# Lie Algebra Representation Theory -SU(3)-Representations in Physics 

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## Contents

1 Introduction ..... 1
1.1 Physical Motivation ..... 1
1.2 Noether Theorem ..... 1
2 SU(3)-Representation Theory ..... 2
2.1 From Group to Algebra ..... 2
2.2 Properties of Gell-Mann Basis ..... 4
2.3 Roots and Weights ..... 6
2.3.1 Roots ..... 6
2.3.2 Weights ..... 8
2.3.3 (2,0)-Representation ..... 12
2.3.4 (1,1)-Representation ..... 13
2.3.5 (3,0)-Representation ..... 15
3 Flavor SU(3) in Particle Physics ..... 15
3.1 From Atoms to Quarks ..... 15
3.2 The Eightfold Way ..... 16
3.3 Quarks and the Flavor $\mathrm{SU}(3)$ ..... 17

## 1 Introduction

The aim of this paper is to draw the connections between elementary particle physics and representation theory of Lie algebras. Generally, the symmetries of physical systems is described by Lie groups, but as it is shown in this paper, the representation theory of groups and algebras are directly related to each other.

After a general discussion of $S U(3)$ and the related algebra $\mathfrak{s u}(3)$, the information about the weight spaces of different representations can be used to order physical (hadronic) particles and furthermore discover a more fundamental scheme for the description of these particles.

This paper uses and applies methods derived in the lecture [Misra, 2013], while the structure and analysis is based on [Georgi, 1999].

### 1.1 Physical Motivation

The main task of a (theoretical) physics is the description or prediction of the behavior nature. One starts with an axiomatic principle of nature followed by specific assumptions for the given system. Using logic and mathematical tools one is able to obtain a possible result of the initial assumptions.

Usually, if the principle of nature can be applied to a broad range of problems and the number of assumptions is relatively small, the corresponding model of nature might be more fundamental.
The first principle which was able to explain Einsteins relativity and classical mechanics was the so called Hamiltonian principle. The equations of motion could be derived using the action $S$ of a physical system for initial time $t_{a}$ and final time $t_{b}$

$$
\begin{equation*}
S\left[\left\{q_{i}(t)\right\},\left\{\dot{q}_{i}(t)\right\}\right]:=\int_{t_{a}}^{t_{b}} \mathrm{~d} t L\left(\left(\left\{q_{i}(t)\right\},\left\{\dot{q}_{i}(t)\right\}, t\right)\right. \tag{1.1}
\end{equation*}
$$

The Lagrange function $L$ is the sum of the kinetic energy of each particle trajectory ${ }^{1} q_{i}(t)$ minus all corresponding potential terms. To obtain the equations of motion for each particle, one has to require that the variational derivative of the action in direction of each particle is stationary for the "true" trajectory $q^{c}$

$$
\begin{equation*}
\left.\frac{\delta S}{\delta q_{i}(\tau)}\right|_{q=q_{c}}:=\left.\frac{\partial}{\partial \epsilon} S\left[\left\{q_{i}^{c}(t)+\epsilon q_{i}(t)\right\},\left\{\dot{q}_{i}^{c}(t)+\epsilon \dot{q}_{i}(t)\right\}\right]\right|_{\epsilon=0}=0 \tag{1.2}
\end{equation*}
$$

At the end of the first half of the 20th century Richard Feynman was able generalize this principle to a even more fundamental form, the so called path integral (for a detailed introduction see [Ramond, 1997]), which was also able to incorporate quantum physics. Instead of finding the stationary path for a particle, one takes all possible paths into account and sums up the probability of all those paths. If one uses a relativistic action for fields, the obtained theories using the path integral formalism, are called Quantum Field Theories (QFT).
To actually understand the flavor model of quarks using the $S U(3)$-representation theory, one does not necessarily have to understand the complete formalism of the according quantum fields, the so called Quantum Chromo Dynamics (QCD). As it is pointed out in the third section of this paper, one just has to recognize that the representations of $S U(3)$ act on the space of particles. Furthermore, to actually understand why groups play such a crucial role in physics, it is useful to review the Noether Theorem.

### 1.2 Noether Theorem

The first theorem by Emmy Noether was published 1918 and is until now a broadly used tool in theoretical physics. Noether proofed that if the action has a continuous symmetry, a corresponding conserved quantity can be found.

[^0]Theorem 1.1. If $\Phi_{\epsilon}: q_{i}(t) \mapsto q_{i}^{\prime}(t, \epsilon)$ is a continuous transformation in $\epsilon$ which does not change the action $S \mapsto S^{\prime}(\epsilon)=S$, then there exists a conserved current given by

$$
\begin{equation*}
J_{i}:=\left.\left(\frac{\partial L}{\partial \dot{q}_{i}(t)}\right)\left(\frac{\partial}{\partial \epsilon} q_{i}^{\prime}(t, \epsilon)\right)\right|_{\epsilon=0} \quad \text { with } \quad \frac{d}{d t} J_{i}=0 \tag{1.3}
\end{equation*}
$$

As an example one could mention that invariance under time corresponds to the conservation of energy, invariance of coordinates to conservation of total momentum and as a further example, invariance under spacial orientation corresponds to conservation of angular momentum.

The invariance under spacial orientation in three dimensions can be expressed by representations of the Lie group $S O(3)$. Therefore one understands the $S O(N)$ groups in physics as groups of external rotations, the rotations which change spacial orientations.
$S O(N)$ also plays an important role in quantum physics. The representations of $S O(N)$ acting on quantum states are corresponding to the so called spin of particles.

Though there could be mentioned many applications of this group in quantum physics, the group analyzed in this paper is the $S U(3)$. In physics one understands this group as a group of internal rotation. As an example, an object which is made out of internal elements can be changed by using $S U(N)$-transformations to transform the internal elements into other internal elements. This interpretation might get more understandable in the third section of this paper.

In quantum physics, several $U(N)$ and $S U(N)$ symmetries are required. The symmetry discussed in this paper is the so called flavor $S U(3)$ of the hadronic particles. But to understand the correspondence of this group to particle physics, one should analyze the properties of $S U(3)$ first.

## 2 SU(3)-Representation Theory

### 2.1 From Group to Algebra

Definition 2.1. The group $S U(3)$ is defined by

$$
\begin{equation*}
S U(3):=\left\{U \in G L(3, \mathbb{C}) \mid U^{\dagger} U=\mathbb{1}, \quad \operatorname{det}(U)=1\right\} \tag{2.1}
\end{equation*}
$$

where $U^{\dagger}$ is the so called adjoint ${ }^{2}$ matrix $\left(U^{T}\right)^{*}=\left(U^{*}\right)^{T}$.
Furthermore the group $S U(3)$ is also a Lie group, thus it fulfills the following properties.
Proposition 2.1. $S U(3)$ is a compact eight parameter ${ }^{3}$ group which is simply-connected through the identity element.

Proof. For simplicity the proof of the properties of a Lie group, that $S U(3)$ is compact and simplyconnected through the identity element, is skipped in this paper.
More important for this paper is the possibility to parameterize $S U(3)$ with a set of eight parameters. One can see this using the following idea. The number of parameters to represent a matrix of $G L(3, \mathbb{C})$ is $9 \times 2=18$. Nine for each component and the two for the real and imaginary part. The property $U^{\dagger} U=\mathbb{1}$ results in nine linear independent equations while $\operatorname{det}(U)=1$ fixes another component. Therefore the number of free parameters is given by $18-9-1=8$.

To be able to identify an algebra corresponding to the group $S U(3)$ one can now use the possibility of parameterizing the group using a set of eight parameters $\vec{\alpha}=\left(\alpha_{1}, \cdots, \alpha_{8}\right)$. As a result, one is able to find a map $\phi: S U(3) \rightarrow U \subset \mathbb{R}^{8}$ through the identification


[^1]Using this definition of the map $\phi$, one can indeed show that $S U(3)$ can be interpreted as a manifold. Since the tangent space of $S U(3)$ defines a linear vector space, one might be able to define a commutation relation on the tangent space which satisfy the properties of a Lie algebra.
Since all elements are simply-connected through the identity, one can find an element of the group $U \in$ $S U(3)$ such that this element, evaluated at the $\alpha_{i}=0$ position, is the identity element $\left.U(\vec{\alpha})\right|_{\vec{\alpha}=0}=e$. Therefore the representation $D$ of this group element should fulfill

$$
\begin{equation*}
\left.D(\vec{\alpha})\right|_{\vec{\alpha}=0}:=\left.D(U(\vec{\alpha}))\right|_{\vec{\alpha}=0}=\mathbb{1} \tag{2.3}
\end{equation*}
$$

Definition 2.2. For a representation of $S U(3)$ defined by (2.3), the derivatives of the representation, with respect to each parameter of the group $\alpha_{i}$, at the identity element are called generators.

$$
\begin{equation*}
g_{\alpha_{j}}:=-\left.i \frac{\partial}{\partial \alpha_{j}} D(\vec{\alpha})\right|_{\vec{\alpha}=0} . \tag{2.4}
\end{equation*}
$$

As it is shown later, those elements are indeed the generators of the algebra.
As an example, one can make the following choice for a representation satisfying equation (2.3) by demanding that $D(\vec{\alpha})=D_{1}\left(\alpha_{1}\right) D_{2}\left(\alpha_{2}\right) \cdots D_{8}\left(\alpha_{8}\right)$ for the matrices

$$
\begin{array}{rlrl}
D_{1} & :=\left(\begin{array}{ccc}
\cos \left(\alpha_{1}\right) & i \sin \left(\alpha_{1}\right) & 0 \\
i \sin \left(\alpha_{1}\right) & \cos \left(\alpha_{1}\right) & 0 \\
0 & 0 & 1
\end{array}\right) & D_{2} & :=\left(\begin{array}{ccc}
\cos \left(\alpha_{2}\right) & \sin \left(\alpha_{2}\right) & 0 \\
-\sin \left(\alpha_{2}\right) & \cos \left(\alpha_{2}\right) & 0 \\
0 & 0 & 1
\end{array}\right)  \tag{2.5}\\
D_{4} & :=\left(\begin{array}{ccc}
\cos \left(\alpha_{4}\right) & 0 & i \sin \left(\alpha_{4}\right) \\
0 & \cos \left(\alpha_{4}\right) & 0 \\
i \sin \left(\alpha_{4}\right) & 0 & 1
\end{array}\right) & D_{5} & :=\left(\begin{array}{ccc}
\cos \left(\alpha_{5}\right) & 0 & \sin \left(\alpha_{5}\right) \\
0 & \cos \left(\alpha_{5}\right) & 0 \\
-\sin \left(\alpha_{5}\right) & 0 & 1
\end{array}\right) \\
D_{6} & :=\left(\begin{array}{ccc}
\cos \left(\alpha_{6}\right) & 0 & 0 \\
0 & \cos \left(\alpha_{6}\right) & i \sin \left(\alpha_{6}\right) \\
0 & i \sin \left(\alpha_{6}\right) & 1
\end{array}\right) & D_{7} & :=\left(\begin{array}{cc}
\cos \left(\alpha_{7}\right) & 0 \\
0 & \cos \left(\alpha_{7}\right) \\
0 & \sin \left(\alpha_{7}\right) \\
0 & -\sin \left(\alpha_{7}\right) \\
1
\end{array}\right) \\
D_{3} & :=\left(\begin{array}{ccc}
\exp \left\{i \alpha_{3}\right\} & 0 & 0 \\
0 & \exp \left\{-i \alpha_{3}\right\} & 0 \\
0 & 0 & 1
\end{array}\right) & D_{8}:=\left(\begin{array}{ccc}
\exp \left\{i \alpha_{8}\right\} & 0 & 0 \\
0 & \exp \left\{i \alpha_{8}\right\} & 0 \\
0 & 0 & \exp \left\{-2 i \alpha_{8}\right\}
\end{array}\right) \frac{1}{\sqrt{3}} .
\end{array}
$$

One can easily verify the condition (2.1) and (2.3). Thie idea for this choice of matrices is the following: take the matrices $O \in S O(3)$ for rotations around each axis and include the corresponding factors of $\pm i$ such that $U^{\dagger} U=\mathbb{1}$ is fulfilled.

The generators for this representation are given by the set $g_{a} 1 \leq a \leq 8$

$$
\begin{array}{llll}
g_{1}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) & g_{2}=\left(\begin{array}{ccc}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{array}\right) & g_{3}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{array}\right)  \tag{2.6}\\
g_{4}=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right) & g_{5}=\left(\begin{array}{ccc}
0 & 0 & -i \\
0 & 0 & 0 \\
i & 0 & 0
\end{array}\right) & g_{6}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right) \\
g_{7}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{array}\right) & g_{8}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right) \frac{1}{\sqrt{3}} .
\end{array}
$$

This set of matrices, the so called the Gell-Mann matrices, span indeed the algebra $\mathfrak{s u}(3)$. The easiest way to proof this is to show that this set of matrices can be identified as the basis of $\mathfrak{s l}(3, \mathbb{C})$. Therefore one can define the ordinary matrix commutator as the Lie bracket for this algebra, knowing that this fulfills the three conditions of an algebra. Before doing so, one defines

$$
\begin{equation*}
T_{a}:=\frac{1}{2} g_{a} \quad 1 \leq a \leq 8 \tag{2.7}
\end{equation*}
$$

Proposition 2.2. The set $T_{a}$ for $1 \leq a \leq 8$ and the matrix commutator bracket define the algebra $\mathfrak{s u}(3)$.

To proof this, one can use that in a small area around the identity element, each matrix in $\mathfrak{s u}(3)$ can be represented by the exponential map (see also [Scherer and Schindler, 2011])

$$
\begin{equation*}
U(\vec{\alpha})=\exp \left\{i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\} \tag{2.8}
\end{equation*}
$$

Proof. The set of $T_{a}$ generates $S U(3)$
(1) All $U(\vec{\alpha})$ satisfy $\operatorname{det}(U(\vec{\alpha}))=1$, since

$$
\operatorname{det}(U(\vec{\alpha}))=\operatorname{det}\left(\exp \left\{i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\}\right)=\exp \left\{\operatorname{Tr}\left[i \sum_{j=1}^{8} \alpha_{j} T_{j}\right]\right\}=\exp \{0\}=1
$$

where Jacobis theorem $\operatorname{det}(\exp \{A\})=\exp \{\operatorname{Tr}[A]\}$ and the fact that all $T_{a}$ 's are traceless are used.
(2) All $U(\vec{\alpha})$ satisfy $U^{\dagger}(\vec{\alpha}) U(\vec{\alpha})=\mathbb{1}$, since

$$
\begin{aligned}
U^{\dagger}(\vec{\alpha}) & =\exp \left\{i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\}^{\dagger}=\sum_{n=0}^{\infty} \frac{1}{n!}\left[\left(i \sum_{j=1}^{8} \alpha_{j} T_{j}\right)^{n}\right]^{\dagger}=\sum_{n=0}^{\infty} \frac{1}{n!}\left(-i \sum_{j=1}^{8} \alpha_{j} T_{j}^{\dagger}\right)^{n} \\
& =\sum_{n=0}^{\infty} \frac{1}{n!}\left(-i \sum_{j=1}^{8} \alpha_{j} T_{j}\right)^{n}=\exp \left\{-i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\}=U(-\vec{\alpha})
\end{aligned}
$$

where it was used that the $T_{j}$ 's are hermitian. Furthermore one gets $U(-\vec{\alpha}) U(\vec{\alpha})=\mathbb{1}$ since

$$
\exp \left\{-i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\} \exp \left\{i \sum_{j=1}^{8} \alpha_{j} T_{j}\right\}=\exp \{0\}=\mathbb{1}
$$

using the Baker-Campbell-Hausdorff formula and that $\sum_{j=1}^{8} \alpha_{j} T_{j}$ obviously commutes with itself.
$S U(3)$ and the matrix commutator bracket fulfill the three axioms of a Lie algebra
As previously mentioned, this can be shown by using that this set of matrices form a basis of $\mathfrak{s l}(3, \mathbb{C})$. Therefore one just has to map the matrices $T_{a}$ on the common basis of $\mathfrak{s l}(3, \mathbb{C})$. This map is given by

$$
\begin{array}{llrl}
H_{1}=2 T_{3}=E_{1,1}-E_{2,2} & H_{2}=\sqrt{3} T_{8}-T_{3}=E_{2,2}-E_{3,3} & \\
E_{1}=T_{1}+i T_{2}=E_{1,2} & E_{2}=T_{6}+i T_{7}=E_{2,3} & E_{3}=T_{4}+i T_{5}=E_{1,3} \\
F_{1}=T_{1}-i T_{2}=E_{2,1} & F_{2}=T_{6}-i T_{7}=E_{3,2} & F_{3}=T_{4}-i T_{5}=E_{3,1}
\end{array}
$$

where $E_{i, j}$ denotes the real $3 \times 3$ matrix with an one in the $i-j$ position while the rest are zeros.
This map is an linear isomorphism and therefore preserves the matrix commutator bracket. Therefore $\mathfrak{s u}(3)=\left(\operatorname{span}\left\{T_{a} \mid 1 \leq a \leq 8\right\},[\cdot, \cdot]\right)$ is the corresponding algebra to the group $S U(3)$.

### 2.2 Properties of Gell-Mann Basis

To be more precise, the properties of $T_{a}=\frac{1}{2} g_{a}$ are discussed here.
Though this set of matrices is equivalent to the algebra of $\mathfrak{s l}(3, \mathbb{C})$, in physics one keeps this definition for the basis vectors for reasons which can be seen later in section 3.2.
Some important properties of these matrices which were already used are

$$
\begin{align*}
\operatorname{Tr}\left[T_{a}\right] & =0  \tag{2.9}\\
T_{a}^{\dagger} & =T_{a} \tag{2.10}
\end{align*}
$$

Furthermore one is also able to introduce a scalar product using the in lecture discussed Killing form

$$
\begin{equation*}
k_{D}\left(T_{a}, T_{b}\right):=\frac{1}{2} \operatorname{Tr}\left[T_{a}^{\dagger} T_{b}\right]=\frac{1}{4} \delta_{a, b} . \tag{2.11}
\end{equation*}
$$

Note that the $\dagger$ plays here no crucial role since all $T_{a}$ 's are hermitian.
Additionally in physics, the commutator relation are defined by the so called structure constants $f_{a b c}$.
Definition 2.3. The coefficients of the linear representation of a commutator bracket for two given basis vectors $T_{a}$ and $T_{b}$ are called structure constants $f_{a b c}$

$$
\begin{equation*}
\left[T_{a}, T_{b}\right]=i \sum_{c=1}^{8} f_{a b c} T_{c} \tag{2.12}
\end{equation*}
$$

Proposition 2.3. The structure constants $f_{a b c}$ are real and totally antisymmetric.

To proof this, one can express the structure constants by evaluating the right-hand-side of (2.12) with another basis vector $T_{c^{\prime}}$ in the Killing form.

$$
\begin{equation*}
\operatorname{Tr}\left[\left[T_{a}, T_{b}\right] T_{c^{\prime}}\right]=\operatorname{Tr}\left[i \sum_{c=1}^{8} f_{a b c} T_{c} T_{c^{\prime}}\right]=i \sum_{c=1}^{8} f_{a b c} \operatorname{Tr}\left[T_{c} T_{c^{\prime}}\right]=\frac{i}{2} f_{a b c^{\prime}} \tag{2.13}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
f_{a b c}=-2 i \operatorname{Tr}\left[\left[T_{a}, T_{b}\right] T_{c}\right] \tag{2.14}
\end{equation*}
$$

Proof. The structure constants are totally antisymmetric
Using equation (2.14) one can see that the interchange of the first two indices results in a minus sign because of the first property of a Lie algebra $[A, B]=-[B, A]$. Furthermore if one uses the cyclic property of the trace

$$
\operatorname{Tr}[A B C]=\operatorname{Tr}[B C A]=\operatorname{Tr}[C A B],
$$

one gets

$$
\begin{aligned}
\operatorname{Tr}\left[\left[T_{a}, T_{b}\right] T_{c}\right] & =\operatorname{Tr}\left[T_{a} T_{b} T_{c}-T_{b} T_{a} T_{c}+T_{a} T_{c} T_{b}-T_{a} T_{c} T_{b}\right] \\
& =\operatorname{Tr}\left[T_{a}\left[T_{b}, T_{c}\right]-T_{b} T_{a} T_{c}+T_{a} T_{c} T_{b}\right] \\
& =\operatorname{Tr}\left[T_{a}\left[T_{b}, T_{c}\right]-T_{a} T_{c} T_{b}+T_{a} T_{c} T_{b}\right] \\
& =\operatorname{Tr}\left[\left[T_{b}, T_{c}\right] T_{a}\right] .
\end{aligned}
$$

And thus one gets $f_{a b c}=f_{b c a}=-f_{c b a}$ and accordingly all other interchanges.
The structure constants are real
One can obtain this fact by conjugating equation (2.12). Using that these matrices are hermitian one gets for the left-hand-side

$$
\left[T_{a}, T_{b}\right]^{\dagger}=\left[T_{b}^{\dagger}, T_{a}^{\dagger}\right]=-\left[T_{a}, T_{b}\right]
$$

and accordingly for the right-hand-side

$$
\left(i \sum_{c=1}^{8} f_{a b c} T_{c}\right)^{\dagger}=-i \sum_{c=1}^{8} f_{a b c}^{*} T_{c}^{\dagger}=-i \sum_{c=1}^{8} f_{a b c}^{*} T_{c}
$$

Thus, one finally obtains

$$
-\left[T_{a}, T_{b}\right]=-i \sum_{c=1}^{8} f_{a b c}^{*} T_{c}
$$

which is only true if $f_{a b c}^{*}=f_{a b c}$ and accordingly the structure constants have to be real.
Later, one also uses the fact that those structure constants are real to identify corresponding representations.

### 2.3 Roots and Weights

As previously mentioned, the set of basis vectors is isomorphic to $L=\mathfrak{s l}(3, \mathbb{C})$. Therefore for a fixed maximal toral subalgebra $T=\operatorname{span}\left\{H_{1}, H_{2}\right\}$, also called Cartan subalgebra (since $T$ is semisimple), the Cartan matrix is given by

$$
C=\left(\begin{array}{cc}
2 & -1  \tag{2.15}\\
-1 & 2
\end{array}\right)
$$

In physics the dimensions of a representation play a crucial role, therefore the studies of algebras concentrated mostly on the studies of weight spaces. To get a general overview it is useful to actually express the roots an weights in a vectorial representation.

According to this the following chapter is mostly a direct application of $\mathfrak{s u}(2)$ representation theory (equivalently to $\mathfrak{s l}(2, \mathbb{C})$ representation theory).

### 2.3.1 Roots

In this subsection, general facts of $\mathfrak{s u}(3)$ are repeated. The focus is hereby on the rootstring-procedure to compute all positive roots out of the number of simple roots in the basis of Gell-Mann matrices. This might be useful since the same procedure is used to abstractly construct the weight spaces.
As a general theorem of representation theory, one can decompose $\mathfrak{s u}(3)$ in its root spaces

$$
\begin{equation*}
L_{\alpha}:=\{x \in L \mid[t, x]=\alpha(t) x \quad \forall t \in T\} \tag{2.16}
\end{equation*}
$$

if one knows the set of all roots $\Phi$

$$
\begin{equation*}
\mathfrak{s u}(3)=\bigoplus_{\alpha \in \Phi} L_{\alpha} \oplus T \tag{2.17}
\end{equation*}
$$

Furthermore one could decompose $\mathfrak{s u}(3)$ into two $\mathfrak{s u}(2)_{\alpha}$ corresponding to the two simple roots $\alpha_{1}$ and $\alpha_{2}$. The $\mathfrak{s u}(2)_{\alpha}$ 's are given by

$$
\begin{equation*}
\mathfrak{s u}(2)_{\alpha}=\left(\operatorname{span}\left(E_{\alpha}, F_{\alpha}=E_{-\alpha}, H_{\alpha}\right),[\cdot, \cdot]\right), \tag{2.18}
\end{equation*}
$$

such that

$$
\begin{equation*}
\left[H_{\alpha}, E_{\alpha}\right]=\alpha\left(H_{\alpha}\right) E_{\alpha} \quad\left[H_{\alpha}, E_{-\alpha}\right]=-\alpha\left(H_{\alpha}\right) E_{-\alpha} \quad\left[E_{\alpha}, E_{-\alpha}\right]=H_{\alpha} \quad \alpha\left(H_{\alpha}\right)=2 . \tag{2.19}
\end{equation*}
$$

Furthermore, for the standard $\mathfrak{s l}(2, \mathbb{C})$ basis

$$
H_{\alpha}=\left(\begin{array}{cc}
1 & 0  \tag{2.20}\\
0 & -1
\end{array}\right) \quad E_{\alpha}=\left(\begin{array}{cc}
0 & 1 \\
0 & 0
\end{array}\right) \quad E_{-\alpha}=\left(\begin{array}{cc}
0 & 0 \\
1 & 0
\end{array}\right)
$$

one can directly verify the following proposition:
Proposition 2.4. The elements of standard $\mathfrak{s u}(2)_{\alpha}$ fulfill the following conditions

1. $H_{\alpha}=H_{\alpha}^{\dagger}$
2. $E_{\alpha}^{\dagger}=E_{-\alpha}$ and $E_{-\alpha}^{\dagger}=E_{\alpha}$,
3. $k_{D}\left(H_{\alpha}, H_{\alpha}\right)=1$ and
4. $k_{D}\left(E_{\alpha}, E_{\alpha}\right)=1 / 2=k_{D}\left(E_{-\alpha}, E_{-\alpha}\right)$.

Proof. Use equation (2.20).
Furthermore, if one looks at the action of both $\mathfrak{s u}(2)_{\alpha}$ one has the additional requirement that $\alpha_{1}\left(H_{\alpha_{2}}\right)=\alpha_{2}\left(H_{\alpha_{1}}\right)=-1$.
Knowing this one can find the representation of $H_{\alpha}$ in the $\left\{t_{1}:=T_{3}, t_{2}:=T_{8}\right\}$ basis. Since the main goal of this section is to express the roots and weights in the $T$-basis, one does not need to find the representation of the $E_{ \pm \alpha_{i}}$ in this basis. The additional computations can be done abstractly.
As a matter of fact, the $H_{\alpha_{i}}$ must be a linear combination of $t_{1}$ and $t_{2}$ such that $\alpha_{i}\left(H_{\alpha_{j}}\right)=C_{i j}$. Therefore the linear coefficients must have the following form.

Proposition 2.5. For a given basis $\left\{t_{1}, t_{2}\right\}$ of $T$, the corresponding $H_{\alpha_{i}}$ of each $\mathfrak{s u}(2)$ are given by

$$
\begin{equation*}
H_{\alpha_{i}}=2 \sum_{n=1}^{2} \frac{t_{n} \alpha_{i}\left(t_{n}\right)}{\sum_{m=1}^{2} \alpha_{i}\left(t_{m}\right) \alpha_{i}\left(t_{m}\right)} \tag{2.21}
\end{equation*}
$$

Proof. To proof this one has to show that $\alpha_{i}\left(H_{\alpha_{j}}\right)=C_{i j}$. It is easy to see that $\alpha_{j}\left(H_{\alpha_{j}}\right)=2$. To solve this system of equation in general, one could also take this condition as the definition for $\alpha_{j}\left(t_{n}\right)$

$$
\begin{equation*}
C_{i j}=\alpha_{i}\left(H_{\alpha_{j}}\right)=2 \sum_{n=1}^{2} \frac{\alpha_{i}\left(t_{n}\right) \alpha_{j}\left(t_{n}\right)}{\sum_{m=1}^{2} \alpha_{j}\left(t_{m}\right) \alpha_{j}\left(t_{m}\right)}=:\left\langle\alpha_{i}, \alpha_{j}\right\rangle . \tag{2.22}
\end{equation*}
$$

To actually obtain a number of four linear independent equations, one can also use the trace identities

$$
1=k_{D}\left(H_{\alpha_{i}}, H_{\alpha_{i}}\right)=4 \sum_{n, m=1}^{2} \frac{k_{D}\left(t_{n}, t_{m}\right) \alpha_{i}\left(t_{n}\right) \alpha_{i}\left(t_{m}\right)}{\left(\sum_{k=1}^{2} \alpha_{i}\left(t_{k}\right) \alpha_{i}\left(t_{k}\right)\right)\left(\sum_{l=1}^{2} \alpha_{i}\left(t_{l}\right) \alpha_{i}\left(t_{l}\right)\right)}=\frac{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}{2 \sum_{l=1}^{2} \alpha_{i}\left(t_{l}\right) \alpha_{i}\left(t_{l}\right)} .
$$

the first $1 / 2$ is for the definition of $k_{D}$, the second for the trace of two Gell-Mann matrices and the third for the definition of the bracket product. Evaluating this results in a two further conditions requiring that $\sum_{l=1}^{2} \alpha_{i}\left(t_{l}\right) \alpha_{i}\left(t_{l}\right)=1$ and making the solutions unique.
Proposition 2.6. The coefficients for the roots acting on Gell-Mann matrices are given by

$$
\begin{array}{ll}
\alpha_{1}\left(t_{1}\right)=\frac{1}{2} & \alpha_{2}\left(t_{1}\right)=\frac{1}{2}  \tag{2.23}\\
\alpha_{1}\left(t_{2}\right)=-\frac{\sqrt{3}}{2} & \alpha_{2}\left(t_{2}\right)=\frac{\sqrt{3}}{2}
\end{array}
$$

Proof. The denominator of (2.22) simplifies to $\alpha_{1}\left(t_{1}\right)^{2}+\alpha_{1}\left(t_{2}\right)^{2}=1=\alpha_{2}\left(t_{1}\right)^{2}+\alpha_{2}\left(t_{2}\right)^{2}$ which results in

$$
\begin{array}{ll}
\alpha_{1}\left(H_{1}\right)=2 \alpha_{1}^{2}\left(t_{1}\right)+2 \alpha_{1}^{2}\left(t_{2}\right)=2 & \alpha_{1}\left(H_{2}\right)=2 \alpha_{1}\left(t_{1}\right) \alpha_{2}\left(t_{1}\right)+2 \alpha_{1}\left(t_{2}\right) \alpha_{2}\left(t_{2}\right)=-1 \\
\alpha_{2}\left(H_{1}\right)=2 \alpha_{2}\left(t_{1}\right) \alpha_{1}\left(t_{1}\right)+2 \alpha_{2}\left(t_{2}\right) \alpha_{1}\left(t_{2}\right)=-1 & \alpha_{2}\left(H_{2}\right)=2 \alpha_{2}^{2}\left(t_{1}\right)+2 \alpha_{2}^{2}\left(t_{2}\right)=2 .
\end{array}
$$

To finally compute all roots, one can use that if $\alpha$ is a root, also $-\alpha$ is a root and that all positive roots can be rewritten as a sum over simple roots

$$
\begin{equation*}
\beta=k_{1} \alpha_{1}+k_{2} \alpha_{2} \quad \text { for } \quad k_{1}, k_{2} \geq 0 \tag{2.24}
\end{equation*}
$$

Furthermore the $\alpha_{j}$-rootstring through a simple root $\alpha_{i}$

$$
\begin{equation*}
\alpha_{i}, \alpha_{i}+\alpha_{j}, \cdots, \alpha_{i}+q \alpha_{j} \tag{2.25}
\end{equation*}
$$

is of length $\left\langle\alpha_{i}, \alpha_{j}\right\rangle=-q$.
Therefore one can use the following technique to obtain all positive roots
(1) Draw in the $k=k_{1}+k_{2}=0$ layer all expectation values for $\left\langle\alpha_{j}, \alpha_{i}\right\rangle$



Figure 1: : Root space of $\mathfrak{s u}(3)$.
(2) Find the negative entries and continue the rootstrings according to length $\left\langle\alpha_{i}, \alpha_{j}\right\rangle$ by adding $q$ times the $\alpha_{j}$ box to the $\alpha_{i}$ box

$$
\begin{equation*}
\ldots \alpha_{\alpha_{2} \text {-string }} \tag{2.27}
\end{equation*}
$$


(3) Continue doing so until each coefficient is positive or zero.

Accordingly for $\mathfrak{s u}(3)$ the roots are given by

$$
\begin{equation*}
\Phi^{+}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{1}+\alpha_{2}\right\} \quad \Phi=\Phi^{+} \cup \Phi^{-} \tag{2.28}
\end{equation*}
$$

In the $T$-basis, the roots have the following form

$$
\begin{equation*}
\left.\alpha_{1}\right|_{T}=\left.\frac{1}{2}\binom{1}{\sqrt{3}} \quad \alpha_{2}\right|_{T}=\left.\frac{1}{2}\binom{1}{-\sqrt{3}} \quad \alpha_{1}\right|_{T}+\left.\alpha_{2}\right|_{T}=\binom{1}{0} . \tag{2.29}
\end{equation*}
$$

Thus the roots are located on a circle with radius 1 and angel $60^{\circ}$ between each root, see also figure 1.

### 2.3.2 Weights

The previous discussion of roots can now be generalized to weights. This is important since one knows that the dimension of a representation is connected to the highest weight of a representation. This section mainly follows [Humphreys, 1980] and [Woit, 2012].
Definition 2.4. For a fixed Cartan subalgebra $T$, one can find a linear form $\lambda \in T^{*}$ according to the following definitions
(1) $\lambda$ is a weight if $\lambda\left(H_{\alpha}\right) \in \mathbb{Z}$ for all simple roots $\alpha$.
(2) $\lambda$ is a dominant weight if $\lambda\left(H_{\alpha}\right) \in \mathbb{Z}_{\geq 0}$ for all simple roots $\alpha$.
(3) $\lambda_{j}$ is a fundamental weight if $\lambda_{j}\left(H_{\alpha_{i}}\right)=\delta_{i, j}$ for all simple roots $\alpha_{i}$.

## Remarks:

1. Differently to the definition in the lecture, in this definition one does not mention if the corresponding weight spaces

$$
\begin{equation*}
\left.V(\lambda):=\left\{|v\rangle \in V\left|H_{\alpha} \cdot\right| v\right\rangle=\lambda\left(H_{\alpha}\right)|v\rangle \quad \forall H_{\alpha} \in T\right\} \quad \text { for the } L \text {-module } L \cdot V \rightarrow V \tag{2.30}
\end{equation*}
$$

are empty or not. But indeed this will be the main concern in this section.
2. One can rewrite $\lambda\left(H_{\alpha}\right)$ as

$$
\begin{equation*}
\lambda\left(H_{\alpha}\right)=2 \sum_{n=1}^{2} \frac{\lambda\left(t_{n}\right) \alpha\left(t_{n}\right)}{\sum_{m} \alpha\left(t_{m}\right)^{2}}=\langle\lambda, \alpha\rangle \tag{2.31}
\end{equation*}
$$

3. According to this definition, each fundamental weight is dominant.
4. The fundamental weights span $T$ and each weight or root can be rewritten as a linear combination of fundamental weights.
5. The fundamental weights will be denoted with $\mu_{i}$ from now on.

Proposition 2.7. The set of simple roots $\left\{\alpha_{1}, \alpha_{2}\right\}$ can be expressed by the Cartan matrix $C$ and the set of fundamental weights $\left\{\mu_{1}, \mu_{2}\right\}$ according to

$$
\begin{equation*}
\alpha_{i}=\sum_{j=1}^{2} C_{i j} \mu_{j} \tag{2.32}
\end{equation*}
$$

Proof. To verify this equation one simply has to compute the action of both sides on $H_{\alpha_{n}}$

$$
\left\langle\alpha_{i}, \alpha_{n}\right\rangle=\alpha_{i}\left(H_{\alpha_{n}}\right)=\sum_{j=1}^{2} C_{i j} \mu_{j}\left(H_{\alpha_{n}}\right)=\sum_{j=1}^{2} C_{i j} \delta_{j, n}=C_{i n}
$$

According to this, we can find the fundamental weights of $\mathfrak{s u}(3)$ by inverting the Cartan matrix.

$$
\begin{align*}
\mu_{i} & =\sum_{j=1}^{2} C_{i j}^{-1} \alpha_{j} \quad \Rightarrow \quad\binom{\mu_{1}}{\mu_{2}}=\frac{1}{3}\left(\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right)\binom{\alpha_{1}}{\alpha_{2}}  \tag{2.33}\\
& \left.\Rightarrow \quad \mu_{1}\right|_{T}=\left.\frac{1}{2 \sqrt{3}}\binom{\sqrt{3}}{1} \quad \mu_{2}\right|_{T}=\frac{1}{2 \sqrt{3}}\binom{\sqrt{3}}{-1} . \tag{2.34}
\end{align*}
$$

Proposition 2.8. For any dominant weight $\lambda$ there exists a (unique) finite-dimensional, irreducible representation $V(\lambda)$ such that $\lambda$ is the highest weight.

Proof. See [Woit, 2012].
Recall: If $\lambda$ is the highest weight of a representation, we can use the usual $\mathfrak{s l}(2, \mathbb{C})$-representation theory and extend it to $\mathfrak{s l}(3, \mathbb{C}) \cong \mathfrak{s u}(3)$, which means that

1. Each weight is of the form $\lambda^{\prime}=\lambda-k_{1} \alpha_{1}-k_{2} \alpha_{2}$ and can be obtained by applying $k_{1}$ times $E_{-\alpha_{1}}$ and $k_{2}$ times $E_{-\alpha_{2}}$ on the maximal weight vector $|\lambda\rangle \in V(\lambda)$
2. $V(\lambda)$ is the sum over all weight spaces

$$
\begin{equation*}
V(\lambda)=\bigoplus_{\lambda^{\prime} \text { is weight }} V\left(\lambda^{\prime}\right), \tag{2.35}
\end{equation*}
$$

such that the dimension of $V(\lambda)$ is given by

$$
\begin{equation*}
\operatorname{dim}(V(\lambda))=\sum_{\lambda^{\prime} \text { is weight }} \operatorname{dim}\left(V\left(\lambda^{\prime}\right)\right) . \tag{2.36}
\end{equation*}
$$

According to this, to finally find the dimension of each representation, one just has to find all existing non-empty weight spaces and add their dimensions.
To find the corresponding non-empty weight spaces one can use a technique similar to the technique of finding the set of positive roots.

Since the weights have the form $\lambda^{\prime}=\lambda-k_{1} \alpha_{1}-k_{2} \alpha_{2}$ for a highest weight $\lambda$, the length of a root-string through $\lambda$ in $\alpha_{i}$-direction is now given by

$$
\begin{equation*}
\lambda\left(H_{\alpha_{i}}\right)=\left\langle\lambda, \alpha_{i}\right\rangle=+k_{i} . \tag{2.37}
\end{equation*}
$$

Now one can apply the same scheme which was used for the roots.
As a choice for the highest weight, it is convenient to choose the fundamental weights $\mu_{i}$ since one directly knows their $H_{\alpha_{i}}$ value.

Definition 2.5. One calls representations with the highest weight $\lambda=n_{1} \mu_{1}+n_{2} \mu_{2}$ the $\left(n_{1}, n_{2}\right)$ fundamental representation.

As an example one can look at the ( 1,0 )-representation.
(1) Again, one starts with the $k=k_{1}+k_{2}=0$ layer which corresponds to the fundamental weight.

(2) Next, one has to find the positive entries and continue the rootstrings according to the length $\left\langle\mu_{1}, \alpha_{j}\right\rangle$ by subtracting $k_{j}$ times the $\alpha_{j}$-box off the $\mu_{1}$-box

$$
\alpha_{1} \text {-box: } \begin{array}{ll}
2 & -1
\end{array} \alpha_{2} \text {-box: } \begin{array}{ll}
-1 & 2
\end{array} \quad \alpha_{1} \text {-string }
$$

Therefore the only possibility is to subtract the $\alpha_{1}$-box one times.

(3) Continuing this procedure one finally obtains


In this case, the representation of highest weight $\mu_{1}$ is of dimension three since each weight space is one dimensional.

$$
\begin{equation*}
V\left(\mu_{1}\right)=V_{\mu_{1}} \oplus V_{\mu_{1}-\alpha_{1}} \oplus V_{\mu_{1}-\alpha_{1}-\alpha_{2}} \tag{2.38}
\end{equation*}
$$



Figure 2: : All weights of the (1,0)-representation (filled) and ( 0,1 )-representation (empty) in $T$ basis.

For the representation with highest weight $\mu_{2}$, the ( 0,1 )-representation, one gets accordingly


If one compares the representation of weights in the $T$-basis, one is able to notice that $(1,0)$ representation has the same weights as the $(0,1)$ representation with the opposite sign (see figure $2)$.

$$
\left.\left.\begin{array}{l}
\left.\mu_{1}\right|_{T}=\frac{1}{2 \sqrt{3}}\binom{\sqrt{3}}{1}  \tag{2.40}\\
\left.\mu_{2}\right|_{T}=\frac{1}{2 \sqrt{3}}\binom{\sqrt{3}}{-1}
\end{array}\left(\mu_{1}-\alpha_{1}\right)\right|_{T}=-\left.\frac{1}{\sqrt{3}}\binom{0}{1} \quad\left(\mu_{1}-\alpha_{1}\right)\right|_{T}=-\left.\left(\alpha_{1}\right)\right|_{T}=-\left.\mu_{2}\right|_{T}\right)\left.\left.\right|_{T} \quad\left(\mu_{2}-\alpha_{2}-\alpha_{1}\right)\right|_{T}=-\left.\mu_{1}\right|_{T} .
$$

With the according definition of the ordering of weights (see also [Humphreys, 1980] chapter 13.2) one is able to see that the smallest weight of $(1,0)$ is minus the highest weight of $(0,1)$ and vice versa. Indeed there is a more general scheme one is using in particle physics to describe those specific representations.
Definition 2.6. For a given representation $D: \mathfrak{s u}(3) \rightarrow G L(n, \mathbb{C})$ one calls $\bar{D}: \mathfrak{s u}(3) \rightarrow G L(n, \mathbb{C})$ the complex-conjugated representation if

$$
\begin{equation*}
\bar{D}\left(T_{a}\right)=D\left(-T_{a}^{*}\right) \quad \forall T_{a} \in \mathfrak{s u}(3) \tag{2.41}
\end{equation*}
$$

Proposition 2.9. The set of matrices $\left\{\left(-T_{a}^{*}\right) \mid 1 \leq a \leq 8\right\}$ with the matrix commutator span $\mathfrak{s u}(3)$ and therefore the complex-conjugated representation is indeed a representation of $\mathfrak{s u}(3)$.

Proof. Since each matrix $T_{a}$ has either only real or only imaginary components, $\left(-T_{a}^{*}\right)= \pm T_{a}$ and therefore they are still a basis for $\mathfrak{s u}(3)$. Additionally they satisfy the same commutator relations. This can be seen by complex-conjugating equation (2.12) and using that the structure constants are real.
$\left(\left[T_{a}, T_{b}\right]\right)^{*}=\left[T_{a}^{*}, T_{b}^{*}\right]=\left[\left(-T_{a}^{*}\right),\left(-T_{b}^{*}\right)\right] \stackrel{(2.12)}{=}\left(i \sum_{c=1}^{8} f_{a b c} T_{c}\right)^{*}=-i \sum_{c=1}^{8} f_{a b c} T_{c}^{*}=i \sum_{c=1}^{8} f_{a b c}\left(-T_{c}^{*}\right)$.

Knowing this, one can relate a given representation to its complex-conjugated representation by their weights.

Proposition 2.10. The weights of the complex-conjugated representation are minus the weights of the according representation and thus the highest weight corresponds to the lowest weight and vice versa.

Proof. For a given weight vector $|\lambda\rangle$ of representation $D$, using the fact that $\left(-H_{\alpha_{i}}^{*}\right)=-H_{\alpha_{i}} \in T$ and that each weight $\lambda$ is a linear functional on $T$, we have the following equation for the complexconjugated representation $\bar{D}$

$$
\bar{D}\left(H_{\alpha_{i}}\right)|\lambda\rangle=D\left(-H_{\alpha_{i}}\right)|\lambda\rangle=\lambda\left(-H_{\alpha_{i}}\right)|\lambda\rangle=-\lambda\left(H_{\alpha_{i}}\right)|\lambda\rangle .
$$

Since this equation holds for all weights of $D, \bar{D}$ has minus the same weights as $D$ and the highest weight of $D$ is the lowest weight of $\bar{D}$ and vice versa.

According to this we can identify that the ( 0,1 )-representation is the complex-conjugated representation of the ( 1,0 )-representation. In physics one makes the following definition to keep track of those low-dimensional representations.

Definition 2.7. A fundamental representation $\left(n_{1}, n_{2}\right)$ is labeled by its dimension $n$. The complexconjugated representation, which has the same dimension, is labeled by $\bar{n}$.

## Remark:

1. In this case we label the (1,0)-representation as the 3 -representation and the ( 0,1 )-representation as the $\overline{3}$-representation.
2. Note that the 3 and the $\overline{3}$-representations of $\mathfrak{s u}(3)$ are indeed the representations $D\left(T_{a}\right)=T_{a}$ and $\bar{D}\left(T_{a}\right)=-T_{a}^{*}$.
3. In general, for a fundamental $\mathfrak{s u}(3)$ representation $\left(n_{1}, n_{2}\right)$, the complex-conjugated representation is given by $\left(n_{2}, n_{1}\right)$.
Before one is able to understand the idea of the so called flavor- $S U(3)$ in physics, it is useful to look at three additional representations of $S U(3)$.

### 2.3.3 (2,0)-Representation

The rootstring diagram for the (2,0)-representation can be seen in figure 3 .
If one counts the number of linear independent weights, one is able to find six independent weight vectors since each weight space is one-dimensional. This is true since each weight vector can be generated by applying the corresponding representations of $E_{-\alpha_{i}}$ to the highest weight vector $\left|2 \mu_{1}\right\rangle$.

The only vector which can be realized in two different ways is corresponding to the weight $2 \mu_{1}-$ $2 \alpha_{1}-\alpha_{2}$.

Proposition 2.11. The two weight vectors $\left(E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle$ and $\left(E_{-\alpha_{2}} E_{-\alpha_{1}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle$ are linearly dependent.

Proof. To show that those two vectors are linear dependent, the following equation has to be true for a $z \in \mathbb{C}$

$$
\begin{aligned}
0 & =\left(E_{-\alpha_{2}} E_{-\alpha_{1}} E_{-\alpha_{1}}-z E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle \\
& =\left(\left[E_{-\alpha_{2}}, E_{-\alpha_{1}}\right] E_{-\alpha_{1}}+(1-z) E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle
\end{aligned}
$$

Using that $\left[E_{-\alpha_{1}}, E_{-\alpha_{2}}\right] \in L_{-\left(\alpha_{1}+\alpha_{2}\right)}$, that $2 \alpha_{1}+\alpha_{2}$ is not a root and therefore $L_{-\left(2 \alpha_{1}+\alpha_{2}\right)}=\emptyset$, one gets that

$$
\left[\left[E_{-\alpha_{2}}, E_{-\alpha_{1}}\right], E_{-\alpha_{1}}\right]=0
$$



Figure 3: : Weight space of the (2,0)-representation in $T$ basis.

Thus one gets

$$
\begin{aligned}
0 & =\left(E_{-\alpha_{1}}\left[E_{-\alpha_{2}}, E_{-\alpha_{1}}\right]+(1-z) E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle \\
& =\left(-E_{-\alpha_{1}} E_{-\alpha_{1}} E_{-\alpha_{2}}+(2-z) E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}}\right) \cdot\left|2 \mu_{1}\right\rangle .
\end{aligned}
$$

Regarding the rootstring diagram, the $2 \mu_{1}-\alpha_{2}$ weight space is empty which means that only the null vector satisfies $H_{\alpha_{2}} \cdot\left|2 \mu_{1}-\alpha_{2}\right\rangle=\left(2 \mu_{1}-\alpha_{2}\right)\left(H_{\alpha_{2}}\right)\left|2 \mu_{1}-\alpha_{2}\right\rangle=-\alpha_{2}\left(H_{\alpha_{2}}\right)\left|2 \mu_{1}-\alpha_{2}\right\rangle$. But therefore one can also use that

$$
-\alpha_{2}\left(H_{\alpha_{2}}\right)\left(E_{-\alpha_{2}} \cdot\left|2 \mu_{1}\right\rangle\right)=\left[H_{\alpha_{2}}, E_{-\alpha_{2}}\right] \cdot\left|2 \mu_{1}\right\rangle=H_{\alpha_{2}} \cdot\left(E_{-\alpha_{2}} \cdot\left|2 \mu_{1}\right\rangle\right)
$$

Comparing this equation to the previous statement one can verify that $E_{-\alpha_{2}} \cdot\left|2 \mu_{1}\right\rangle=|0\rangle$. Thus those two vectors are linearly dependent if

$$
0=(2-z) E_{-\alpha_{1}} E_{-\alpha_{2}} E_{-\alpha_{1}} \cdot\left|2 \mu_{1}\right\rangle,
$$

which is obviously true for $z=2$.

Therefore the (2,0)-representation is six-dimensional and will be labeled as the 6 -representation.

### 2.3.4 (1,1)-Representation

The rootstring diagram of the $(1,1)$-representation can be found in figure 4 .
As the diagram is already suggesting, the weight $\lambda=\mu_{1}+\mu_{2}-\alpha_{1}-\alpha_{2}$ is different from the other weights.

Proposition 2.12. The $V(\lambda)$-weight space of weight $\lambda=\mu_{1}+\mu_{2}-\alpha_{1}-\alpha_{2}$ is two-dimensional.

Proof. If $V(\lambda)$ is two-dimensional, then the two vectors $\left|\lambda_{1}\right\rangle:=E_{-\alpha_{1}} E_{-\alpha_{2}} \cdot\left|\mu_{1}+\mu_{2}\right\rangle$ and $\left|\lambda_{2}\right\rangle:=$ $E_{-\alpha_{2}} E_{-\alpha_{1}} \cdot\left|\mu_{1}+\mu_{2}\right\rangle$ must fulfill the following condition

$$
\begin{equation*}
\left|\lambda_{2}\right\rangle=z_{1}\left|\lambda_{1}\right\rangle+z_{2}\left|\lambda^{\prime}\right\rangle \tag{2.42}
\end{equation*}
$$

for a $z_{1}, z_{2} \in \mathbb{C}$ and a vector $\left|\lambda^{\prime}\right\rangle \in V(\lambda)$ such that $\left\langle\lambda_{1} \mid \lambda^{\prime}\right\rangle=0$. Therefore the following scalar products must be different

$$
\left\langle\lambda_{1} \mid \lambda_{2}\right\rangle=z_{1}\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle \quad\left\langle\lambda_{2} \mid \lambda_{1}\right\rangle=z_{1}^{*}\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle \quad\left\langle\lambda_{2} \mid \lambda_{2}\right\rangle=\left|z_{1}\right|^{2}\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle+\left|z_{2}\right|^{2}\left\langle\lambda^{\prime} \mid \lambda^{\prime}\right\rangle .
$$



Figure 4: : Weight space of the (1,1)-representation in $T$ basis.

Accordingly the following equation can only be fulfilled if $z_{2}=0$, which means $\left|\lambda_{1}\right\rangle$ is linearly dependent on $\left|\lambda_{2}\right\rangle$.

$$
\left\langle\lambda_{1} \mid \lambda_{2}\right\rangle\left\langle\lambda_{2} \mid \lambda_{1}\right\rangle=\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle\left\langle\lambda_{2} \mid \lambda_{2}\right\rangle .
$$

Thus, to proof this proposition, one has to compute the according scalar products.
Using that $E_{\alpha_{i}}^{\dagger}=E_{-\alpha_{i}}$ one gets for

$$
\begin{aligned}
\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle & =\left\langle\mu_{1}+\mu_{2}\right| E_{\alpha_{2}} E_{\alpha_{1}} E_{-\alpha_{1}} E_{-\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle \\
& =\left\langle\mu_{1}+\mu_{2}\right| E_{\alpha_{2}}\left(\left[E_{\alpha_{1}}, E_{-\alpha_{1}}\right]+E_{-\alpha_{1}} E_{\alpha_{1}}\right) E_{-\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle .
\end{aligned}
$$

Furthermore $\left[E_{\alpha_{1}}, E_{-\alpha_{1}}\right]=H_{\alpha_{1}}$ and according to the root-string diagram $E_{\alpha_{1}}$ kills the $E_{-\alpha_{2}}$. $\left|\mu_{1}+\mu_{2}\right\rangle$ vector, while the $H_{\alpha_{1}}$-expectation value for this vector is given by 2 . Thus on gets
$\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle=2\left\langle\mu_{1}+\mu_{2}\right| E_{\alpha_{2}} E_{-\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle=2\left\langle\mu_{1}+\mu_{2}\right| H_{\alpha_{2}}+E_{-\alpha_{2}} E_{\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle=2\left\langle\mu_{1}+\mu_{2} \mid \mu_{1}+\mu_{2}\right\rangle$.
Accordingly one also gets

$$
\left\langle\lambda_{2} \mid \lambda_{2}\right\rangle=2\left\langle\mu_{1}+\mu_{2} \mid \mu_{1}+\mu_{2}\right\rangle .
$$

For the other scalar product one has to compute

$$
\begin{aligned}
\left\langle\lambda_{1} \mid \lambda_{2}\right\rangle & =\left\langle\mu_{1}+\mu_{2}\right| E_{\alpha_{1}} E_{\alpha_{2}} E_{-\alpha_{1}} E_{-\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle \\
& =\left\langle\mu_{1}+\mu_{2}\right| E_{\alpha_{1}}\left(\left[E_{\alpha_{2}}, E_{-\alpha_{1}}\right]+E_{-\alpha_{1}} E_{\alpha_{2}}\right) E_{-\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle .
\end{aligned}
$$

Now one has to use that $\alpha_{2}-\alpha_{1}$ is not a root and therefore $\left[E_{\alpha_{2}}, E_{-\alpha_{1}}\right]=0$. Additionally $E_{\alpha_{1}}$ and $E_{\alpha_{2}}$ kill the $\left|\mu_{1}+\mu_{2}\right\rangle$ state.

$$
\left\langle\lambda_{1} \mid \lambda_{2}\right\rangle=\left\langle\mu_{1}+\mu_{2}\right| H_{\alpha_{1}} H_{\alpha_{2}}\left|\mu_{1}+\mu_{2}\right\rangle=\left\langle\mu_{1}+\mu_{2} \mid \mu_{1}+\mu_{2}\right\rangle .
$$

And therefore one can show that $\left\langle\lambda_{1} \mid \lambda_{1}\right\rangle\left\langle\lambda_{2} \mid \lambda_{2}\right\rangle /\left\langle\lambda_{1} \mid \lambda_{2}\right\rangle\left\langle\lambda_{2} \mid \lambda_{1}\right\rangle=4 \neq 1$ which means those two vectors are linearly independent and therefore the according weight space has to be two-dimensional.

Therefore the (1,1)-representation is eight-dimensional and according the labeling scheme, it can be identified as the 8 -representation.


Figure 5: : Weight space of the (3,0)-representation in $T$ basis.

### 2.3.5 (3,0)-Representation

The rootstring diagram for the (3,0)-representation can be found in figure 5 .
To proof that each weight space is indeed one dimensional, one has to apply the same proof as used for the 6 -representation of $\mathfrak{s u}(3)$. Therefore the representation is ten-dimensional and will be labeled as the 10 -representation.

## 3 Flavor SU(3) in Particle Physics

To better understand how to relate the representation theory to particle physics, it is useful to review the history of particle physics. The next section gives a brief summary of the knowledge particle physicist had at the mid of the 20th century. To get an actual impression of the evolution of particle physics, I can recommend to read the first chapter in [Griffiths, 1987].

### 3.1 From Atoms to Quarks

While the first serious experiments of elementary particle physics started before the 20th century, Murray Gell-Mann (1964) was able to use the $S U(3)$-representation theory to explain the appearance and connection between many new particles. With the notion of quarks, Gell-Mann was able to reduce the large number of possible baryonic elementary particles to just three quarks ${ }^{4}$.

The main goal or idea of elementary particle physics is to describe all the matter one observes with just combinations of a finite set of such elementary particles.

Starting with the notion of atoms, going over electrons, protons and neutrons, one is now able to classify every massive object ${ }^{5}$ in three categories.

- Leptons (light particles) like the electron, muon and neutrinos

[^2]- Mesons (middle-weight) like the pion, the exchange particle of strong interactions
- Baryons (heavy particles) like the protons and neutrons

At the time Gell-Mann introduced his $S U(3)$-representation model of hadronic particles (mesons and baryons), the following "laws of nature" were observed during a large number of particle experiments.

- each particle which was classified as a lepton has a quantum number ${ }^{6} L=1$.
- each particle which was classified as a baryon has a quantum number $B=1$.
- each particle $A$ has an antiparticle $\bar{A}$ with the negative quantum numbers $-B$ or $-L$.
- during particle reactions and scattering processes, those two quantum numbers are always conserved.

One was able to do such classifications by the following idea (attributed to Richard Feynman) "whatever is not expressly forbidden is mandatory". As an example: the neutron can decay to a proton, electron and antineutrino ( $B=1$ and $L=0$ on both sides of (3.1)), while a proton (luckily) can not decay to a positron and a photon $(B=1$ and $L=0$ on the left-hand-side but $B=0$ and $L=1$ on the right-hand-side of (3.2)).

$$
\begin{align*}
& n \rightarrow p+e^{-}+\bar{\nu}_{e}  \tag{3.1}\\
& p \nrightarrow e^{+}+\gamma \tag{3.2}
\end{align*}
$$

Furthermore several new hadronic particles were discovered after the end of the second world war. In comparison to the already discovered particles, those particles were "strange". Though they were created as other hadronic particles, their live times was of several magnitudes longer. To keep track of the additional difference of hadronic particles, another conserved quantity was introduced. The so called Strangeness $S$.

At this point of time, more than thirty possible elementary particles were known, while half a century before, the number of those elementary particles were assumed to be two. If one reads of this chaotic period of particle physics one may hear the term the "particle zoo". The number of hadronic elementary particles was finally reduced to three essential ones by Gell-Mann in 1964.

The light Baryons

|  | Q |  |  |
| :---: | :---: | :---: | :---: |
| S | -1 | 0 | 1 |
| 0 |  | $n$ | $p$ |
| -1 | $\Sigma^{-}$ | $\Sigma^{-}, \Lambda$ | $\Sigma^{+}$ |
| -2 | $\Xi^{-}$ | $\Xi^{0}$ |  |

The heavy Baryons

|  | Q |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| S | -1 | 0 | 1 | 2 |
| 0 | $\Delta^{-}$ | $\Delta^{0}$ | $\Delta^{+}$ | $\Delta^{++}$ |
| -1 | $\Sigma^{*-}$ | $\Sigma^{* 0}$ | $\Sigma^{*+}$ |  |
| -2 | $\Xi^{*-}$ | $\Xi^{*+}$ |  |  |

The Mesons

|  | Q |  |  |
| :---: | :---: | :---: | :---: |
| S | -1 | 0 | 1 |
| 1 |  | $K^{0}$ | $K^{+}$ |
| 0 | $\pi^{-}$ | $\pi^{0}, \eta, \eta^{\prime}$ | $\pi^{+}$ |
| -1 | $K^{-}$ | $\bar{K}^{0}$ |  |

Figure 6: List of known hadrons at 1953. Organized by charge $Q$ and strangeness $S$. Furthermore baryons carry the baryon number $B=1$ while mesons have the baryon number $B=0$.

### 3.2 The Eightfold Way

In 1961, Gell-Mann introduced a geometrical scheme to order the already known hadronic particles. As one can see in figures 7 and 8 , one gets a geometric symmetrical array if one orders the strangeness in vertical and the charge in diagonal direction.

Doing so one can recognize that the so called baryon and meson octet look like the 8 -representation of $\mathfrak{s u}(3)$, while the baryon decuplet seems to correspond to the 10 -representation.
To finally connect the representation theory to the particle physics, one has to couple the strangeness $S$ and the charge $Q$ to the elements of the Cartan algebra $T_{3}$ and $T_{8}$.

By looking at the coordinates of the weights in the $T$-basis, one is able to find out that the scaling of

[^3]

Figure 7: The baryon and the meson octet.


Figure 8: The baryon decuplet.
the $T_{8}$ or $S$ direction for mesons is given by ${ }^{7}$

$$
\begin{equation*}
S=\frac{2}{\sqrt{3}} T_{8} \tag{3.3}
\end{equation*}
$$

The baryons somehow have an offset of one compared to the mesons. But knowing that the baryon number for baryons is equal to one, while mesons carry the number $B=0$, one can generalize the $T_{8}$ coordinate of hadrons by a simple shift to

$$
\begin{equation*}
T_{8}=\frac{\sqrt{3}}{2}(B+S) \tag{3.4}
\end{equation*}
$$

Since the $Q$-direction is oriented diagonal, to be more precise in $\left(-\alpha_{2}\right)$-direction, the $T_{3}$-coordinate of particles also depends on $T_{8}$. Finally the identification is given by

$$
\begin{equation*}
T_{3}=Q-\frac{B+S}{2}=Q-\frac{1}{\sqrt{3}} T_{8} \tag{3.5}
\end{equation*}
$$

The first successes of this model was that Gell-Mann actually predicted the existence and the mass of the not yet discovered $\Omega^{-}$particle.

But even more important this model was able to predict a set of even more elementary particles, the so called quarks, which are the generating set of all known hadronic particles.

### 3.3 Quarks and the Flavor SU(3)

As Gell-Mann has realized, the known hadronic particles correspond to representations of $\mathfrak{s u}(3)$. To be more precise the 10 and the 8 -representation. But why is one not able to find particles of the 6 or even the 3 -representation?

[^4]If there was the possibility of even more fundamental particles, each hadron has to be a combination of such elementary particles. This corresponds to the idea that an internal change of such elementary particle for a given hadron transforms this hadron into another hadron. As mentioned in the beginning, this is why $S U(N)$ seems to represent such internal rotations.
The only way to actual obtain these representation out of more fundamental representations is to build tensor products of smaller representations. To start, one can choose the 3 and the the $\overline{3}$-representation of $\mathfrak{s u}(3)$.


Figure 9: : The 3 and $\overline{3}$-representation of $\mathfrak{s u}(3)$ corresponding to quarks and anti-quarks.
There are several things one should mention here:

- Using equation (3.5) and (3.4) one can now compute the charge and strangeness of those quarks. But there is one decision one has to make before - are quarks baryons or not?
- Since quarks should also be able to explain baryons, we require that the coordinate system is shifted by the $B$-quantum number of quarks.
- If one chooses the $\mu_{1}-\alpha_{1}=(0,-1 / \sqrt{3})^{T}$ weight to have the strangeness-coordinate ${ }^{8} S=-1$, one obtains by using equation (3.4) that the baryonic offset of quarks is given by $B=1 / 3$. Thus, the strangeness for the other two weights is equal to zero. Therefore the $\left(\mu_{1}-\alpha_{1}\right)$-weight is named the strange quark.
- Indeed all the quantum numbers of the complex-conjugated representation are equal to minus the quantum numbers of the ordinary representation (also the baryon number). Therefore the complex-conjugated representation can be identified as the representation of the anti-quarks.
Summarizing this, the possible elementary particles should have the following attributes.

$$
\text { Quarks }-B=1 / 3
$$

| Flavor | Symbol | $Q$ | $S$ |
| :--- | :---: | ---: | ---: |
| up | $u$ | $2 / 3$ | 0 |
| down | $d$ | $-1 / 3$ | 0 |
| strange | $s$ | $-1 / 3$ | -1 |

$$
\text { Anti-Quarks }-B=-1 / 3
$$

| Flavor | Symbol | $Q$ | $S$ |
| :--- | :---: | ---: | :---: |
| anti-up | $\bar{u}$ | $-2 / 3$ | 0 |
| anti-down | $\bar{d}$ | $1 / 3$ | 0 |
| anti-strange | $\bar{s}$ | $1 / 3$ | 1 |

The different weights or particles are called flavors. This is the reason one calls this $S U(3)$-representation the flavor $S U(3)$.
To calculate the tensor spaces of the representation using Young-tableaux, one has introduce an ordering such that the weights correspond to a partition. One can realize this by requiring that the fundamental weight $\mu_{i}$ corresponds to

$$
\begin{equation*}
\mu_{i}=\sum_{n=1}^{i} \epsilon_{i} . \tag{3.6}
\end{equation*}
$$

To acknowledge that the trace of each generator is zero, the sum over all $\epsilon$ has to be zero as well. In this case $\epsilon_{1}+\epsilon_{2}+\epsilon_{3}=0$. Thus, the quark representations can be expressed by $\mu_{1}=\epsilon_{1}$ and $\mu_{2}=\epsilon_{1}+\epsilon_{2}$ which correspond to the following box diagrams.

[^5]

The first tensor products are given by


While the $3 \otimes 3$ corresponds to the $2 \epsilon_{1}=2 \mu_{1}=(2,0)=6$-representation and the $\overline{3}$-representation, the $\overline{3} \otimes 3$ corresponds to the $2 \epsilon_{1}+\epsilon_{2}=(1,1)=8$-representation and the $\epsilon_{1}+\epsilon_{2}+\epsilon_{3}=0$ which is a one-dimensional representation. Thus the result is

$$
\begin{equation*}
\overline{3} \otimes 3=8 \oplus 1 \tag{3.8}
\end{equation*}
$$

Accordingly $3 \otimes 3 \otimes 3$ actually corresponds to $(6 \oplus \overline{3}) \otimes 3=(6 \otimes 3) \oplus 8 \oplus 1$

$$
\begin{equation*}
6 \otimes 3=\square \square \square 1 \tag{3.9}
\end{equation*}
$$

which is nothing else than $10 \oplus 8$. Thus one finally gets

$$
\begin{equation*}
3 \otimes 3 \otimes 3=10 \oplus 8 \oplus 8 \oplus 1 \tag{3.10}
\end{equation*}
$$

Concluding this, the tensor product of $\overline{3} \otimes 3$, corresponding to the combination of a quark and a anti-quark, results in an octet and a singlet. This is exactly what could describe the nine already discovered mesons ( $\eta^{\prime}$ is the meson in the singlet). As one could verify, adding the quantum numbers of a quark and a anti-quark will always result in charge equal one ore zero, strangeness equal to one or zero and baryon number equal to zero. Thus one can also identify which meson is build out of which quark and anti-quark. Furthermore, since the complex conjugated representation of $\overline{3} \otimes 3$ is $3 \otimes \overline{3}$ which is the same representation, the complex-conjugated weights in the 8 -representation are their antiparticles, which was also discovered at this point of time. Also the $\eta$ and the $\pi^{0}$ mesons are their own antiparticle.

This also works for the baryons. A combination of three quarks has a baryon number of one, charge of minus one, zero, one and also two in the 10-representation and accordingly for the strangeness. Furthermore one is able to understand the meaning of each representation. The higher the dimension of the representation, the more excited the system. The singlet allows the $Q=0=B+S$ states, which means the only possible combination is given by the $u d s$-quarks - all quarks are different. In the octet, one is able to have the same quarks twice, while in the decuplet one can also have three times the same quark. The reason why the singlet state is not observed is due to quantum statistics ${ }^{9}$ which prohibit such a (not-excited) combination. The reason for having two 8 -representations can be understand as a consequence of creating the 8 out of the $6 \otimes 3$ and one out of $\overline{3} \otimes 3$. The main difference of these two 8 s are the symmetries under the exchange of quarks. While one 8 is symmetric under a few permutations, the other 8 is antisymmetric under a few permutations (see also chapter 2.7.2 of [Chýla, ]).

With the discovery of new particles, the number of quarks is presently six - which is the same number of the present elementary leptons. To explain further elementary processes, one has introduced the color $S U(3)$ for quarks, which let each quark appear in three different versions - reg, green and blue (this has nothing to do with the "visual" color of quarks). Including the Higgs-mechanism, the theory of colored quarks, the so called Quantum Chromo Dynamics, combined with the theory for Leptons, the electroweak Yang-Mills theory, form the so called Standard Model of particle physics.

[^6]
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[^0]:    ${ }^{1}$ For relativistic theories one introduces the Lagrange function for fields. Since fields are also dependent on spacial coordinates the Lagrange function therefore becomes an integral over the Lagrange density which is the equivalent to the classical Lagrange function of trajectories.

[^1]:    ${ }^{2}$ One should not confuse this with the adjoint-representation of a matrix for a given commutator $[\cdot, \cdot]$.
    ${ }^{3}$ In general, the dimension of $S U(N)=N^{2}-1$.

[^2]:    ${ }^{4}$ To describe further particles, three additional quarks where introduced later. But the idea stays the same
    ${ }^{5}$ Note that current theories are not able to explain the so called dark matter. This may be due to the fact that, currently, one is not able to connect curved space time theories to the standard model and make actual predictions.

[^3]:    ${ }^{6}$ In principal one has even three different Lepton numbers depending on the "generation" of Leptons, but since the Gell-Mann model deals with hadronic particles, this is not relevant for this paper.

[^4]:    ${ }^{7}$ Strange particles do not want to be proper normalized.

[^5]:    ${ }^{8}$ As one can see later on, one actually chooses $B$ to be $1 / 3$, since baryons are build out of three quarks and therefore have the baryon number one.

[^6]:    ${ }^{9}$ Quarks must obey the Fermi statistics which require the tensor state of flavor and spin to be totally antisymmetric under the exchange of particles. Since the spin component for this state can only be symmetric, the flavor part has to be antisymmetric. This can only be achieved if the flavor state is zero. The symmetric excited flavor states are allowed. See also chapter 11.4 of [Georgi, 1999].

