Gravomagnetism in special relativity

H. Kolbenstvedt
Department of Physics, University of Trondheim, Dragvoll N-7055, Norway

(Received 30 March 1987; accepted for publication 1 September 1987)

The existence of a gravomagnetic field from a uniformly moving mass is predicted from special relativity and gravitational time dilation. Except for a factor of one-half, the field is identical to the field following from the linearized theory of gravitation.

I. INTRODUCTION

The Maxwellian features of the Einstein equations in the linear weak field approximation are demonstrated in several textbooks on general relativity. As a consequence, it is possible to introduce the analog of the magnetic field, the gravomagnetic field. This field provides a simple way for the discussion of many interesting effects, like geodesic deviation of spinning particles, precession of gyroscopes orbiting the Earth, and "dragging" of inertial frames by rotating masses, by leaning on well-known effects from classical electromagnetism and atomic physics involving spin–orbit and spin–spin coupling. These applications of the gravomagnetic field are rarely discussed in the literature; an exception is Rindler's book,[1] but even here there is only a short final paragraph in the chapter on general relativity.

For the benefit of undergraduate students, or students not taking courses in general relativity, Bedford and Krumm[2] have demonstrated the existence of the gravomagnetic field from a moving infinite line of constant mass density, using arguments from special relativity only, in close analogy to the derivation of the magnetic field from a straight current found in several recent textbooks on electromagnetism. With the same audience in mind, we here "derive" the gravomagnetic field from a point mass moving with a constant linear velocity much smaller than the velocity of light, using arguments mainly from special relativity. In contrast to the infinite line field, this point-mass field can by superposition give the field from, e.g., an extended rotating body (star), and is thus of considerable practical interest.

II. THE FIELDS

To perform our derivation, we need one piece of extra information. A clock situated in a weak gravitational potential $\phi \ll 1$ is slowed down (ϕ is assumed negative) by a factor of

$$S = 1 + \phi\,, \tag{1}$$

compared to a clock situated where $\phi = 0$ (we use units where the velocity of light is unity). This effect of gravitational time dilation is usually considered to be an effect of general relativity, but can be derived from a falling elevator thought experiment and requires only the principle of equivalence combined with the Doppler formula.[1]

We also make the assumption that the slowing down of clocks is the only effect of the gravitational potential. This assumption cannot be justified from special relativity and is really part of the basic postulate of general relativity (where the potential modifies measuring rods as well).

Consider an object of mass M moving with a constant velocity $V \ll 1$ along the positive x axis of the laboratory frame of reference, its position at time t being Vt.

A small test particle of mass m is moving under the influence of the gravitational field from M.

In the rest frame of the source particle M, the motion of the test particle is determined by the variation principle

$$\delta \int (-m\,ds) = \delta \int \mathscr{L}_0\, dt_0 = 0\,, \tag{2}$$

where ds is the four-dimensional line element and indices "0" indicate the rest frame.

The integrand is given by

$$\mathscr{L}_0\, dt_0 = -m[(1 + 2\phi_0)dt_0^2 - dr_0^2]^{1/2} \tag{3}$$

and is, apart from the factor of $(-m)$, just the free particle line element modified by Eq. (1) to account for the effect of the potential

$$\phi_0 = -GM/r_0\,. \tag{4}$$

We now perform a Lorentz transformation to the laboratory frame. Disregarding terms of relative order V^2, the transformation equations are

$$x_0 \approx x - Vt, \quad y_0 = y, \quad z_0 = z, \quad t_0 \approx t - Vx\,. \tag{5}$$

Special relativity cannot predict the transformation properties of ϕ but, whether it transforms as a scalar, a component of a four vector, or as a tensor, we assume that it is at least approximately invariant under the low-velocity transformation (5). The line element ds, however, is invariant, and the transformed Eq. (3) becomes (when $\phi \ll 1$):

$$\mathscr{L}\, dt = -m[(1 + 2\phi)(dt - V\,dx)^2 - (dx - V\,dt)^2 - dy^2 - dz^2]^{1/2}\,, \tag{6}$$

where now, of course, $\phi = \phi(x,y,z,t)$. Neglecting terms containing V^2 and dividing by dt, we find the Lagrangian

$$\mathscr{L} = -m[1 - v^2 + 2\phi - 4\phi \mathbf{V}\cdot\mathbf{v}]^{1/2}\,, \tag{7}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the test particle velocity in the laboratory frame.

From here on, we restrict ourselves to

$$V \ll v \ll 1\,,$$
$$\phi \sim v^2 \ll 1\,, \tag{8}$$

i.e., the case where the source particle moves very slowly compared to the now nonrelativistic test particle, and the kinetic and potential energies of the test particle are of the same orders of magnitude (common in bound systems).

A series expansion of the Lagrangian of Eq. (7) in the small quantities of Eq. (8) gives

$$\mathscr{L} \approx \tfrac{1}{2}mv^2 - m\phi + 2m\phi \mathbf{V}\cdot\mathbf{v}\,, \tag{9}$$

where we have omitted a trivial constant term of $(-m)$

and neglected relativistic corrections to the standard nonrelativistic kinetic and potential energies.

We now compare this result with the familiar Lagrangian for a nonrelativistic particle of mass m and charge q moving in an external electromagnetic field with potentials U and $\mathbf{A}$.

$$\mathscr{L}_{em} = \tfrac{1}{2}mv^2 - qU + q\mathbf{A}\cdot\mathbf{v}\,. \quad (10)$$

We see that Eqs. (9) and (10) have exactly the same form and that we have the formal correspondence

$$qU \leftrightarrow m\phi\,, \qquad q\mathbf{A} \leftrightarrow 2m\phi\mathbf{V}\,, \quad (11)$$

between the interaction terms.

The existence of an analog to the vector potential immediately implies the gravitational analog of the magnetic field

$$\mathbf{b} = \nabla\times(2\phi\mathbf{V}) = -2\mathbf{V}\times(\nabla\phi)\,. \quad (12)$$

Expressed by the potential ϕ, we write the "gravielectric" and "gravimagnetic" fields,

$$\mathbf{g} = -\nabla\phi\,, \qquad \mathbf{b} = 2(\mathbf{V}\times\mathbf{g})\,. \quad (13)$$

Since retardation effects are unimportant in our approximation, $\mathbf{g}$ takes the standard form

$$\mathbf{g} = -(GM/r_0^2)\hat{\mathbf{r}}_0\,. \quad (14)$$

We see that except for the factor of 2 in $\mathbf{b}$, the fields from a moving mass M are formally identical to the electromagnetic fields from a moving charge of magnitude $q = GM$.

The equation of motion of the test particle becomes

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} + (\mathbf{v}\times\mathbf{b})\,, \quad (15)$$

where we have omitted an "induction" term (containing V^2) analogous to $\partial\mathbf{A}/\partial t$ in electromagnetism.

III. COMPARISON WITH GR

In the full linearized theory of gravitation, the rest system Lagrangian is given by[3]

$$\mathscr{L}_0 = -m[(1+2\phi_0)dt_0^2 - (1-2\phi_0)dr_0^2]^{1/2} \quad (16)$$

differing from Eq. (3) by the space curvature factor of $(1-2\phi_0)$. Repeating our calculations with this Lagrangian, we find that the effects of space curvature and gravitational time dilation contribute equally to the gravomagnetic field. The second part of Eq. (13) must be modified accordingly to

$$\mathbf{b} = 4(\mathbf{V}\times\mathbf{g})\,. \quad (17)$$

Since there is no reliable way of introducing the space curvature from special relativity, the result (17) must strictly be considered as a consequence of general relativity. However, special relativity, combined with the principle of equivalence, and extended by the postulate following Eq. (1), can predict the existence and the qualitative features of the gravomagnetic effect. The situation is, in this sense, completely analogous to the case of light deflection by a mass, where special relativity explains only one-half of the deflection.

As mentioned in the introduction, the concept of the gravomagnetic field provides an elementary method for a qualitative discussion of many of the classical effects of gravitation. One should also pay attention to the fact that Eq. (9), modified by a factor of 2 in the last term, is the starting point for recent speculations on quantization of gravomagnetic flux and on the existence of so-called gravitopoles. The gravitopoles imply the quantization of mass, much in the same way as their electromagnetic counterparts, the magnetic monopoles, imply quantization of charge.[4]

[1]W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977), 2nd ed., pp. 188–192.
[2]D. Bedford and P. Krumm, Am. J. Phys. **53**, 889 (1985).
[3]H. C. Ohanian, *Gravitation and Spacetime* (Norton, New York, 1976), p. 208.
[4]A. Zee, Phys. Rev. Lett. **55**, 2379 (1985).

Experiments with Fourier transforms at radio frequencies

Carl A. Kocher
Department of Physics, Oregon State University, Corvallis, Oregon 97331

(Received 18 June 1987; accepted for publication 18 September 1987)

Quantitative experiments in spectrum analysis can be performed with a radio receiver and a nonmonochromatic signal source. Construction details are given for a generator of interrupted radio-frequency oscillations in a repeated sequence of N equally spaced bursts, where $N = 1, 2, 3, \ldots$. The resulting continuous spectrum is analogous to the spatial distribution of light in Fraunhofer diffraction by an N-slit grating. Variation of the burst parameters illustrates the frequency-time uncertainty relation $\Delta\nu\Delta t \approx 1$ (for $N = 1$) and, more generally, the correspondence between time-dependent signals and their Fourier transforms.

I. INTRODUCTION

The Fourier integral or transform, representing a superposition of harmonic oscillations or waves, is a valuable tool in the study of linear systems, in quantum mechanics, and in the analysis of acoustical and optical spectra. A Fourier integral is interesting because it can reveal the spectrum of a time-dependent signal, showing, for exam-

