$ is connected, the subspace $\mathbf{T}_o$ is invariant under the linear isotropy representation of $(G_c)_o$. Hence, there exists a G_c-invariant distribution $\mathbf{T}$ on $\mathbf{V}$ which assigns to the point o the subspace $\mathbf{T}_o$. As $\mathfrak{t} + (\mathfrak{g}_c)_o$ is not a subalgebra, $\mathbf{T}$ is not completely integrable.

Consider now $\mathbf{V}$ as the quotient space G/K. The tangent space at o to the fibre $\mathbf{V}(e_0)$ is $\mathfrak{h}/\mathfrak{k} = \mathfrak{f} + (\mathfrak{g}_c)_o/(\mathfrak{g}_c)_o$, on which the linear isotropic representation of K induces the dual of the $\mathrm{U}(n)$-action on the space of all complex skew-symmetric (n, n)-matrices. The K-action leaves $\mathbf{T}_\sigma$ invariant and its representation on $\mathbf{T}_o$ is equivalent to the dual of the $\mathbf{U}(n)$-action on $\mathbf{C}^n$. Clealy these two representations of $\mathbf{U}(n)$ are inequivalent.

With respect to any G-invariant Riemann metric on $G/K = \mathbf{V}$, the subspaces $\mathfrak{h}/\mathfrak{k}$ and $\mathbf{T}_o$ are mutually orthogonal, and the distribution $\mathbf{T}$ assigns to each point V on $\mathbf{V}$ the orthogonal complement $\mathbf{T}_V$ of the tangent space of the fibre $\mathbf{V}(\pi(V))$ ih the tangent space to $\mathbf{V}$.

Let π_* be the differential of the projection $\pi : \mathbf{V} \to S^{2n}$. At each point $V \in \mathbf{V}$. the restriction to $\mathbf{T}_V$ of $(\pi_*)_V$ is an isomorphism onto the tangent space of S^{2n} at $\pi(V)$. If we choose the G-invariant Kähler metric on $\mathbf{V}$ associated to $-1/(4n-2)$ times the Killing form of $\mathfrak{g}$, this isomorphism becomes an isometry.

1.8. LEMMA 1.1. *On the open subset $\mathbf{V}_0$, the distribution $\mathbf{T}$ is defined by the following $n(n+1)/2$ equations*:

(1.8.1) $$d\xi_{ij} + (1/2)(\xi_j d\xi_i - \xi_i d\xi_j) = 0, \quad 1 \leq i < j \leq n.$$

Proof. We denote by $\mathbf{D}$ the distribution on $\mathbf{V}_0$ which assings to each point the subspace of solutions of the equations (1.8.1), and show that $\mathbf{D}=\mathbf{T}$. The n vector fields

$$\partial/\partial\xi_k-(1/2)\textstyle\sum_l\xi_l\,\partial/\partial\xi_{kl},\quad 1\le k\le n, \tag{1.8.2}$$

are solutions of (1.8.1) and span the distribution $\mathbf{D}$. Since the subspace $\mathbf{T}_o$ is spanned by $(\partial/\partial\xi_1)_o,\dots,(\partial/\partial\xi_n)_o$, $\mathbf{T}_o=\mathbf{D}_o$. Therefore, in order to verify $\mathbf{T}=\mathbf{D}$, it suffices to show that $\mathbf{D}$ is invariant under the G_c-action. We will show that if $g.o\in\mathbf{V}_0$ for some $g\in G_c$, the image of $\mathbf{T}_o$ under the differential of the translation L_g coincides with $\mathbf{D}_{g.o}$ $(1\le k\le n)$.

Take an arbitrary vector $X^*\in\mathbf{T}_o$. X^* is the tangent vector at $o=V_0$ of the path $t\mapsto(\exp tX).o$ for some $X\in\mathfrak{t}$**(1.7)**. We put $\sigma_t=(g.\exp tX)$,

$$(d/dt)_{t=o}(\xi_i(\sigma_t.V_0))=\dot{\xi}_i \text{ and } (d/dt)_{t=0}(\xi_{ij}(\sigma_t.V_0))=\dot{\xi}_{ij},$$

where $o=V_0$.
Then,

$$(L_g)_*(X^*)=\textstyle\sum_i\dot{\xi}_i(\partial/\partial\xi_i)_{g.o}+\sum_{ik}\dot{\xi}_{ik}(\partial/\partial\xi_{ik})_{g.o}.$$

This vector belongs to $\mathbf{D}_{g.o}$, namely, written as a linear combination of n vector fields given by (1.8.2) at $g.o$, if and only if its coefficients satisfy the equations

$$\dot{\xi}_{ji}+(1/2)\dot{\xi}_j\xi_i(g.o)-(1/2)\dot{\xi}_j\xi_j(g.o)=0,\quad 1\le j<i\le n. \tag{1.8.3}$$

On account of Cartan's equations (1.5.2),

$$x_0(\sigma_t.e_l)-\textstyle\sum_i\xi_i(\sigma_t.V_0)x_i(\sigma_t.e_l)=0,$$
$$x_{j'}(\sigma_t.e_l)+(1/2)\xi_j(\sigma_t.V_0)x_0(\sigma_t.e_l)-\textstyle\sum_i\xi_{ji}(\sigma_t.V_0)x_i(\sigma_t.e_l)=0,$$

$1\le j,\ l\le n$. We differentiate both sides of each equation at $t=0$ and obtain the equality

$$\textstyle\sum_i\{\dot{\xi}_{ji}+(1/2)\dot{\xi}_j\xi_i(g.o)-(1/2)\dot{\xi}_i\xi_j(g.o)\}g_{il}=\xi_j(g.o)(g.X)_{0l}+ \tag{1.8.4}$$
$$(g.X)_{j'l}+\textstyle\sum_i\{-(1/2)\xi_i(g.o)\xi_j(g.o)+\xi_{ij}(g.o)\}(g.X)_{il},\ 1\le j,\ l\le n.$$

Since $X\in\mathfrak{t}$, $(g.X)_{\lambda l}=(g)_{\lambda 0}X_{0l}$. Take $\xi\in\mathfrak{n}$ such that $(\exp\xi).V_0=\exp\xi.o=g.o$. By (1.5.1), the components of the column vector $\exp\xi.e_j$ are

$$(\xi_j(g.o),\dots,\delta_{ij},\dots,\dots,-(1/2)\xi_i(g.o)\xi_j(g.o)+\xi_{ij}(g.o),\dots,).$$

Thus, the right hand side of the equality (1.8.4) is equal to

$$B(\exp \xi.e_j,\ g.e_0)X_{0l},$$

where B is the symmetric bilinear form defined in **1.1**. Since $\exp \xi.e_j \in g.V_0$, $B(\exp \xi.e_j,\ g.e_0) = 0$ and the left hand side of the equality (1.8.4) is zero. From the assumption that $g.V_0$ belongs to $\mathbf{V}_0$, it follows that the determinant of the $(n,\ n)$-minor $((g)_{ij})$ is not zero. Thus, we have verified the equalities (1.8.3) for an arbitrary vector X^* in $\mathbf{T}_o$, completing the proof.

2. Cartan's projective imbedding

2.1. Here, we summarize what we need from the spin representation theory ([9] Ch.II.§XI). Let us denote by N the set of integers $\{1, \ldots, n\}$ and by $\mathbf{N}$ the collection of all subsets in N, consisting of 2^n subsets including the empty set $\emptyset$. For $A \in \mathbf{N}$, $\#(A)$ denotes the number of integers in A, A^c the complement of A. For $A,\ B \in \mathbf{N}$, $A + B$ is the subset of those integers which belong to $A \cup B$ but not to $A \cap B$. Given $A,\ B \in \mathbf{N}$, we denote by $p(A,\ B)$ the number of pairs $(i,\ j)$ such that $i \in A$, $j \in B$ and $i \geq j$, and put $\varepsilon(A,\ B) = (-1)^{p(A,B)}$.

Let $\mathfrak{C}$ be the Clifford algebra over $\mathbf{C}^{2n+1}$ with the symmetric bilinear form B, the quotient algebra of the tensor algebra over $\mathbf{C}^{2n+1}$ modulo the ideal generated by $v \otimes v + B(v,\ v).1$, $v \in \mathbf{C}^{2n+1}$. The subspace $\mathfrak{C}_2$ spanned by $[u,\ v] = u,v - v.u (u,\ v \in \mathbf{C}^{2n+1})$ is closed under the bracket product, and is a Lie algebra. To each $[u,\ v]$, we assign the linear map $l([u,\ v])$ of $\mathbf{C}^{2n+1}$ given by $w \mapsto [[u,\ v],\ w] = 4(B(u,\ w)v + B(v,\ w)u)$. Then l defines a Lie algebra isomorphism $\mathfrak{C}_2 \to \mathfrak{g}_c$.

Using the basis $\{e_\lambda\}$ given by (1.1.1), we put

$$a_i = (1/4)[e_0,\ e_i] \text{ and } a_{i'} = (1/4)[e_0,\ e_{i'}], \quad 1 \leq i \leq n,$$

then, $l(a_i) = E_{i0} - E_{0i'}$ and $l(a_{i'}) = E_{i'0} - E_{0i}$, where $E_{\lambda\mu}$ is the matrix whose $(\lambda,\ \mu)$-entry is 1 and others are all 0, $(\lambda,\ \mu \in \{0,1, \ldots, n,\ 1', \ldots, n'\})$. Thus, the Lie algebra $\mathfrak{C}_2$ is generated by a_i and $a_{i'}$, $1 \leq i \leq n$. Indeed,

$$(2.1.1) \qquad l(-[a_i,\ a_j]) = E_{ij'} - E_{ji'},\ l(-[a_i,\ a_{j'}]) = E_{ij} - E_{j'i'}, \text{ and}$$
$$l(-[a_{i'},\ a_{j'}]) = E_{i'j} - E_{j'i},\ 1 \leq i, j \leq n.$$

In the associative algebra $\mathfrak{C}$,

$$(2.1.2) \qquad a_i a_j + a_j a_i = a_{i'} a_{j'} + a_{j'} a_{i'} = 0 \text{ and } a_i a_{j'} + a_{j'} a_i + (1/2)\delta_{ij} = 0.$$

Hence,

$$(2.1.3) \qquad [a_i,\ a_j] = 2a_i a_j,\ [a_i,\ a_{j'}] = 2a_i a_{j'} + (1/2)\delta_{ij}$$
$$\text{and } [a_{i'},\ a_{j'}] = 2a_{i'} a_{j'},\ 1 \leq i, \quad j \leq n.$$

For each $A = \{i_1, \ldots, i_\nu\} \in \mathbf{N}$ $(1 \leq i_1 < \cdots < i_\nu \leq n)$, we put

$$\Lambda_A = (\sqrt{2})^\nu a_{i_1} \ldots a_{i_\nu} a_{1'} \ldots a_{n'}. \tag{2.1.4}$$

Then, by (2.1.2), we have

$$a_i . \Lambda_A = \begin{cases} 0, \text{ if } i \in A, \\ (1/\sqrt{2})\varepsilon(i, A)\Lambda_{A+\{i\}}, \text{ if } i \notin A, \end{cases} \tag{2.1.5}$$

$$a_{i'} . \Lambda_A = \begin{cases} (1/\sqrt{2})\varepsilon(i, A)\Lambda_{A+\{i\}}, \text{ if } i \in A, \\ 0, \text{ if } i \notin A. \end{cases} \tag{2.1.6}$$

Thus, the subspace Λ in $\mathfrak{C}$ spanned by these 2^n elements Λ_A, $A \in \mathbf{N}$, is a right ideal in the associative subalgebra $\mathfrak{C}^+$ generated by 1 and $\mathfrak{C}_2$. By assigning to each element a in the subalgebra $\mathfrak{C}^+$ (resp. the Lie algebra $\mathfrak{C}_2$), the restriction $r(a)$ to Λ of the right multiplication by a, we obtain a representation of the associative algebra $\mathfrak{C}^+$ (resp. the Lie algebra $\mathfrak{C}_2$) on Λ. We denote by ρ the homomorphism $r \circ l^{-1}$ from $\mathfrak{g}_c$ into the general linear Lie algebra $\mathfrak{gl}(\Lambda)$.

We denote by $H(\lambda)$ the diagonal matrix $\sum \lambda_i(E_{ii} - E_{i'i'}) = l(-\sum \lambda_i[\alpha_i, a_{i'}])$, $\lambda_i \in \mathbf{C}$, $1 \leq i \leq n$. These diagonal matrices form a Cartan subalgebra of $\mathfrak{g}_c$. Using the equalities (2.1.2 and 3), we obtain

$$\rho(H(\lambda)). \Lambda_A = ((-1/2)\textstyle\sum_1^n \lambda_k + \sum_1^\nu \lambda_{i_\nu})\Lambda_A, \quad \text{for } A = \{i_1, \ldots, i_\nu\} \in \mathbf{N}.$$

Thus, $(1/2)\sum_1^n \lambda_k$ is the highest weight of the representation ρ and Λ_N is a highest weight vector. The representation ρ on Λ is the spin representation of $\mathfrak{g}_c$. (With respect to the basis $\{\Lambda_A, A \in \mathbf{N}\}$, the matrix representations of $r(a_i + a_{i'})$ and $r((\sqrt{-1})(a_i - a_{i'}))$, are skew-hermitian.)

2.2 We denote by $(G_c)^*$ the connected Lie subgroup in the general linear group $GL(\Lambda)$ corresponding to the Lie algebra $\mathfrak{C}_2$. The center Z of $(G_c)^*$ is $\{\pm I\}$ and hence the group $(G_c)^*$ is Spin $(2n+1, \mathbf{C})$, the universal covering group of $SO(2n+1, \mathbf{C})$. Obviously, $G_c \cong (G_c)^*/Z$ induces the isomorphism ρ.

Let us denote by $\mathbf{P}^{2^n-1}$ the complex projective space of all complex lines through the origin in the 2^n-dimensional complex vector space Λ, and by o^* the point in $\mathbf{P}^{2^n-1}$ determined by the line along the highest weight vector Λ_N. The complex spin group $(G_c)^*$ acts on the projective space modulo the center Z, and *the* $(G_c)^*$*-orbit through the point* o^* *can be identified with the complex manifold* $\mathbf{V} = G_c/(G_c)_o$.

The Lie subalgebra $(\mathfrak{g}_c)_o$ is spanned by

$$E_{i0} - E_{0i'} \ (1 \leq i \leq n), \ E_{ij'} - E_{ji'} \ (1 \leq i < j \leq n) \text{ and}$$

$$E_{ij} - E_{j'i'} \ (1 \leq i, j \leq n),$$

and $l^{-1}((\mathfrak{g}_c)_o)$ is spanned by a_i $(1 \leq i \leq n)$, $a_i a_j$ $(1 \leq i < j \leq n)$ and $a_{j'} a_i + \delta_{ij}$ $(1 \leq i < j \leq n)$ by (2.1.1–3). Hence, $\rho((\mathfrak{g}_c)_o)$ is contained in the subalgebra of matrices X such that $\rho(X).\Lambda_N$ is a scalar multiple of Λ_N by (2.1.4.6). Moreover, one can verify easily that these two subalgebras coincide. Thus, the isotropy subgroup of $(G_c)^*/Z$ at the point o^* contains a connected subgroup isomorphic to $(G_c)_o$ as its connected component. As is mentioned in **I.4**, the normalizer of $(G_c)_o$ in G_c is itself and hence the isotropy subgroup at o^* is isomorphic to $(G_c)_o$. Therefore, the $(G_c)^*$-orbit through the point o^* can be identified with $G_c/(G_c)_o = \mathbf{V}$.

We denote by ι this imbedding of $\mathbf{V}$ into $\mathbf{P}^{2^n-1}$. Given $g \in G_c$, take $g^* \in (G_c)^*$ lying over g. Then, $\iota(g.V) = g^*.\iota(V)$ for $V \in \mathbf{V}$. Particularly, if $X \in \mathfrak{g}_c$,

$$\iota((\exp X).V) = (\exp \rho(X)).\iota(V) \text{ for } V \in \mathbf{V}. \tag{2.2.1}$$

2.3. Our purpose is to describe the imbedding ι in terms of the coordinates (ξ_i, ξ_{jk}) on the open subset $\mathbf{V}_0$ defined in 1.5 and of appropriate homogeneous coordinates on the projective space $\mathbf{P}^{2^n-1}$.

We adopt some notational conventions following É. Cartan [7]. Let $i_1, \ldots, i_{2k}$ be an arbitrary choice of $2k$ integers in $N = \{1, \ldots, n\}$. We put

$$\xi_{i_1 \ldots i_{2k}} = (1/2^k k!) \sum \varepsilon(j_1 \ldots j_{2k})(\xi_{j_1 j_2}) \ldots (\xi_{j_{2k-1} j_{2k}})$$

where in the summation $\{j_1, \ldots, j_{2k}\}$ runs over all permutations of $i_1, \ldots, i_{2k}$, and $\varepsilon(j_1, \ldots, j_{2k})$ denotes the sign of the permutation $j_1, \ldots, j_{2k}$. Obviously, $\xi_{i_1 \ldots i_{2k}}$ is skew-symmetric with respect to the indecies. If $i_1, \ldots, i_{2k}$ are all distinct, $\xi_{i_1 \ldots i_{2k}}$ is equal to

$$\sum_{j_{2a-1} < j_{2a};\, j_2 < \ldots < j_{2k}} \varepsilon(j_1 \ldots j_{2k})(\xi_{j_1 j_2}) \ldots (\xi_{j_{2k-1} j_{2k}}).$$

One can verify easily the equality

$$\xi_{i_1 \ldots i_{2k}} = \textstyle\sum_{a=1}^{2k-1} (-1)^{a-1} \xi_{i_a i_{2k}} \xi_{i_1 \ldots \hat{i}_a \ldots i_{2k-1}}. \tag{2.3.1}$$

For any choice of $2k-1$ integers $i_1, \ldots, i_{2k-1}$ from N, we put

$$\xi_{i_1 \ldots i_{2k-1}} = (1/2^{k-1}(k-1)!) \sum \varepsilon(j_1 \ldots j_{2k-1})(\xi_{j_1})(\xi_{j_2 j_3}) \ldots (\xi_{j_{2k-2} j_{2k-1}})$$

as in the previous case. We have the equality

$$\xi_{i_1 \ldots i_{2k-1}} = \textstyle\sum_{a=1}^{2k-1} (-1)^{a-1} \xi_{i_a} \xi_{i_1 \ldots \hat{i}_a \ldots i_{2k-1}}. \tag{2.3.2}$$

Again, $\xi_{i_1\dots i_{2k-1}}$ is skew-symmetric in indeces.

If $A = \{i_1, \dots, i_k\}$ and $1 \leq i_1 < \dots < i_k \leq n$, we also denote by ξ_A the function $\xi_{i_1 \dots i_k}$, and if $A = \emptyset$, we put $\xi_\emptyset = 1$.

2.4. For later convenience, we prepare a new basis for the representation space Λ. Given $A = \{i_1, \dots, i_k\}$, $1 \leq i_1 < \dots < i_k \leq n$, we put $A^c = \{j_1, \dots, j_l\}$, $1 \leq j_1 < \dots < j_l \leq n$, and

$$(2.4.1) \qquad {}^*\Lambda_A = \begin{cases} (-1)^k \varepsilon(A, N)\Lambda_{A^c}, & \text{if } \#(A) = 2k, \\ (-1)^k (1/\sqrt{2})\varepsilon(A, N)\Lambda_{A^c}, & \text{if } \#(A) = 2k-1, \end{cases}$$

where $\Lambda_{A^c} = (\sqrt{2})^l a_{j_1} \dots a_{j_l} a_{1'} \dots a_{n'}$ by (2.1.4).

LEMMA 2.1. *Let (ξ_i, ξ_{jk}) be the coordinates on the open subset $\mathbf{V}_0$ defined in* **1.5** *and let $[z_A]$ be the homogeneous coordinates on $\mathbf{P}^{2^n-1}$ associated to the basis $\{{}^*\Lambda_A$, $A \in \mathbf{N}\}$ of Λ defined by* (2.4.1).

Then, on the open subset $\mathbf{V}_0$, the immersion $\iota : \mathbf{V} \to \mathbf{P}^{2^n-1}$ maps the point with coordinates (ξ_i, ξ_{jk}) to the point $[\xi_A]$.

The result coincides with the projective imbedding defined by Cartan [7].

Proof. Take an arbitrary point in $\mathbf{V}_0$ and let (ξ_i, ξ_{ik}) be the coordinates of the point. The point is written as $\exp \xi . o$ for some $\xi = (X_{\lambda\mu}) \in \mathfrak{n}$ where $\xi_i = X_{0i} = -X_{i'0}$ and $\xi_{ij} = X_{i'j} = -X_{j'i}$ **(1.5)**. By (2.2.1), $\tau(\exp \xi . o) = \exp \rho(\xi) . o^*$.

By definition, $\rho = r \circ l^{-1}$, and $\exp \rho(\xi) = \exp r(l^{-1}(\xi))$. By (2.1.2) and (2.1.2),

$$l^{-1}(\xi) = -\sum_i \xi_i a_{i'} - \sum_{ij} \xi_{ij} a_{i'} a_{j'}.$$

Since r is an associative algebra homomorphism of $\mathfrak{C}_+$, one can easily verify that $r(\exp a) = \exp r(a)$ for any $a \in \mathfrak{C}_2$. Thus,

$$(2.4.1) \qquad \tau(\exp \xi . o) = \exp(-(\textstyle\sum_i \xi_i a_{i'} + \sum_{ij} \xi_{ij} a_{i'} a_{j'})).o^*.$$

What left is to compute the left hand side of the above equality. For this, it is helpful to notice that the subalgebra generated by $a_{1'}, \dots, a_{n'}$ is isomorphic to the exterior algebra over the vector space spanned by these vectors. The exponential in the right hand side of the equality is a finite sum.

$$\exp l^{-1}(\xi) =$$
$$\sum_k (1/k!)(-1)^k\{k(\sum \xi_i a_{i'})(\sum \xi_{ij} a_{i'} a_{j'})^{k-1} + (\sum \xi_{ij} a_{i'} a_{j'})^k\} =$$
$$\sum_k \{(k!)2^{k-1} \sum \xi_{i_1 \dots i_{2k-1}} a_{i'_1} \dots a_{i'_{2k-1}}\} + \sum_k (k!)2^k \sum \xi_{i_1 \dots i_{2k}} a_{i'_1} \dots a_{i'_{2k}}.$$

For $A = \{i_1, \dots, i_\nu\}$, $1 \le i_1 < \dots < i_\nu \le n$, by (2.1.5-6),

$$(a_{i'_1} \dots a_{i'_\nu}).\Lambda_N = (1/\sqrt{2})^\nu \varepsilon(A, N)\Lambda_{A^c}.$$

Thus,

$$(\exp l^{-1}(\xi)).\ \Lambda_N = \sum_{A \in \mathbf{N}} C_A.\xi_A \Lambda_{A^c},$$

where the constant $C_A = (-1)^k (1/\sqrt{2})\varepsilon(A, N)$ if $\#(A) = 2k-1$, and $C_A = (-1)^k \varepsilon(A, N)$ if $\#(A) = 2k$.

By (2.4.1), $^*\Lambda_A = C_A \Lambda_{A^c}$, and

$$(\exp l^{-1}(\xi)).\ \Lambda_N = \sum_{A \in \mathbf{N}} C_A.\xi_A {}^*\Lambda_A.$$

Finally we have $z_A(\iota(\exp \xi.\ o)) = \xi_A$, completing the proof.

3. A class of surfaces in S^{2n}

3.1. In this section, we study local properties of an oriented surface M immersed in S^{2n} ($n \ge 2$). A complex structure is uniquely determined on M by the orientation and the first fundamental form. Without loss of generality, we may assume that a surface is sufficiently small and imbedded as a submanifold in S^{2n}.

Let $(\mathbf{V}, \pi, S^{2n})$ be the twistor bundle, and let $\Pi : G \to \mathbf{V}$ be the quotient map defined in **1.2**. Given an immersion of a surface M into S^{2n}, we call a map $\phi : M \to \mathbf{V}$ a lift of the immersion, if $\pi.\phi$ is the given immersion. If m is a G-valued function $(E_0, E_1, \dots, E_n, E_{1'}, \dots, E_{n'})$ on M such that E_0 is the immersion, then the map $\Pi.m$, which assigns to a point $p \in M$ the maximal isotropic subspace spanned by $E_1(p), \dots, E_n(p)$, is a lift of M. Conversely, any lift is locally obtained in this form. We say that a G-valued moving frame m determines a lift $\Pi.m$.

We put

(3.1.1) $$dE_\lambda = \sum_\mu E_\mu \Omega_{\mu\lambda},\ (\lambda, \mu = 0, 1, \dots, n, 1', \dots, n').$$

As the matrix $(\Omega_{\lambda\mu})$ is $\mathfrak{g}$-valued,

(3.1.2) $$\Omega_{00} = 0,\ \Omega_{0i} = -\Omega_{i'0},\ \Omega_{0i'} = -\Omega_{i0},\ \Omega_{i'j'} = -\Omega_{ji},$$
$$\Omega_{ij'} = -\Omega_{ji'},\ \Omega_{i'j} = -\Omega_{j'i},\ (i, j = 1, \dots, n),\text{ and}$$

$$\Omega_{\mu\lambda} = -\bar{\Omega}_{\mu\lambda}, \ (\lambda., \mu = 0,1,\ldots, n, 1',\ldots, n').$$

LEMMA 3.1. *Let $\phi : M \rightarrow \mathbf{V}$ be a lift of an oriented surface M immersed in S^{2n} $(n \geq 2)$. Then, the image of M under ϕ is tangent to the distribution $\mathbf{T}$ at each point if and only if, on a neighborhood of each point on M, ϕ is determined by a G-valued moving frame m satisfying the equalities*

$$(3.1.3) \qquad \Omega_{ij'} = \Omega_{i'j} = 0, \text{ for } 1 \leq i, j \leq n.$$

A lift ϕ is further anti-holomorphic if and only if, m satisfies both (3.1.3) *and*

$$(3.1.4) \qquad \Omega_{0i} = \bar{\Omega}_{0i'} \text{ is of bidegree } (0,1) \text{ for } 1 \leq i \leq n.$$

Proof. Take a point $p \in M$, and a tangent vector X at p. Let X' be the tangent vector at the identity of the group G corresponding to the matrix $((\Omega_{\lambda\mu})p(X))$. The equalities (3.1.1) means that the image $(m_*)p(X)$ of X under the differential of m is the image of X' under the differential of the left translation $L_{m(p)}$. (That is, the matrix $(\Omega_{\lambda\mu})$ of 1-forms is the reciprocal image of the Maurer-Cartan form on the group G under the differential of m.) Thus, $(\phi_*)p(X) = (\Pi_* m_*)p(X) = \{(L_{m(p)})_*\}_o\{(\Pi_*)e(X')\}$.

By (1.5.3), the (1,0)-component of $(\Pi_*)e(X')$ is

$$(3.1.5) \qquad \textstyle\sum_j \Omega_{0i}(X)(\partial/\partial\xi_i)_o + \sum_{i<j}\Omega_{ij'}(X)(\partial/\partial\xi_{ij})_o.$$

On account of Lemma 1.1, $(\phi_*)p(X)$ is tangent to $\mathbf{T}$ at $\phi(p)$ if and only if $(d\xi_{ij})_o((\Pi_*)e(X')) = 0$ $(1 \leq i \leq j \leq n)$, and hence if and only if $(\Omega_{ij'})p(X) = 0$ $(1 \leq i < j \leq n)$. We have seen that $\phi(M)$ is tangent to $\mathbf{T}$ if and only if (3.1.3) holds. Suppose that this is the case. Again, from the expression (3.1.5) of $(\Pi_*)e(X')$, it follows that ϕ is anti-holomorphic if and only if (3.1.4) is valid.

3.2. Let us impose an additional condition on a G-valued moving frame m on M that E_1 is a tangent vector field of bidegree (1,0) of M. Let $(\Omega_1, \Omega_{1'})$ be the dual basis of $(E_1, E_{1'})$. With respect to the complex structure on the surface, Ω_1 and $\Omega_{1'}$ are of bidegree (1,0) and (0,1) respectively.

As before, we put

$$dE_\lambda = \textstyle\sum_\mu E_\mu \Omega_{\mu\lambda}, \ (\lambda, \mu = 0,1,\ldots, n, 1',\ldots, n').$$

Since

$$(3.2.1) \qquad dE_0 = E_1\Omega_1 + E_{1'}\Omega_{1'},$$

(3.2.2) $\Omega_{10} = \Omega_1$, $\Omega_{1'0} = \Omega_{1'}$, and $\Omega_{\mu 0} = 0$ for $\mu = 2,\ldots,n, 2',\ldots n'$.

LEMMA 3.2. *Let $\phi : M \to \mathbf{V}$ be a lift of an oriented surface M immersed in S^{2n} $(n \geq 2)$. Then, ϕ is anti-holomorphic and the image $\phi(M)$ is tangent to $\mathbf{T}$ if and only if, on a neighborhood of each point on M, ϕ is determined by a G-valued moving frame m, such that E_0 is the immersion and E_1 is a tangent vector field of bidegree* (1,0) *of M and that the condition* (3.1.3) *is satisfied, namely, $\Omega_{ij'} = \Omega_{i'j} = 0$, for $1 \leq i, j \leq n$.*

Proof. The condition is sufficient. Indeed, for such a moving frame m, (3.1.3) and (3.2.2) are valid, and hence the second condition (3.1.4) in Lemma 3.1 is satisfied.

Next, we show that the condition is necessary. By Lemma 3.1, there exists locally a G-valued moving frame m satisfying (3.1.3) and (3.1.4). Let F_1 be a (1,0)-tangent vector field of unit length on M, and let $F_{1'}$ be the complex conjugate of F_1. Then, $dE_0 = F_1\Theta_1 + F_{1'}\Theta_{1'}$, where $(\Theta_1, \Theta_{1'})$ is the dual basis of $(F_1, F_{1'})$ and Θ_1 and $\Theta_{1'}$ are of bidegree (1,0) and (0,1) respectively.

On the other hand, $dE_0 = \sum E_i\Omega_{i0} + \sum E_{i'}\Omega_{i'0}$ by (3.1.1). From (3.1.2) and (3.1.4), it follows that the 1-form $\sum E_i\Omega_{i0}$ is of bidegree (1,0) and the 1-form $\sum E_{i'}\Omega_{i'0}$ is of bidegree (0,1). Therefore, $F_1\Theta_1 = \sum E_i\Omega_{i0}$. This implies that $F_1(p)$ belongs to the maximal isotropic subspace $\phi(p)$ spanned by $E_1(p), \ldots, E_n(p)$ at each point p. Thus, on a neighborhood of each point in M, we can choose a G-valued moving frame m' such that its second column is F_1 and that $\Pi.m' = \phi$. Hence, m' satisfies the condition (3.1.3).

3.3. Let M be an oriented surface immersed in S^{2n}. We denote by $T(M)$ the tangent bundle over M, and by $S(M)$ the restriction to M of the tangent bundle over S^{2n}. Obviously, $T(M)$ is a sub-bundle of $S(M)$. With respect to the complex structure on M, $T(M)$ is a holomorphic vector bundle.

Let $\mathbf{F}$ be the subset of the group G consisting of matrices whose 0-th column, regarded as a point in S^{2n}, belongs to M. The right action by the subgroup H, consisting of all matrices in G leaving e_0 fixed, leaves $\mathbf{F}$ invariant and $F/H = M$. Thus, $\mathbf{F}$ is the principal bundle of $S(M)$ with the structure group H.

We denote by $\Omega^* = (\Omega^*{}_{\lambda\mu})$ the restriction of the left invariant Maurer-Cartan form on G to $\mathbf{F}$, and by ω the $\mathbf{h}$-valued 1-form $(\omega_{\lambda\mu})$ given by $\omega_{\lambda\mu} = 0$ if either $\lambda = 0$ or $\mu = 0$, and $\omega_{\lambda\mu} = \Omega^*{}_{\lambda\mu}$ otherwise. The form ω defines a connection on the pricipal bundle $\mathbf{F}$.

Let E be a (smooth) section of the vector bundle $S(M)$ defined on M, and let

Y be a tangent vector field on M. The covariant differentiation $\nabla_Y(E)$ of E along Y with respect to the connection ω is given by the equality

$$(dE)(Y)p = a(p)p + \nabla_Y(E)_p, \tag{3.3.1}$$

where a is a scalar and $B(p, \nabla_Y(E)_p) = 0$ ([11], Chap. VII).

For later use, we prepare the following

LEMMA 3.3. *Suppose that F is an $S(M)$-valued section on M such that $F(p)$ is orthogonal to $T(M)_p$ at each point $p \in M$. Then, $\nabla F = 0$ if and only if F is a constant $\mathbf{R}^{2n+1}$-valued function.*

Proof. If F is costant, obviously, $\nabla F = 0$ by (3.3.1). Conversely, suppose that $\nabla F = 0$. Since $B(p, F(p)) = 0$ and $B(T(M)_p, F(p)) = 0$ by assumption, $a(p) = B(p, (dE)(Y)p) = 0$ for any p and Y. Hence, $dF = 0$ by (3.3.1) and F is constant.

3.4. Here, we regard a point in $\mathbf{V}$ as a complex structure J_p on the tangent space S_p to S^{2n} at p **(1.3)**. We recall that the subgroup K in H consists of matrices leaving the subspace V_0 spanned by $e_1, \ldots, e_n$ invariant. The tangent space to S^{2n} at e_0 is spanned by $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{1'}, \ldots, \varepsilon_{n'}$. The point V_0 in $\mathbf{V}$ is the complex structure J_0 defined by $J_0.\varepsilon_i = \varepsilon_{i'}$, $J_0.\varepsilon_{i'} = -\varepsilon_i$ $(1 \leq i \leq n)$. The group K is the subgroup of matrices in H which commute with J_0.

Let M be an oriented surface immersed in S^{2n}. A necessary and sufficient condition for the structure group H of the vector bundle $S(M)$ to reduce to its subgroup K is that each fibre S_p of the vector bundle $S(M)$ admits an orthogonal complex structure J_p so that $S(M)$ is a complex vector bundle. If that is the case, we denote by J the smooth section $p \mapsto J_p$. By replacing J_p with $-J_p$, if necessary, we can always assume that J_p belongs to the fibre $\mathbf{V}(p)$ over p **(1.3)**.

Suppose that $S(M)$ is a complex vector bundle with a complex vector bundle structure J. Then, the map $p \mapsto J_p \in \mathbf{V}$ is a lift ϕ of M. Conversely, to a lift ϕ of M into $\mathbf{V}$, there corresponds a complex vector bundle structure J on $S(M)$ such that $\phi(p) = J_p$.

A reduction of the structure group H of $S(M)$ to its subgroup K preserves the connection ω in **3.3**, if and only if the complex structure J is parallel, that is, $\nabla J = 0$ ([11] Chap. II, Prop. 7.4). If J is parallel, $S(M)$ is a holomorphic vector bundle over M by a theorem of Koszul-Malgrange [12].

LEMMA 3.4. *Let $\phi : M \to \mathbf{V}$ be a lift of an oriented surface M immersed in*

$S^{2n}(n \geq 2)$. *The image* $\phi(M)$ *is tangent to* $\mathbf{T}$ *if and only if the reduction of the structure group* H *of the real vector bundle* $S(M)$ *to the subgroup* K *associated to* ϕ *preserves the connection* ω. *The lift* ϕ *is further anti-holomorphic if and only if the tangent vector bundle* $T(M)$ *is a complex sub-bundle of the holomorphic vector bundle* $S(M)$ *associated to* ϕ.

Proof. Let ϕ be an arbitrary lift of an oriented sufrace M, and let J be the complex vector bundle structure on $S(M)$ associated to ϕ. We take a local G-valued moving frame $m = (E_0, E_1, \ldots, E_n, E_{1'}, \ldots, E_{n'})$ such that $\Pi.m = \phi$.

Applying (3.3.1) to each E_i, we have

$$\nabla E_i = dE_i - E_0\Omega_{0i} = \textstyle\sum_1^n E_j\Omega_{ji} + \sum_1^n E_{j'}\Omega_{j'i},\ (1 \leq i \leq n).$$

The complex vector bundle structure J being parallel with respect to the connection ω means that the bidegree of a section is preserved by the covariant differentiation. Thus, $\nabla J = 0$ if and only if

$$\nabla E_i = \textstyle\sum_1^n E_j\Omega_{ji}\ (1 \leq i \leq n), \text{ and } \Omega_{j'i} = 0, \text{ for } 1 \leq i, j \leq n,$$

or equivalently, M admits a lift whose image is tangent to $\mathbf{T}$, in virtue of Lemma 3.1.

Suppose that M admits a lift ϕ whose image is tangent to $\mathbf{T}$. Then, by Lemma 3.2, ϕ is anti-holomorphic if and only if we can choose a local G-valued moving frame m such that $\Pi.m = \phi$ and that E_1 is a tangent vector field of bidegree (1,0), which amounts to that the tangent bundle $T(M)$ is a complex sub-bundle of $S(M)$. We have finished the proof.

3.5. Let $(E_0, g_1, \ldots, g_n, \ldots, g_{1'}, \ldots, g_{n'})$ be an orthonormal moving frame on M such that E_0 is the position vector and that g_1 and $g_{1'}$ form an orthonormal frame of the tangent space to the surface, adapted to the orientation. We denote by $\{\omega_1, \omega_{1'}\}$ the dual basis of $\{g_1, g_{1'}\}$. The second fundamental form II is given by

$$\text{(3.5.1)} \qquad \textstyle\sum' g_\lambda(h_{\lambda 11}\omega_1\omega_1 + h_{\lambda 11'}\omega_1\omega_{1'} + h_{\lambda 1'1}\omega_{1'}\omega_1 + h_{\lambda 1'1'}\omega_{1'}\omega_{1'}),$$

where in the summation the index λ runs through $2, \ldots, n, 2', \ldots, n'$, and $h_{\lambda 1'1} = h_{\lambda 11'}$.

We put

$$E_1 = (1/\sqrt{2})(g_1 - \sqrt{-1}g_{1'}), \ldots, E_n = (1/\sqrt{2})(g_n - \sqrt{-1}g_{n'}),$$
$$E_{1'} = \bar{E}_1, \ldots, E_{n'} = \bar{E}_n.$$

Then, $m = (E_0, E_1, \ldots, E_n, E_{1'}, \ldots, E_{n'})$ is a G-valued moving frame on M.

Obviously, any G-valued moving frame $m = (E_0, E_1, \dots, E_n, E_{1'}, \dots, E_{n'})$ on M such that $E_0(p) = p$ and that E_1 is a tangent vector field of bidegree (1.0), is constructed in the above manner.

We use the same notations as in **3.1** and **3.2**. From $ddE_0 = 0$, it follows that

$$d\Omega_{\lambda 0} + \textstyle\sum_\mu \Omega_{\lambda\mu} \wedge \Omega_{\mu 0} = 0 \text{ for } \lambda = 1, 1', \text{ and}$$

$$\Omega_{\lambda 1} \wedge \Omega_{10} + \Omega_{\lambda 1'} \wedge \Omega_{1'0} = 0 \text{ for } \lambda = 2, \dots, n, 2', \dots, n'. \tag{3.5.2}$$

Put

$$\Omega_{\lambda 1} = H_{\lambda 11}\Omega_{10} + H_{\lambda 11'}\Omega_{1'0}, \ \Omega_{\lambda 1'} = H_{\lambda 1'1}\Omega_{10} + H_{\lambda 1'1'}\Omega_{1'0}.$$

By (3.5.2), $H_{\lambda 11'} = H_{\lambda 1'1}$, $(\lambda. = 2, \dots, n, 2', \dots, n')$.

In terms of the moving frame m, the second fundamental form II is written as

$$\textstyle\sum' E_\lambda (H_{\lambda 11}\Omega_1\Omega_1 + H_{\lambda 11'}\Omega_{1'}\Omega_1 + H_{\lambda 1'1}\Omega_1\Omega_{1'} + H_{\lambda 1'1'}\Omega_{1'}\Omega_{1'}).$$

Comparing this expression of the second fundamental form with (3.5.1), we have

$$H_{j11'} = (1/\sqrt{2})\{h_{j11} + h_{j1'1'}) + \sqrt{-1}(h_{j'11} + h_{j'1'1'})\}, \ H_{j'11'} = \bar{H}_{j11'}$$

for $j = 2, \dots, n$.

Thus, a surface is minimal, that is, the mean curvature vector

$$\textstyle\sum' (1/2)(h_{\lambda 11} + h_{\lambda 1'1'})g_\lambda$$

vanishes, if and only if $H_{\lambda 11'} = H_{\lambda 1'1} = 0$ for $\lambda = 2, \dots, n, 2', \dots, n'$, or equivalently if and only if, the 1-form $\Omega_{\lambda 1}$ is of bidegree (1,0) for $j = 2, \dots, n, 2', \dots, n'$ ([8]).

By Lemma 3.1, *if an oriented surface M immersed in S^{2n} $(n \geq 2)$ admits an anti-holomorphic lift ϕ whose image $\phi(M)$ is tangent to* $\mathbf{T}$, *then M is minimal* ([2]).

The quartic form Q defined by Bryant [2] is the (4,0)-component of the covariant symmetric 4-tensor $B(\mathrm{II}, \mathrm{II})$ and is written as

$$Q = B(\mathrm{II}(E_1, E_1), \mathrm{II}(E_1, E_1))\Omega_1\Omega_1\Omega_1\Omega_1.$$

He shows that if a surface is oriented and minimal, Q is a holomorphic tensor field with respect to the complex structure on the surface. He calls a minimal surface in S^{2n} with vanishing Q superminimal [2]. From the above expression, it is clear that the superminimality means that the vector $\mathrm{II}(E_1, E_1)$ is isotropic with respect to B.

By definition,

$$\mathrm{II}(E_1, E_1) = \textstyle\sum' E_\lambda \Omega_{\lambda 1}(E_1).$$

Thus, *if M admits an anti-holomorphic lift ϕ whose image $\phi(M)$ is tangent to* $\mathbf{T}$, *then M is not only minimal but also superminimal in virtue of* Lemma 3.1 ([2]).

LEMMA 3.5 *Suppose that a minimal surface on S^{2n} is contained a hyperplane of dimension $2n-1$ in $\mathbf{R}^{2n+1}$. Then, the hyperplane must contain the origin of $\mathbf{R}^{2n+1}$*

Proof. Let $\{e_\lambda\}$ be the basis of $\mathbf{C}^{2n+1}$ defined in **1.1**. In virtue of homogeneity of the Riemann metric on S^{2n}, it suffices to prove the lemma in the case where the hyperplane is perpendicular to the vectors e_n and $e_{n'}$. As before, we choose a local moving frame m, in which E_1 and $E_{1'}$ are tangent to the surface and hence orthogonal to e_n and $e_{n'}$. Since the surface is minimal. Ω_j and $\Omega_{j'1}$ are of bidegree (1,0) for $j=2,\dots,n$. Thus,

$$dE_1(E_{1'}) = -E_0 + E_1\Omega_{11}(E_{1'}).$$

Since E_1 is orthogonal to e_n and $e_{n'}$, the n-th and the n'-th components of E_1, as well as, of dE_1 are zero. From the above equality, it follows that the n-th and the n'-th components of the position vector E_0 are zero. Thus, the surface is lying on the hyperplane $x_n = x_{n'} = 0$.

3.6. DEFINITION. A surface immersed in S^{2n} is said to be in general position if no $(2n-1)$-plane contains the surface.

LEMMA 3.6. *Suppose that a surface M immersed in S^{2n} admits an anti-holomorphic lift ϕ of the immersion such that the image under ϕ is tangent to the distribution* $\mathbf{T}$.

(1) *The image of M in S^{2n} is not in general position if and only if there is a non-zero isotropic vector contained in all the maximal isotropic subspaces $\phi(p)$, $p \in M$.*

(2) *If the surface M is not in general position in S^{2n}, then $\iota(\phi(M))$ is also not in general position in $\mathbf{P}^{2^n-1}$.*

Proof. (1) Suppose that M lies in a $(2n-1)$-plane P. Since a surface satisfying the assumption is minimal (**3.5**), the plane P passes the origin of $\mathbf{R}^{2n+1}$ by Lemma 3.5. Let U be the 2-plane perpendicular to P. Clealy, at each point $p \in M$, $U \subset S_p$. Thus, for any $u \in U$, the section $p \mapsto u \in S_p$ is parallel by Lemma 3.3.

Let F be an $S(M)$-valued section on M such that $F(p)$ is orthogonal to $T_p(M)$ at each point $p \in M$. Then, F is parallel if and only if F is a constant $\mathbf{R}^{2n+1}$-valued function by Lemma 3.3. Thus, the vector space Γ of all parallel $S(M)$-valued sections on M orthogonal to $T(M)$ may be regarded as a subspace in $\mathbf{R}^{2n+1}$.

Under the assumption, the complex vector bundle structure J on $S(M)$

associated to the lift ϕ is parallel with respect to the connection ω, and $T(M)$ is a complex sub-bundle of $S(M)$ by Lemma 3.4. Hence, J commutes with the covariant differentiation ∇ and leaves $T(M)$ invariant. As a consequence, the vector space Γ of sections is invariant by J. If a vector $v \in \mathbf{R}^{2n+1}$ belongs to Γ, $p \mapsto J_p(v)$ is again a constant vector belonging to Γ, that is, the restriction of J_p to Γ is a complex structure J' independent on p.

The subspace U is contained in Γ, but may not be invariant by the complex structure J'. Choose a subspace U' of real dimension 2 in Γ which is invariant under J'. Since $U' \subset S_p$ at every $p \in M$, M is contained in the $(2n-1)$-plane through the origin, perpendicular to U'.

The maximal isotropy subspace $\phi(p)$ is the (1,0)-component of $(S_p)_c$, and contains an isotropic non-zero vector $(u - J'.u)$, $u \in U'$, which is common for all points p in M.

The converse is obvious. Indeed, if a non-zero isotropic vector v is contained in $\phi(p)$ for all $p \in M$, v and its complex conjugate are orthogonal to p. Therefore, M is contained in the hyperplane perpendicular to the real and imaginary components of v, which are linealy independent.

(2) If M is not in general position, there is an isotropic vector of unit length contained in all $\phi(p)$, $p \in M$ by the above result (1). By homogeneity, we may assume that this isotropic vector is e_n.

Consider the subset $\mathbf{V}'$ of $\mathbf{V}$ consisting of all maximal isotropic subspaces containing e_n. From **1.5**, it follows easily that in the open subset $\mathbf{V}_0$, $\mathbf{V}_0 \cap \mathbf{V}'$ is defined by the equations $\xi_n = 0$, $\xi_{in} = 0$ $(i = 1, \ldots, n-1)$. Therefore, the image of $\mathbf{V}'$ under the imbedding ι is contained in the linear submanifold in $\mathbf{P}^{2^n-1}$ defined by the homogeneous linear equations

$$z_{A^c} = 0, \text{ where } A = \{i_1, \ldots, i_{k-1}, n\} \in \mathbf{N},$$

and is not in general position.

Since $\phi(M) \subset \mathbf{V}'$, $\iota(\phi(M))$ is not in general position. We have finished the proof of the statement (2).

4. Conformal immersions

4.1. THEOREM. *Given a compact Riemann surface, there always exists conformal and minimal immersion into* S^{2n} $(n \geq 2)$, *whose image is in general position, i.e. not contained in any* $2n-1$ *dimensional hyperplane.*

In the rest of the paper, we prove the theorem. We begin with the following remark: Suppose that a Riemann surface M admits an anti-holomorphic immersion φ into $\mathbf{V}$ and that the image $\varphi(M)$ is tangent to the distribution $\mathbf{T}$. Then, $\pi.\varphi: M \to S^{2n}$ is an immersion. Obviously, φ is a lift of the immersion $\pi.\varphi$, and hence the immersion $\pi.\varphi$ is minimal by **3.5**. Moreover, the new complex structure on M determined by the orientation of M and the first fundamental form induced by $\pi.\varphi$ coincides with the original one.

Indeed, since φ is anti-holomorphic, φ is conformal with respect to any hermitian metric on M and the G-invariant hermitian metric on $\mathbf{V}$ indroduced in **1.7**. As mentioned in **1.7**, the differential of π is isometric on $\mathbf{T}$ at each point. Hence, the immersion $\pi.\varphi$ is conformal and the conclusion follows.

Thus, in order to construct a conformal and minimal immersion of a given compact Riemann surface M into S^{2n}, it suffices to find an anti-holomorphic immersion of M into the complex manifold $\mathbf{V}$ such that the image of M is tangent to the distribution $\mathbf{T}$ at each point ([2]). In what follows, we will work on the Riemann surface $\bar{M}$, the real manifold M endowed with its conjugate complex structure, and find a holomorphic immersion of $\bar{M}$ into $\mathbf{V}$ tangent to the distribution $\mathbf{T}$.

4.2. Suppose that a set of $n(n+1)/2$ meromorphic functions f_i $(1 \leq i \leq n)$, $f_{jk}(1 \leq j,\ k \leq n)$ on a compact Riemann surface $\bar{M}$ satisfies equalities

$$df_{ij} - (1/2)(f_i\, dfj - fj\, df_i) = 0 \ (1 \leq i < j \leq n). \tag{4.2.1}$$

We define $f_{i_1 \dots i_k}$ in terms of the f_i's and f_{jk}'s in the same way as in **2.3**, and denote by φ the holomorphic map of $\bar{M}$ into $\mathbf{P}^{2^n-1}$ given by

$$p \mapsto [1, \dots, f_i(p), \dots, f_{ik}(p), \dots, f_{i_1} \dots {}_{i_k}(p), \dots].$$

If f_i $(1 \leq i \leq n)$, f_{jk} $(1 \leq j,\ k \leq n)$ are all holomorphic at a point $p \in \bar{M}$, the point $\varphi(p)$ in the complex projective space $\mathbf{P}^{2^n-1}$ belongs to the submanifold $\mathbf{V}$ by Lemma 2.1, and the image of the differential $(\varphi_*)_p$ is contained in $\mathbf{T}_{\varphi(p)}$ by Lemma 1.1. Since the set of points where these $n(n+1)/2$ functions are all holomorphic is dense in $\bar{M}$, $\varphi(\bar{M})$ is contained in $\mathbf{V}$ and tangent to the distribution $\mathbf{T}$.

Next, we require that

(4.2.2) φ is an immersion.

On account of Lemma 3.6. (2), in order that the immersed surface $\pi.\varphi(M)$ in S^{2n} is in general position, it is sufficient that the image $\varphi(\bar{M})$ is in general position in $\mathbf{P}^{2^n-1}$, namely that the 2^n functions $1, \dots, f_i, \dots, f_{ij}, \dots, f_{i_1 \dots i_k}, \dots$ are linearly independent over $\mathbf{C}$. For this purpose, we impose the following condition:

(4.2.3) There is a point p_0 on $\bar{M}$ where $\mathrm{ord}_{p_0}(f_{i_1\dots i_k}) = \sum \mathrm{ord}_{p_0}(f_{i_a})$ and the orders of these 2^n functions $1,\dots, f_i,\dots, f_{ij},\dots, f_{i_1\dots i_k},\dots$ at p_0 are all distinct.

Thus, the proof is reduced to find a set of $n(n+1)/2$ meromorphic function f_i $(1 \le i \le n)$, $f_{jk}(1 \le j < k \le n)$ on $\bar{M}$ satisfying the above three conditions (4.2.1), (4.2.2) and (4.2.3). This will be done by induction on n (≥ 2). (It is obvious that we have to exclude the case where $n = 1$.)

4.3. Before we proceed further, we formulate some criteria for the differential of a holomorphic curve in the complex projective space not to vanish at a point, which will be used frequently.

Take arbitrary meromorphic functions $z_1,\dots, z_m$ on a Riemann surface and denote by φ the holomorphic map into the projective space $\mathbf{P}^m$ which assigns to a point p the point in $\mathbf{P}^m$ with homogeneous coordinates $[1, z_1(p),\dots, z_m(p)]$. We assume that at least one of $z_1,\dots, z_m$ is non-constant so that the map is not trivial. We are concerned with the differential $(\varphi_*)_p$ at a point p.

Let ν be the minimum of the orders of $1 = z_0, z_1,\dots, z_m$ at p. Let ζ be a local holomorphic coordinate centered at p, $\zeta(p) = 0$. Put $w_i = z_i\zeta^{-\nu}$, $i = 0,\dots, m$ then $[w_0,\dots, w_m]$ defines φ in a neighborhood of p. We denote by $\varphi^\wedge(p)$ the point $(w_0(p),\dots, w_m(p))$ in $\mathbf{C}^{m+1}$.

The image of $(d/d\zeta)_p$ under the differential $(\varphi_*)p$ is the tangent vector to $\mathbf{P}^m$ given by the projection of the vector $\varphi^{\wedge\prime}(p) = (w_0'(p),\dots, w_m'(p))$ in $\mathbf{C}^{m+1}$. Thus, $(\varphi_*)p = 0$ if and only if

$$(w_0'(p),\dots, w_m'(p)) = \lambda(w_0(p),\dots, w_m(p)) \text{ for some } \lambda \in \mathbf{C}.$$

If one of $z_0, z_1,\dots, z_m$ is of order $\nu + 1$ at p, no such λ exists and $(\varphi_*)_p$ is injective.

4.4. When $n = 2$, Bryant shows the existence of a holomorphic map $\varphi: \bar{M} \to \mathbf{P}^3$ whose image is in general position. His holomorphic map is not only immersion but also imbedding ([2], Theorem G). Nevertheless, as the first step of induction, we shall construct a holomorphic immersion φ of $\bar{M}$ into $\mathbf{P}^3$ subject to (4.2.1-3).

Take a finite number of distinct points $p_1,\dots, p_k$ on an arbitrary Riemann surface M, and assign to each point a non-zero integer μ_i. Then, there exists a meromorphic function f on M whose order at p_i is μ_i $(1 \le i \le k)$. To see this, write $\mu_i = \nu_i - \nu_i'$ with integers $\nu_i, \nu_i' \le -2$. In virtue of the existence theorem of abelian differentials on a Riemann surface ([10] II. 5.), we can choose meromorphic 1-forms ω_i and ω_i' holomorphic everywhere except p_i and of the orders ν_i

and ν_i' at p_i respectively, for each i. The meromorphic function f determined by $f(\sum \omega' i) = \sum \omega_i$ serves the purpose.

We start with a meromorphic function f_1 on a compact Riemann surface $\bar{M}$, having a zero of order 2 at a point p_0. Let $p_0, \ldots, p_k$ be the distinct zeros of the differential df_1, and let $\nu_0(=2), \nu_2, \ldots, \nu_k$ be the orders of df_1 at these zeros. Let $q_1, \ldots, q_m$ be the distinct poles of f_1, and let $\mu_1, \ldots, \mu_m$ be the orders of f_1 at these poles. We choose a meromorphic function F on $\bar{M}$ such that the orders of F at p_0, $\ldots, p_k$ are $\nu_0, \ldots, \nu_k$ and that the orders at $q_1, \ldots, q_m$ are $\mu_1 - 1, \ldots, \mu_m - 1$.

We put $f_2 = dF/df_1$. Then, $\mathrm{ord}_{p_i}(f_2) = -1$ for $i = 0, \ldots, k$, and $\mathrm{ord}_{q_j}(f_2) = -1$ for $j = 1, \ldots, m$. If we put $f_{12} = F + (1/2)f_1 f_2$, the relation (4.2.1) is satisfied.

At the point p_0, $\mathrm{ord}_{p_0}(f_1) = 2$, $\mathrm{ord}_{p_0}(F) = 1$, $\mathrm{ord}_{p_0}(f_2) = -1$. In terms of a local holomorphic coordinate ζ such that $\zeta(p_0) = 0$, $f_1 = a_2\zeta^2 + \ldots$ $(a_2 \neq 0)$, $f_2 = b_{-1}\zeta^{-1} + \cdots (b_{-1} \neq 0)$, $F = 2a_2 b_{-1}\zeta + \cdots$ and $f_{12} = (3/2)a_2 b_{-1}\zeta + \cdots$. Thus $\mathrm{ord}_{p_0}(f_{12}) = 1$. We have shown that p_0 is the point satisfying (4.2.3).

The next step is to show that the map φ defined by (4.2.1) is regular at each point p. We divide the proof into three cases, depending on the order of df_1 at p. First, suppose that (i) $\mathrm{ord}_p(df_1) = 0$. If further $\mathrm{ord}_p(f_2) \geq 0$, then $\mathrm{ord}_p(f_{12}) \geq 0$. Therefore, $(df_1)_p \neq 0$ implies that $(\varphi_*)_p$ does not vanish.

Suppose that $\mathrm{ord}_p(df_1) = 0$ and $\mathrm{ord}_p(f_2) < 0$. Then $\mathrm{ord}_p(f_2) \geq -1$, as $\mathrm{ord}_p(dF) = \mathrm{ord}_p(f_2)$. In terms of a local holomorphic coordinate ζ vanishing at p,

$$
\begin{aligned}
&f_1 = a_0 + a_1\zeta + \cdots, \ (a_1 \neq 0),\\
&f_2 = b_{-\nu}\zeta^{-\nu} + b_{-\nu+1}\zeta^{-\nu+1} + \cdots, \ (b_{-\nu} \neq 0, \ \nu > 1),\\
&F = \{1/(-\nu+1)\}a_1 b_{-\nu}\zeta^{-\nu+1} + \cdots \text{ and}\\
&f_{12} = -(1/2)a_0 b_{-\nu}\zeta^{-\nu}\\
&\qquad + (1/2)[\{(1+\nu)/(1-\nu)\}a_1 b_{-\nu} - a_0 b_{-\nu+1}]\zeta^{-\nu+1} + \cdots.
\end{aligned}
$$

From these, one concludes that ϕ_* does not vanish at p. Indeed, if $a_0 = 0$, the minimum of the orders of $1, f_1, f_2$, and f_{12} at the point is $-\nu$ and the order of f_{12} is $-\nu + 1$ and hence ϕ_* does not vanish at p by (**4.2**). If $a_0 \neq 0$,

$$
\begin{aligned}
&\varphi^\wedge(p) = (0,0, b_{-\nu}, (1/2)a_0 b_{-\nu}),\\
&\varphi^{\wedge\prime}(p) = (0,0, b_{-\nu+1}, (1/2)[\{(1+\nu)/(1-\nu)\}a_1 b_{-\nu} - a_0 b_{-\nu+1}]).
\end{aligned}
$$

The latter is not a scalar multiple of the former.

Suppose that (ii) $\mathrm{ord}_p(df_1)_p > 0$. The point p is not of $p_0, \ldots, p_k$. By our choice, $\mathrm{ord}_p(F) \geq \mathrm{ord}_p(df_1) = \nu_i > 0$. As is mentioned above, $\mathrm{ord}_p(f_2) = -1$. We put $\nu_i = \nu$. In terms of a local holomorphic coordinate ζ such that $\zeta(p) = 0$,

$$f_1 = a_0 + a_{\nu+1}\zeta^{\nu+1} + \cdots, \ (\nu > 0, \ a_{\nu+1} \neq 0),$$
$$f_2 = b_{-1}\zeta^{-1} + \cdots \ (b_{-1} \neq 0),$$
$$F = \{(\nu+1)/\nu\}a_{\nu+1}b_{-1}\zeta^{\nu} +$$
$$f_{12} = -(1/2)a_0 b_{-1}\zeta^{-1} + \cdots, \text{ if } a_0 \neq 0. \text{ and}$$
$$f_{12} = a_{\nu+1}b_{-1}\{(2+\nu)/2\nu\}\zeta^{\nu} + \cdots, \text{ if } a_0 = 0.$$

In both cases, the minimum of the orders of 1, f_1, f_2, and f_{12} at the point is -1 and the order of 1 is 0 at the point. By **4.3.** ϕ_* does not vanish at p.

Finally, suppose that (iii) $\mathrm{ord}_p(df_1) < 0$. Obviously, $\mathrm{ord}_p(f_1) < 0$, and the point p is one of $q_1, \ldots, q_m$. By our choice of F, $\mathrm{ord}_p(F) = \mathrm{ord}_p(f_1) - 1 < 0$ and $\mathrm{ord}_p(f_2) = -1$.

$$f_1 = a_\nu\zeta^{\nu} + \cdots, \ (a_\nu \neq 0, \ \nu < 0),$$
$$f_2 = b_{-1}\zeta^{-\nu} + \cdots, \ (b_{-1} \neq 0),$$
$$F = \{1/(\nu-1)\}a_\nu b_{-1}\zeta^{\nu-1} +$$
$$f_{12} = a_\nu b_{-1}\{(-\nu+3)/2(\nu-1)\}\zeta^{\nu-1} + \cdots.$$

Clearly, $\nu - 1$ is the minimum value of the orders of 1, f_1, f_2 and f_{12} at p and $\mathrm{ord}_p(f_1) \geq \nu$. Again by **4.3**, we conclude that φ_* does not vanish at p. We have completed the case where $n = 2$.

4.5. The induction hypothesis is that we have a set of $(n-1)n/2$ meromorphic functions f_i, f_{jk} $(1 \leq i, j < k \leq n-1)$ on $\bar{M}$ satisfying (4.2.1-3). Let p_0 be the point asserted in (4.2.3). The first task is to find a meromorphic function f_n suth that the differential form $f_n df_i$ is exact for every $i = 1, \ldots, n-1$.

From $\bar{M}$, we exclude the point p_0 and all zeros and poles of these functions and their differentials and obtain an open dense subset. In this open dense subset, we choose a finite number of distinct points $p_1, \ldots, p_\rho$.

Let $\mathfrak{D}$ be a divisor on $\bar{M}$ given by $p_0^{-\nu}p_1^{\nu} \ldots . p_1^{\nu}$ with a positive integer ν. The integers ρ and ν will be determined later. Let $L(\mathfrak{D}^{-1})$ be the vector space spanned by meromorphic functions f on $\bar{M}$ such that $\mathrm{div}\,(f) \geq \mathfrak{D}^{-1}$. If f is not identcally zero and belongs to $L(\mathfrak{D}^{-1})$, f has a zero of order at least ν at p_0, and all poles of f are in the subset $\{p_1, \ldots, p_\rho\}$, and their orders are at least $-\nu$.

Let $\{p_{\rho+1}, \ldots, p_{\rho+\sigma}\}$ be the subset of points in $\bar{M}$ each of which is a pole of one of the functions f_i $(1 \leq i \leq n-1)$. By choice, the points $p_1, \ldots, p_\rho, p_{\rho+1}, \ldots, p_{\rho+\sigma}$ are all distinct. We take $\rho + \sigma$ small circles γ_k centered at p_k $(k = 1, \ldots, \rho + \sigma)$ so that the disks encircled by them are mutually disjoint.

We denote by g the genus of the Riemann surface $\bar{M}$. Let $\{\alpha_l, \beta_l\,; l = 1, \ldots,$

$g\}$ be a set of loops forming a system of generators for the fundamental group of $\bar{M}$. We choose these loops not intersecting with any circle γ_k.

To each $f \in L(\mathfrak{D}^{-1})$, we assign

$$\int_{\alpha_l} f df_i, \int_{\beta_l} f df_i, \int_{\gamma_k} f df_i,$$

for each $i\ (1 \leq i \leq n-1)$, $l (1 \leq l \leq g)$ and $k\ (1 \leq k \leq \rho + \sigma)$, and obtain

$$(\rho + \sigma + 2g)(n-1)$$

linear functions on $L(\mathfrak{D}^{-1})$.

The Riemann-Roch theorem implies that

$$\dim L(\mathfrak{D}^{-1}) \geq \deg \mathfrak{D} - g + 1$$

([10] III. 4). In our case, $\deg \mathfrak{D} = \nu(\rho - 1)$. We will choose ν and ρ sufficiently large so that $\dim L(\mathfrak{D}^{-1})$ is larger than the number of the linear functions above, and consequently there exists a non-constant meromorphic function f_n annihilated by all these linear functions.

The inequality in question is $\nu(\rho - 1) - g + 1 > (\rho + \sigma + 2g)(n-1)$, or equivalently, $\nu > (n-1) + \{(\sigma + 1)(n-1) + g(2n-1) - 1\}(\rho - 1)^{-1}$. It suffices to choose $\nu > n$ and $\rho > (\sigma + 1)(n-1) + g(2n-1)$.

If this is done, $f_n df_i = dF_i$ with a meromorphic function F_i on $\bar{M}$ for each $i\ (1 \leq i \leq n-1)$ where F_i is unique up to an additional constant. Put $f_{in} = -F_i + (1/2) f_i f_n$ for each $i\ (1 \leq i \leq n-1)$. Then, the relations in (4.2.1) are valid. Next, we will choose ν so large that the condition (4.2.3) is satisfied.

4.6. By induction hypothesis, at the point p_0, the orders of the meromorphic functions $f_{i_1 \ldots i_k}$, $1 \leq i_1 < \ldots < i_k \leq n-1$, are all distinct. First, we choose ν larger than the absolute value of the order at p_0 of any one of these functions. Put $\nu' = \mathrm{ord}_p(f_n)$, which is larger than or equal to ν.

In terms of a local holomorphic coordinate ζ vanishing at p_0,

$$f_i = c_i \zeta^{\nu_i} + \cdots (\nu_i = \mathrm{ord}_{p_0}(f_i) \neq 0,\ c_i \neq 0,\ 1 \leq i \leq n-1),$$
$$f_n = c_n \zeta^{\nu'} + \cdots (c_n \neq 0).$$

Hence, the order of dF_i at p_0 is $\nu_i + \nu' - 1$. If the power series expansion of F_i at p_0 has the non-zero constant term, we subtract the constant from F_i and use the result as F_i without affecting our argument. Then,

$$F_i = \{\nu_i (\nu_i + \nu')^{-1}\} c_i c_n \zeta^{\nu_i + \nu'}$$

$$f_{in} = \{(\nu_i - \nu')/2(\nu_i + \nu')\}c_i c_n \zeta^{\nu_i+\nu'} + \cdots,$$

and $\mathrm{ord}_{p_0}(f_{in}) = \nu_i + \nu'$.

By induction hypothesis, if $1 \leq i_1 < \cdots < i_k \leq n-1$, the order of $f_{i_1 \ldots i_k}$ at p_0 is $\sum \nu_{i_a}$. We denote by $c_{i_1 \ldots i_k}$ the leading coefficient in its power series expansion in ζ. The non-zero constants $c_{i_1 \ldots i_k} (1 \leq i_1 < \ldots < i_k \leq n-1)$ are subject to the relations (2.3.1) and (2.3.2).

Now, we examine the order of $f_{i_1 \ldots i_k n}$ at p_0. From the definition of $f_{i_1 \ldots i_k n}$ given in **2.3**, it is obvious that

$$\mathrm{ord}_{p_0}(f_{i_1 \ldots i_k n}) \geq \nu' + \sum \nu_{i_a}.$$

Using the formulas (2.3.1) and (2.3.2), we determine the coefficient $c_{i_1 \ldots i_k n}$ of the $(\nu' + \sum \nu_{i_a})$-th power of ζ in the power series expansion of $f_{i_1 \ldots i_k n}$ at p_0. If k is odd,

$$c_{i_1 \ldots i_k n} = \sum_{b=1}^{k} B_{i_b}(\nu' + \nu_{i_b})^{-1} - (1/2) c_n c_{i_1 \ldots i_k}.$$

with some constants $B_{i_b} \neq 0$ $(1 \leq b \leq k)$.

If k is even,

$$c_{i_1 \ldots i_k n} = \sum_{b<c} B_{i_b i_c}\{\nu_{i_b}/(\nu' + \nu_{i_b}) - \nu_{i_c}/(\nu' + \nu_{i_c})\} + c_n c_{i_1 \ldots i_k},$$

with $B_{i_b i_c} \neq 0$ $(1 \leq b < c \leq k)$.

In both cases, the constant term $-(1/2) c_n c_{i_1 \ldots i_k}$ if k is odd, $c_n c_{i_1 \ldots i_k}$ if k is even, is not zero by induction hypothesis and by the inequality $c_n \neq 0$. Therefore, we can choose a large positive integer ν so that if $\nu' > \nu$, the coefficient $c_{i_1 \ldots i_k n}$ does not vanish for every $f_{i_1 \ldots i_k n}$.

We have seen that

$$\mathrm{ord}_{p_0}(f_{i_1 \ldots i_k n}) = \nu' + \sum \nu_{i_a} \text{ for } \{i_1, \ldots, i_k, n\} \in \mathbf{N},$$

and hence the condition (4.2.3) is verified at the point p_0.

4.7. We shall show that the holomorphic map φ defined by (4.2.2) is regular at each point. Take an arbitrary point p in $\bar{M}$.

We first take up the case where $\mathrm{ord}_p(f_n) \geq 0$. If $\mathrm{ord}_p(F_i) \neq 0$, form the equality $f_n df_i = dF_i$, $\mathrm{ord}_p(F_i) \geq \mathrm{ord}_p(f_i)$, and hence $\mathrm{ord}_p(f_{in}) \geq \mathrm{ord}_p(f_i)$. If $\mathrm{ord}_p(F_i) = 0$, either $\mathrm{ord}_p(f_{in}) \geq 0$ and $\mathrm{ord}_p(f_i) \geq 0$, or $0 > \mathrm{ord}_p(f_{in}) \geq \mathrm{ord}_p(f_i)$. Therefore, the minimum of the orders of the functions at p does not decrease by adding the $f_{i_1 \ldots i_k n}$'s to the old family $\{f_{i_1 \ldots i_k}\}$, which contains the constant function 1. The induction hypothesis immediately yields that φ is regular at p.

Suppose that $\mathrm{ord}_p(f_n) < 0$. Then, p is one of $p_1, \ldots, p_\rho$ and $-\nu \le \mathrm{ord}_p(f_n) < 0$. Moreover, by the choice of the point p, $\mathrm{ord}_p(f_i) = \mathrm{ord}_p(df_i) = 0$ $(1 \le i \le n-1)$, and f_{ij} is holomorphic $(1 \le i < j \le n-1)$ at the point. Since $f_n df_i = dF_i$. $\mathrm{ord}_p(dF_i) = \mathrm{ord}_p(f_n) < -1$, and $\mathrm{ord}_p(F_i) \le -1$. In terms of a local holomorphic coordinate ζ vanishing at p,

$$f_i = a_{i0} + a_{i1}\zeta + \cdots (a_{i0}, a_{i1} \neq 0),$$
$$f_n = b_{-\mu}\zeta^{-\mu} + b_{-\mu+1}\zeta^{-\mu+1} + \cdots (b_{-\mu} \neq 0, \mu \ge 2) \text{ and}$$
$$f_{in} = -(1/2)(a_{i0}b_{-\mu}\zeta^{-\mu} + [a_{i1}b_{-\mu}\{(\mu+1)/(\mu-1)\} + a_{i0}b_{-\mu+1}]\zeta^{-\mu+1} + \cdots).$$

It follows that $-\mu$ is the minimum of the orders of the functions $f_{i_1\ldots i_k}$, $\{i_1, \ldots, i_k\} \in \mathbf{N}$, at p. As in **4.3**, we multiply each function by ζ^μ and form $\varphi^\wedge(p)$ and $\varphi^{\wedge\prime}(p)$. We look at the $\{n\}$-th and the $\{i, n\}$-th coordinates of these two vectors in $\mathbf{C}^{2^n}$.

$$\varphi^\wedge(p) = [0, \ldots, b_{-\mu}, \ldots, -(1/2)a_{i0}b_{-\mu}, \ldots], \text{ and}$$
$$\varphi^{\wedge\prime}(p) = [0, \ldots, b_{-\mu+1}, \ldots, -(1/2)(a_{i1}b_{-\mu}\{(\mu+1)/(\mu-1)\} + a_{i0}b_{-\mu+1}), \ldots].$$

Suppose that $\varphi_*(p) = 0$. Then, $\lambda\varphi^\wedge(p) = \varphi^{\wedge\prime}(p)$ for some $\lambda \in \mathbf{C}$. Thus,

$$\lambda b_{-\mu} = b_{-\mu+1}, \text{ and } \lambda(a_{i0}b_{-\mu}) = a_{i1}b_{-\mu}\{(\mu+1)/(\mu-1)\} + a_{i0}b_{-\mu+1}$$
$$= a_{i1}b_{-\mu}\{(\mu+1)/(\mu-1)\} + \lambda(a_{i0}b_{-\mu}),$$

yielding that $a_{i1}b_{-\mu}\{(\mu+1)/(\mu-1)\} = 0$. This is a contradiction. We have shown that φ satisfies (4.2.2), completing the proof of the theorem.

REFERENCES

[1] R. L. Bryant, Submanifolds and special structures in the octonians, J. Diff. Geom., **17** (1982), 185–232.

[2] ——, Conformal and minimal immersions of compact surfaces into the 4-sphere, J. Diff. Geom., **17** (1982), 455–473.

[3] ——, Lie groups and twistor spaces, Duke Math. J., **52** (1985), 223–261.

[4] F. E. Burstall and J. H. Rawnsley, Twistor theory for Riemannian symmetric spaces, Lect. Notes in Math., **1424** (1990) Springer-Verlag, Berlin, Heidelberg, New York, Tokyo.

[5] E. Calabi, Minimal immersions of surfaces in Euclidean spheres, J. Diff. Geom., **1** (1967), 111–125.

[6] ——, Quelques applications de l'analyse complexe aux surfaces d'aire minima, Topics in complex manifolds, University of Montreal, (1968), 59–81.

[7] É. Cartan, Leçons sur la théorie des spineurs I, II, Hermann, (1938) Paris, (English translation, The theory of spinors, Dover, New York).

[8] S.-S. Chern and J. G. Wolfson, Minimal surfaces by moving frames, Amer. J. Math.,

15 (1983) 59–83.
[9] C. Chevalley, Theory of Lie groups I, (1946) Princeton. Princeton University Press.
[10] H. M. Farkas and I. Kra, Riemann surfaces, Springer, New York (1980).
[11] S. Kobayashi and K. Nomizu, Foundations of differential geometry I, II, John Wiley & Sons, New York (1969).
[12] J.-L. Koszul, and B. Malgrange, Sur certaines structures fibrées complexes, Arch. Math., **9** (1958), 102–109.
[13] H. B. Lawson, Jr. and M.-L. Michelson, Spin Geometry, Princeton Univ. Press, Princeton (1989).

Washington University
(jihano@artsci.wustl.edu)