Post-Newtonian Theory

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1 Introduction

In the universe, two inspiralling compact objects will steadily lose their gravitational binding energy by emission of gravitational radiation, which can be detectd nowadays by LIGO and VIRGO. The gravitational waves from this kind of binaries, like neutron stars binaries or black holes binaries, are also expected to be detectable by LISA in the future. These inspiralling binaries systems are very relativistic and require a high-order post-Newtonian correction as their orbital velocities as high as 0.5c in the last rotations.

Recent studies [2] have demonstrated that the results correspond grossly with the post-Newtonian precision which is required to implement successively the optimal filtering technique in the LIGO/VIRGO detectors.

As large number of orbital rotations are monitored by detectors in their frequency bandwidth, the orbital phase of the binary can be measured accurately. The post-Newtonian approximation is valid under the assumptions of a weak gravitational field inside the source, and of slow internal motions. When assuming a source is post-Newtonian, it mainly means that the source is at once slowly moving and weakly stressed. This article will show how we come to the post-Newtonian expansion and present some important mathematical equations with the proof.

2 Post-Newtonian Sources

2.1 Einstein Field Equations

Consider the Einstein's Equation coupled to matter: $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$. Because $T_{\mu\nu}$ doesn't vanish, we need to keep the metric perturbation. Define the gravitational-field amplitude:

$$h^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} - \eta^{\alpha\beta} \tag{1}$$

where g the determinant of the covariant metric, $g = \det(g_{\alpha\beta})$ and $\eta^{\alpha\beta}$ represents an auxiliary Minkowskian metric.

Using the conservation of the matter tensor, we will get four equations, which are called the harmonic-coordinate condition:

$$\partial_\mu h^{\alpha\mu} = 0 \tag{2}$$

Plugging (1) into the Einstein's tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R$ and using (2), the Einstein field equations in harmonic coordinates can be written in the form of inhomogeneous flat d'Alembertian equations:

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta} \tag{3}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, is the d'Alembert operator, and $\tau^{\alpha\beta} = |g|T^{\alpha\beta} + \frac{c^4}{16\pi G}\Lambda^{\alpha\beta}$. $\Lambda^{\alpha\beta}$ is the gravitational source term, whose exact expression, including all non-linearities, is[2]

$$\begin{aligned}
\Lambda^{\alpha\beta} =& - h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda} \\
&- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu} \\
&+ \frac{1}{8}(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu})(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi})\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon} \\
&= N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + L^{\alpha\beta}(h,h,h,h) + \mathcal{O}(h^5)
\end{aligned} \tag{4}$$

Also, we have $\partial_\mu\tau^{\alpha\mu} = 0$. Outside the domain of the source, the gravitational source term also has the property: $\partial_\mu\Lambda^{\alpha\mu} = 0$.

Assume that the gravitational field is independent of time (stationary) in some remote past and the metric was stationary in the past, which means

$$\begin{aligned}
&\frac{\partial}{\partial t}h^{\alpha\beta}(\mathbf{x},t) = 0 \\
&\lim_{r\to\infty} h^{\alpha\beta}(x,t) = 0
\end{aligned} \tag{5}$$

when t is smaller than some finite negative constant $-T$, ensuring that the matter source is isolated from the rest of the Universe and does not receive any radiation from infinity.

The solution of Eq.(3) can be obtained by using a Green function

$$\Box_x G(x^\mu - x'^\mu) = \delta^{(4)}(x^\mu - x'^\mu) \tag{6}$$

where $\Box_x$ denotes the d' Alembertian with respect to coordinate x^μ, because the general solution of Eq.(3) can be written in the form as

$$h^{\alpha\beta}(x^\mu) = -\frac{16\pi G}{c^4}\int G(x^\mu - x'^\mu)\tau^{\alpha\beta}(x'^\mu)d^4x' \tag{7}$$

As the waves travel towards in time, we use the retarded Green function, which is given by

$$G(x^\mu - x'^\mu) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(|\mathbf{x}-\mathbf{x}'| - (x^0 - x'^0))\theta(x^0 - x'^0) \tag{8}$$

Here the boldface is the space vectors $\mathbf{x} = (x^1, x^2, x^3)$.

Plugging (8) into (7), the Einstein differential field equations (3) can be written equivalently into the form of the integro-differential equations:

$$h^{\alpha\beta} = \frac{16\pi G}{c^4}\Box^{-1}_{ret}\tau^{\alpha\beta} \tag{9}$$

where the usual retarded inverse d'Alembertian operator is defined as

$$(\Box^{-1}_{ret}f)(\mathbf{x}, t) = -\frac{1}{4\pi}\int\int\int \frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}f(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c) \tag{10}$$

2.2 post-Newtonian expansion

Now we have the Einstein's Equations, but the problem is that it is difficult to solve the equation exactly and analytically. Only a little special cases which have many geometric symmetries allow us to get exact solution. Currently, scientists use perturbation to get the approximate solutions, one of which is the post-Newtonian expansion.

The post-Newtonian expansion expands the motion of matter or metric field configuration in powers of 1/c. First, let's assume that the matter source is expanded as

$$T^{\mu\nu} = \sum_{m}^{+\infty}\frac{1}{c^n}T^{\mu\nu}_{(n)} \tag{11}$$

Since we are talking about post-Newtonian source (slow and weak self gravitation), metric field is generated from very slow matter. As $v/c \ll 1$, which indicates that the matter source is non-relativistic, the metric field is given by instant potential field as Newton gravity. The change of metric field in time is smaller than spatial gradient of metric field due to the slow matter, making the d'Alembert operator become a Laplacian operator.

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \nabla^2(1 + \mathcal{O}(v/c)^2) \tag{12}$$

Moreover, it is natural to think that the retarded effect is also very small, so post-Newtonain expansion includes

$$\overline{F(t - \frac{r}{c})} = F(t) - \frac{r}{c}F^{(1)}(t) + \frac{r^2}{2c^2}F^{(2)}(t) + ... \tag{13}$$

From the above equation, we can find that the radius r is required to be small for the validity of post-Newtonian expansion, which the near zone is te region surrounding the source of small extent with respect to the wavelength of the emitted radiation: $r \ll \lambda$.

Then we do the expansions in the powers of c in the region where r is small

$$\begin{aligned} h^{\mu\nu} &= \sum_{n=2}^{+\infty} \frac{1}{c^n} h^{\mu\nu}_{(n)} \\ \tau^{\mu\nu} &= \sum_{n=-2}^{+\infty} \frac{1}{c^n} \tau^{\mu\nu}_{(n)} \end{aligned} \tag{14}$$

Put these expansions into Eq.(3), and then equate terms with the same power of c, we will get a recursive series of Poisson-type equations

$$\nabla^2 h^{\mu\nu}_{(n)} = 16\pi G \tau^{\mu\nu}_{(n-4)} + \partial_t^2 h^{\mu\nu}_{(n-2)} \tag{15}$$

The solution, with general homogeneous part requring only non-singular term at $r = 0$, can be expressed as[6][10]

$$h^{\mu\nu}_{(n)} = 16\pi G \Delta^{-1} \tau^{\mu\nu}_{(n-4)} + \partial_t^2 \Delta^{-1} \tau^{\mu\nu}_{(n-2)} + \sum_{l=0}^{+\infty} B^{\mu\nu}_{(n),L}(t) \hat{x}_L \tag{16}$$

where $\Delta^{-1}\rho(x) = -\frac{1}{4\pi} \int \frac{d^3x'}{|x-x'|} \rho(x')$ is the Poisson Integration. After proceeding iteration, we will get

$$h^{\mu\nu}_{(n)} = 16\pi G \sum_{k=0}^{n/2-1} \partial_t^{2k} \Delta^{-k-1} \tau^{\mu\nu}_{(n-4-2k)} + \sum_{l=0}^{+\infty} \sum_{k=0}^{n/2-1} \partial_t^{2k} B^{\mu\nu}_{(n-2k),L} \Delta^{-k} \hat{x}_L \tag{17}$$

Put (17) into (14), we get

$$
\begin{aligned}
h^{\mu\nu} &= \sum_{n=2}^{+\infty} \frac{1}{c^n} h^{\mu\nu}_{(n)} \\
&= 16\pi G \sum_{n=2}^{+\infty} \frac{1}{c^n} \sum_{k=0}^{n/2-1} \partial_t^{2k} \Delta^{-k-1} \tau^{\mu\nu}_{(n-4-2k)} + \sum_{l=0}^{+\infty} \sum_{n=2}^{+\infty} \sum_{k=0}^{n/2-1} \frac{1}{c^n} \partial_t^{2k} B^{\mu\nu}_{(n-2k),L} \Delta^{-k} \hat{x}_L \\
&= 16\pi G \sum_{k=0}^{+\infty} \sum_{n=2+2k}^{+\infty} \frac{1}{c^n} \partial_t^{2k} \Delta^{-k-1} \tau^{\mu\nu}_{(n-4-2k)} + \sum_{k=0}^{+\infty} \sum_{n=2+2k}^{+\infty} \sum_{l=0}^{+\infty} \frac{1}{c^n} \partial_t^{2k} B^{\mu\nu}_{(n-2k),L} \Delta^{-k} \hat{x}_L \\
&= 16\pi G \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{1}{c^{(n+2k+2)}} \partial_t^{2k} \Delta^{-k-1} \tau^{\mu\nu}_{(n-2)} + \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \frac{1}{c^{n+2k+2}} \partial_t^{2k} B^{\mu\nu}_{(n+2),L} \Delta^{-k} \hat{x}_L \\
&= \frac{16\pi G}{c^4} \sum_{k=0}^{+\infty} (\frac{\partial}{c\partial t})^{2k} \Delta^{-k-1} (\sum_{n=0}^{+\infty} \frac{1}{c^{(n-2)}} \tau^{\mu\nu}_{(n-2)}) + \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} (\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}}) \Delta^{-k} \hat{x}_L \\
&= \frac{16\pi G}{c^4} \sum_{n=0}^{+\infty} \frac{1}{c^{(n-2)}} \Box^{-1}_{inst} \tau^{\mu\nu}_{(n-2)} + \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} (\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}}) \Delta^{-k} \hat{x}_L \\
&= \frac{16\pi G}{c^4} \sum_{n=-2}^{+\infty} \frac{1}{c^n} \Box^{-1}_{inst} \tau^{\mu\nu}_{(n)} + \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} (\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}}) \Delta^{-k} \hat{x}_L \\
&= \frac{16\pi G}{c^4} \Box^{-1}_{inst} \tau^{\mu\nu} + \sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} (\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}}) \Delta^{-k} \hat{x}_L
\end{aligned}
\tag{18}
$$

where

$$
\Box^{-1}_{inst} = \sum_{k=0}^{+\infty} (\frac{\partial}{c\partial t})^{2k} \Delta^{-k-1} \tag{19}
$$

The homogeneous part with the function $B^{\mu\nu}_{(n)}(t)$ is not in the best form for the purpose. So a new function $A^{\mu\nu}_L(t)$ is introduced. we change the variable B to A by

$$
\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}} = -\frac{\partial_t^{2l+1} A^{\mu\nu}_L(t)}{c^{2l+1}(2l+1)!!} \tag{20}
$$

Using the identity[1]

$$
\Delta^{-k}(\hat{x}_L) = \frac{(-)^l}{l!} \frac{(2l+1)!!}{(2l)!!(2l+2k+1)!!} r^{2k+l} \hat{n}_L \tag{21}
$$

and

$$\hat{\partial}_L = \sum_{k=0}^{l/2} (-)^k \frac{(2l-2k-1)!!}{(2l-1)!!} \delta_{\{i_1 i_2 \ldots} \delta_{i_{2k-1} i_{2k} \ldots} \partial_{i_{2k+1} \ldots i_l\}} \Delta^k$$
$$\hat{\partial}_L (r^\lambda) = \hat{n}_L r^{\lambda - l} \frac{\lambda!!}{(2l)!!}, (\text{if } \lambda \geq 2l, \text{ otherwise } 0), \tag{22}$$

we have

$$\sum_{l=0}^{+\infty} \sum_{k=0}^{+\infty} \frac{1}{c^{2k}} \partial_t^{2k} (\sum_{n=0}^{+\infty} \frac{B^{\mu\nu}_{(n+2),L}}{c^{n+2}}) \Delta^{-k} (\hat{x}_L) = - \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{\partial_t^{2k+1} A^{\mu\nu}_L(t)}{c^{2k+1}} \frac{\hat{\partial}_L r^{2k}}{(2k+1)!} \tag{23}$$

By Taylor expansions, we know that

$$\begin{aligned} A^{\mu\nu}_L(t - \frac{r}{c}) - A^{\mu\nu}_L(t + \frac{r}{c}) &= \sum_{k=0}^{+\infty} \frac{1}{k!} (-)^k (\frac{r}{c})^k \partial_t^k A^{\mu\nu}_L(t) - \sum_{k=0}^{+\infty} \frac{1}{k!} (\frac{r}{c})^k \partial_t^k A^{\mu\nu}_L(t) \\ &= -2 \sum_{k=0}^{+\infty} \frac{1}{(2k+1)!} (\frac{r}{c})^{2k+1} \partial_t^{2k+1} A^{\mu\nu}_L(t) \end{aligned} \tag{24}$$

as only odd terms remain.

Then the solution (18) will become

$$\begin{aligned} h^{\mu\nu} &= \frac{16\pi G}{c^4} \Box^{-1}_{inst} \tau^{\mu\nu} - \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{\partial_t^{2k+1} A^{\mu\nu}_L(t)}{c^{2k+1}} \frac{\hat{\partial}_L r^{2k}}{(2k+1)!} \\ &= \frac{16\pi G}{c^4} \Box^{-1}_{inst} \tau^{\mu\nu} - \frac{4G}{c^4} \sum_{l=0}^{\infty} \frac{(-)^l}{l!} \hat{\partial}_L [\frac{A^{\mu\nu}_L(t - \frac{r}{c}) - A^{\mu\nu}_L(t + \frac{r}{c})}{2r}] \end{aligned} \tag{25}$$

Even though the post-Newtonian approximation works well in the domain of its validity, it does not work at far region. So one need to develop new construction of approximate expression of exterior field, which leads to the post-Minkoswkian approximation. Usually this method equips multipole expansion, so it is also called MPM expansion.

2.3 Post-Minkowskian Approximation

In the vacuum region outside the compact-support source, the field equations (2) and (3) can be solved in the form of a formal non-linearity or post-Minkowskian expansion, considering the field variable $h^{\alpha\beta}$ as a non-linear metric perturbation of Minkowski space-time. We don't need to consider any physical constraints of matter. Expand the exact solution

$$h^{\alpha\beta}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\alpha\beta}_{(n)} \tag{26}$$

At the linearized level (or first-post-Minkowskian approximation), we wirte:

$$h^{\alpha\beta}_{ext} = G h^{\alpha\beta}_{(1)} + \mathcal{O}(G^2) \tag{27}$$

where the subscript "ext" means that the solution is valid only in the exterior of the source. $h^{\alpha\beta}_{(1)}$ is the linear coefficient with the dimension of the inverse of G. In vacuum, it satisfies

$$\Box h^{\alpha\beta}_{(1)} = 0 \tag{28}$$

$$\partial_\mu h^{\alpha\mu}_{(1)} = 0 \tag{29}$$

2.3.1 General solution of the linearized vacuum eqautions

From (28), we can obtain

$$(-\frac{1}{c^2}\partial_t^2 + \partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2})h^{\alpha\beta}_{(1)}(r,t) = 0 \tag{30}$$

Using the retarded and advanced time variables, $u = t - \frac{r}{c}$ and $v = t + \frac{r}{c}$, define

$$f(u,v) = (v-u)^{-l} h^{\alpha\beta}_{(1)}(r,t) \tag{31}$$

Then Eq.(30) can be transformed into

$$[(v-u)\partial_{uv} + (l+1)\partial_u - (l+1)\partial_v]f = 0 \tag{32}$$

where $\partial_{uv} = \frac{\partial^2}{\partial_u \partial_v}$.

This equation is called Euler-Poisson-Darboux equation $E_{m,n}$, defined as

$$E_{m,n}(f) = [(v-u)\partial_{uv} + (l+1)\partial_u - (l+1)\partial_v]f = 0 \tag{33}$$

It can easily proved by checking both sides that

$$\partial_u E_{m,n}(f) = E_{m,n+1}(\partial_u f) \tag{34}$$

Thus, if f is the solution of $E_{m,n}$, then ∂_u is a solution of $E_{m,n+1}$, which means that if the general solution of $E_{m,n}$ is known as f, then the general solution of $E_{m,n+1}$ can also be obtained. Also, we can get the general solution $E_{m+1,n}$ by exchanging u and v, m and n.

First, consider

$$
\begin{aligned}
E_{1,1} &= [(v-u)\partial_{uv} + \partial_u - \partial_v]f \\
&= \partial_{uv}[(v-u)f] = 0
\end{aligned}
\tag{35}
$$

So, the general solution of $E_{1,1}$ is

$$
f_{1,1} = \frac{U(u)+V(v)}{v-u} \tag{36}
$$

where $U(u)$ and $V(v)$ are arbitary functions.

As long as we have the general solution of $E_{1,1}$, by using (34), we can have the general solution of $E_{l+1,l+1}$[1]

$$
f = \frac{2}{l!c^{l+1}} \frac{\partial^{2l}}{\partial u^l \partial v^l}[\frac{U(u)+V(v)}{v-u}] \tag{37}
$$

where U and V are arbitary functions of order c^{l+1} and here introduces a the factor $\frac{2}{l!c^{l+1}}$ for later convenience. Then back to the definition of f, Eq.(31), we will get the general solution of $h^{\alpha\beta}_{(1)}(r,t)$:

$$
\begin{aligned}
h^{\alpha\beta}_{(1)_L} &= \frac{2}{l!c^{l+1}}(v-u)^l \frac{\partial^{2l}}{\partial u^l \partial v^l}[\frac{U^{\alpha\beta}_L(u)+V^{\alpha\beta}_L(v)}{v-u}] \\
&= \frac{(-)^l}{2^l} \sum_{j=0}^{l} \frac{2^j(2l-j)!}{j!(l-j)!} \frac{U^{(j)\alpha\beta}_L(r-r/c)+(-)^j V^{(j)\alpha\beta}_L(t+r/c)}{c^j r^{l-j+1}}
\end{aligned}
\tag{38}
$$

Using the expression of $\hat{\partial_L}$, we will have[1]

$$
\hat{n}^L h^{\alpha\beta}_{(1)_L} = \hat{\partial}_L[\frac{U^{\alpha\beta}_L(r-r/c)+V^{\alpha\beta}_L(t+r/c)}{r}] \tag{39}
$$

where $\hat{\partial_L}$ denotes the tracefree part of $\partial_{i_1 i_2 \ldots i_l} = \partial_{i_1}\partial_{i_2} \ldots \partial_{i_l}$.

From $\hat{\partial_L} f(r) = \hat{n}^L (2r)^l (\frac{\partial}{\partial(r^2)})^l f(r)$[1], and $\partial_t h^{\alpha\mu}_{(1)} = 0 (t \leq -T)$, we can know that for $t \leq -T$, $\partial_t U^{\alpha\beta}_L(r-r/c) + \partial_t V^{\alpha\beta}_L(r+r/c)$ must be an odd polynomial in r of maximum degree $2l-1$, with coeffient depending on t. So, for $t < -T$, we can write

$$
\partial_t U^{\alpha\beta}_L(r-r/c) + \partial_t V^{\alpha\beta}_L(r+r/c) = \sum_{j=1}^{2l-1} jC_j[(t-r/c)^j - (t+r/c)^j] \tag{40}
$$

Separtating $\partial_t U(t-r/c)$ and $\partial_t V(t+r/c)$, we will get that, for $r < -T$

$$
\begin{aligned}
U_L^{\alpha\beta}(t-r/c) &= A + \sum_{j=0}^{2l-1} C_j (t-r/c)^{j+1} = A + \sum_{j=1}^{2l+1} C_j (t-r/c)^j \\
V_L^{\alpha\beta}(t+r/c) &= B + \sum_{j=0}^{2l-1} C_j (t+r/c)^{j+1} = B - \sum_{j=1}^{2l+1} C_j (t+r/c)^j
\end{aligned}
\tag{41}
$$

From $\hat{\partial}_L(r^\lambda) = \hat{n}_L r^{\lambda-l}\frac{\lambda!!}{(2l)!!}$, (if $\lambda \geq 2l$, otherwise 0) (Eq.(22)), we have

$$
\hat{\partial}_L(r^{2j}) = 0, (j = 0, 1, 2, ..., l-1) \tag{42}
$$

Also we know that in $(t+r)^i - (t-r)^i$, only odd terms of r remain, where $u = t - r, v = t + r$. Thus we have

$$
\hat{\partial}_L\left(\frac{(t+r)^i - (t-r)^i}{r}\right) = 0, (i = 0, 1, 2, ..., 2i) \tag{43}
$$

Therefore, when $t \leq -T$, only the term with degree $2l+1$ remains, which will give

$$
\hat{n}_L h_{(1)L}^{\alpha\beta} = \hat{\partial}_L[(A+B)/r - 2(C_{2l+1}/c^{2l+1})r^{2l}] \tag{44}
$$

Now consider $\lim_{r\to\infty} h_{(1)}^{\alpha\beta}(x,t) = 0$, which ensures that C_{2l+1} must be zero. As $C_{2l+1} = 0$ and $\hat{\partial}_L(\frac{(t+r)^i-(t-r)^i}{r}) = 0, (i = 0, 1, 2, ..., 2i)$, the value of $h_{(1)L}^{\alpha\beta}$ will not change even if the function V changes as below

$$
V_L^{\alpha\beta} = B - \sum_{j=1}^{2l} C_j (t-r/c)^j \tag{45}
$$

Then the general soluton of $h_{(1)}^{\alpha\beta}$ can be represented as

$$
h_1^{\alpha\beta} = \sum_l \hat{\partial}_L\left(\frac{U_L^{\alpha\beta}(t-r/c)}{r}\right) \tag{46}
$$

We will find that evidently a monopolar wave satisfies $\Box(\frac{U_L(u)}{r}) = 0$.

Following Ref.[11] to algebraically decompose $U_L^{\alpha\beta}$ and then construsting new functions $A(u), ..., J(u)$, $h_{(1)}^{\alpha\beta}$ can be reprensented as [1]

$$
\begin{aligned}
h_1^{00} =& \sum_{l\geq 0} \partial_L(r^{-1}A_L(u)) \\
h_1^{0i} =& \sum_{l\geq 0} \partial_{iL}(r^{-1}B_L(u)) + \sum_{l\geq 1}[\partial_{L-1}(r^{-1}C_{iL-1}(u)) + \epsilon_{iab}\partial_{aL-1}(r^{-1}D_{bL-1}(u))] \\
h_1^{ij} =& \sum_{l\geq 0}[\partial_{ijL}(r^{-1}E_L(u)) + \delta_{ij}\partial_L(r^{-1}F_L(u))] \\
&+ \sum_{l\geq 1}[\partial_{L-1(i}(r^{-1}G_{j)L-1}(u)) + \epsilon_{ab(i}\partial_{j)aL-1}(r^{-1}H_{bL-1}u))] \\
&+ \sum_{l\geq 2}[\partial_{L-2}(r^{-1}I_{ijL-2}(u)) + \partial_{aL-1}(r^{-1}\epsilon_{ab(i}J_{j)bL-2}(u))]
\end{aligned}
\tag{47}
$$

Eq.(29), the harmonicity condition gives algebraic and differential constraints among $A, B, ...$ and J. To express the constraints simply, define

$$
\begin{aligned}
(l\geq 0)M_L(u) &= A_L + 2B_L^{(1)} + E_L^{(2)} + F_L \\
(l\geq 1)S_L(u) &= -D_L - \frac{1}{2}H_L^{(1)} \\
(l\geq 0)W_L(u) &= B_L + \frac{1}{2}E_L^{(1)} \\
(l\geq 0)X_L(u) &= \frac{1}{2}E_L \\
(l\geq 0)Y_L(u) &= -B_L^{(1)} - E_L^{(2)} - F_L \\
(l\geq 1)Z_L(u) &= \frac{1}{2}H_L
\end{aligned}
\tag{48}
$$

where $F^{(n)}(u)$ denotes d^nF/du^n. Then the constrains are [1]

$$
\begin{aligned}
M_L^{(1)} =& 0 \\
M_L^{(2)} =& 0 \\
S_L^{(1)} =& 0
\end{aligned}
\tag{49}
$$

and

$$
\begin{aligned}
C_L &= M_L^{(1)} - Y_L^{(1)} \\
G_L &= 2Y_L \\
I_L &= M_L^{(2)} \\
J_L &= 2S_L^{(1)}
\end{aligned}
\tag{50}
$$

The general solution of $h_{(1)}^{\alpha\beta}$ can also be expressed uniquely in terms of M, S, W, X, Y and Z, which satisfy the above constraints. In previous studies[2], the most general solution of the linearized field equations (28, 29) has been found in the form of

$$
h_{(1)}^{\alpha\beta} = k_{(1)}^{\alpha\beta} + \partial^{\alpha}\varphi_{(1)}^{\beta} + \partial^{\beta}\varphi_{(1)}^{\alpha} - \eta^{\alpha\beta}\partial_{\mu}\varphi_{(1)}^{\mu}
\tag{51}
$$

The first term depends on two STF-tensorial multipole moments, $M_L(u)$ and $S_L(u)$, which are arbitrary functions of time, but obeythe laws of conservation of the monopole:$M = const$ and dipoles: $M_i = const, S_i = const$.

The conservation of these two lowest-order moments ensures the following constancy: the total mass of the source, $M = const$, center-of-mass position, $X_i \equiv M_i/M = const$, total linear momentum $P_i \equiv M_i^{(1)} = 0$, and total angular momentum, $S_i = const$. By translating the origin of our coordinates to the center of mass, we can easily have $X_i = 0$

The first term is given by the following equations:

$$
\begin{aligned}
k_1^{00} &= -\frac{4}{c^2}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L\left(\frac{M_L(u)}{r}\right) \\
k_1^{0i} &= -\frac{4}{c^3}\sum_{l\geq 1}\frac{(-)^l}{l!}\left\{\partial_{L-1}\left(\frac{M_{iL-1}^{(1)}(u)}{r}\right) + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}\left(\frac{S_{bL-1}(u)}{r}\right)\right\} \\
k_1^{ij} &= -\frac{4}{c^4}\sum_{l\geq 2}\frac{(-)^l}{l!}\left\{\partial_{L-2}\left(\frac{M_{iL-2}^{(2)}(u)}{r}\right) + \frac{2l}{l+1}\partial_{aL-2}\left(\epsilon_{ab(i}S_{j)bL-1}^{(1)}(u)\left(\frac{1}{r}\right)\right)\right\}
\end{aligned}
\tag{52}
$$

The components of the gauge-vector φ_1^α are given by:

$$
\begin{aligned}
\varphi_1^0 &= -\frac{4}{c^3}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_L(\frac{W_L(u)}{r}) \\
\varphi_1^i &= -\frac{4}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l!}\partial_{iL}(\frac{X_L(u)}{r}) \\
&- \frac{4}{c^4}\sum_{l\geq 1}\frac{(-)^l}{l!}\{\partial_{L-1}(\frac{Y_{iL-1}(u)}{r}) + \frac{l}{l+1}\epsilon_{iab}\partial_{aL-1}(\frac{Z_{bL-1}(u)}{r})\}
\end{aligned}
\tag{53}
$$

The detailed expressions of M, S, W, X, Y and Z can be found in Ref.[2].

2.3.2 Full post-Minkowskian solution

Consider the full post-Minkowskian series $h^{\alpha\beta}_{ext} = \sum_{n=1}^{+\infty} G^n h^{\alpha\beta}_{(n)}$, and plugge it into the vacuum Einstein field equations (2,3); with $\tau^{\alpha\beta} = \frac{c^4}{16\pi G}\Lambda^{\alpha\beta}$, as $T^{\alpha\beta} = 0$, we can get the set of equations:

$$\Box h^{\alpha\beta}_{(n)} = \Lambda^{\alpha\beta}_{(n)}(h_{(1)}, h_{(2)}, ..., h_{(n-1)}) \tag{54}$$

$$\partial_\mu h^{\alpha\mu}_{(n)} = 0 \tag{55}$$

In detail, we have [3]

$$
\begin{aligned}
\Box h_2^{\alpha\beta} =& N^{\alpha\beta}(h_1, h_1) \\
\Box h_3^{\alpha\beta} =& M^{\alpha\beta}(h_1, h_1, h_1) + N^{\alpha\beta}(h_1, h_2) + N^{\alpha\beta}(h_2, h_1) \\
\Box h_4^{\alpha\beta} =& L^{\alpha\beta}(h_1, h_1, h_1, h_1) \\
&+ M^{\alpha\beta}(h_1, h_1, h_2) + M^{\alpha\beta}(h_1, h_2, h_1) + M^{\alpha\beta}(h_2, h_1, h_1) \\
&+ N^{\alpha\beta}(h_2, h_2) + N^{\alpha\beta}(h_3, h_1) + N^{\alpha\beta}(h_1, h_3)
\end{aligned}
\tag{56}
$$

Ref.[5] gives more explicit expressions for $N^{\alpha\beta}$, $M^{\alpha\beta}$ and $L^{\alpha\beta}$.

Define that

$$\Box^{-1}_{ret}\rho(t, x) = \mathbf{FP}_{B=0}[-\frac{1}{4\pi}\int \frac{d^3x'}{|x - x'|}|\frac{x'}{r_0}|^B \rho(t - |x' - x|/c, x)] \tag{57}$$

Using $\nabla^2(\frac{1}{r-r'}) = 4\pi\delta(r - r')$, we can easily find that

$$\Box[-\frac{1}{4\pi}\int \frac{d^3x'}{|x - x'|}|\frac{x'}{r_0}|^B \rho(t - |x' - x|/c, x)] = (\frac{r}{r_0})^B \rho(t, x) \tag{58}$$

Thus $\Box \cdot \Box_{ret}^{-1}\rho(t,x) = \rho(t,x)$. The general solution with a homogeneous solution is

$$h_{(m)}^{\mu\nu}(t,x) = \Box_{ret}^{-1}\Lambda_{(m)}^{\mu\nu}(t,x) + \sum_{l=0}^{+\infty}\hat{\partial}_L\left(\frac{U_{(m)L}^{\mu\nu}(t-r/c)}{r}\right) \tag{59}$$

where $r = |\mathbf{x}|$ and the functions $U_{(m)}^{\mu\nu}$ are smooth functions of retarded time $u = t - r/c$, which become constant in the past. We can find that a monopolar wave satisfies $\Box(U_{(0)}^{\mu\nu}(u)/r) = 0$.

2.3.3 Near-zone and far-zone structures

Ref.[1] proves that the general structure of the expansion of the post-Minkowskian exterior metric in the near-zone (when $r \to 0$) can be written in the form as

$$h_{(m)}^{\mu\nu}(\mathbf{x},t) = \sum \hat{n}_L r^q (\ln r)^p F_{L,q,p,m}(t) + O(r^N), \forall N \in \mathbb{N} \tag{60}$$

where $q \in \mathbb{Z}$ with $q_0 \leq q \leq N$ (q_0 is a negative integer and $p \leq m-1$), and functions $F_{L,q,p,m}$ are multilinear functionals of the source multipole moments. Paralleling the structure of the near-zone expansion, we have a similar result concerning the structure of the far-zone expansion at Minkowskian future null infinity ($r \to +\infty$)

$$h_{(m)}^{\mu\nu}(\mathbf{x},t) = \sum \frac{\hat{n}_L (\ln r)^p}{r^k} G_{L,k,p,m}(u) + O\left(\frac{1}{r^N}\right), \forall N \in \mathbb{N} \tag{61}$$

where $u = t - r/c$.

3 The matching equation

Now, we have solution of Einstein Equations in the Near-zone and Far-zone expansion by the post-Newtonian expansions and post-Minkowskian expansions respectively and each solution have different domain of validity. Fortunately, there is a fundamental fact that there always exist a "matching" region around post-Newtonian sources, in which both the expansions are valid.

In the matching region $d < r < \lambda$, let h be the exact solution. Then the one from post-Minkowskian expansion $\mathcal{M}(h)$ is numerically equal to h.

$$h = \mathcal{M}(h), (r > d) \tag{62}$$

The solution from post-Newtonian expansion $\bar{h}$ is also numerically equal to h in $r < \lambda$.

$$h = \bar{h}, (r < \lambda) \tag{63}$$

Thus we have

$$\bar{h} = \mathcal{M}(h), (d < r < \lambda) \tag{64}$$

To change Eq.(64) into a matching equation, we replace $\mathcal{M}(h)$ by its near zone expansion $\overline{\mathcal{M}(h)}$, and replace $\overline{h}$ by $\mathcal{M}(\overline{h})$, its multipole expansion. The matching equation is

$$\overline{\mathcal{M}(h)} = \mathcal{M}(\overline{h}) \tag{65}$$

3.1 General expression of the multipole expansion

Taking the hypothesis of matching, Eq.(65), the multipole expansion of the solution of Einstein field equation outside the post-Newtonian source is

$$\mathcal{M}(h^{\mu\nu}) = \Box^{-1}_{ret}\mathcal{M}(\Lambda^{\mu\nu}) - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{\mathcal{H}^{\mu\nu}_L(t-r/c)}{r}) \tag{66}$$

where the "multipole moments" are given by

$$\mathcal{H}^{\mu\nu}_L(u) = \mathcal{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B x_L \overline{\tau}^{\mu\nu}(\mathbf{x}, u) \tag{67}$$

Here, $\overline{\tau}^{\mu\nu}$ is the post-Newtonian expansion of the stress-energy pseudo-tensor. The proof is given below.

Consider $\Delta^{\mu\nu}$ as the difference between $h^{\mu\nu}$, the solution of the field equation everywhere both inside and outside the source, and the first term of (66), defined as

$$\Delta^{\mu\nu} = h^{\mu\nu} - \Box^{-1}_{ret}\mathcal{M}(\Lambda^{\mu\nu}) \tag{68}$$

According to Eq.(9), it becomes

$$\begin{aligned}\Delta^{\mu\nu} =& \frac{16\pi G}{c^4}\Box^{-1}_{ret}\tau^{\mu\nu} - \Box^{-1}_{ret}\mathcal{M}(\Lambda^{\mu\nu})\\ =& \frac{16\pi G}{c^4}\Box^{-1}_{ret}[\tau^{\mu\nu} - \mathcal{M}(\tau^{\mu\nu})]\end{aligned} \tag{69}$$

where we used $\mathcal{M}(\Lambda^{\mu\nu}) = \frac{16\pi G}{c^4}\mathcal{M}(\tau^{\mu\nu})$ because $T^{\mu\nu}$ has a compact support. Here we can see from the above equation that $\Delta^{\mu\nu}$ is a retarded integral of a source with spatially compact support. Therefore, $\mathcal{M}(\tau^{\mu\nu})$ can be obtained from the formula for the multipole expansion of the retarded solution of a wave equation with compact-support source.

To get the solution of $\mathcal{M}(\Delta^{\mu\nu})$, let us consider a general situation first. Following Ref.[4], let $S(\mathbf{x},t)$ be some source and $S(\mathbf{x},t)=0$ when $|\mathbf{x}|>r_0$, where r_0 is some finite fixed radius. V is the slution of

$$\Box V=-4\pi S \tag{70}$$

Thus, we have

$$V(\mathbf{x},t)=-4\pi\Box^{-1}_{ret}S=\int\frac{d^3\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}S(\mathbf{x}',t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}) \tag{71}$$

To slove Eq.(70), we expand the source $S(\mathbf{x},t)$ in spherical harmonics

$$S(\mathbf{x},t)=\sum_{l\geq 0}\hat{n}_L(\theta,\varphi)S_L(r,t) \tag{72}$$

For each term in Eq.(72), we apply the $\Box^{-1}_{ret}$ and have [4]

$$\begin{aligned}\Box^{-1}_{ret}[(\hat{n}_LS_L)(\mathbf{x},t)]=&\hat{\partial_L}\{\frac{1}{r}\int_{-\infty}^{t-r}d\tau R_L^{[a]}(\frac{t-r-\tau}{2},\tau)\}\\&-\int_{-\infty}^{t-r}d\tau\hat{\partial}_L\{\frac{1}{r}R_L^{[a]}(\frac{t+r-\tau}{2},\tau)\}\end{aligned} \tag{73}$$

where (with c=1)$R_L^{[a]}(\rho,\tau)$ is related to $S_L(r,t)$

$$R_L^{[a]}(\rho,\tau)=\rho^l\int_a^\rho\frac{(\rho-x)^l}{l!}(\frac{2}{x})^{l-1}S_L(x,\tau+x). \tag{74}$$

This result is valid everywhere both inside and outside the source for any source (with compact or unbounded support), and it is independent of choice of a. Now, consider the case that a field point is outside a compact source ($r>r_0$), if we choose $a=r_0$, the second term of Eq.(73) will vanish when $r>r_0$. Then we have

$$-4\pi\Box^{-1}_{ret}(\hat{n}_LS_L)=\frac{(-)^l}{l!}\hat{\partial}_L(\frac{F_L(t-r/c)}{r}) \tag{75}$$

where

$$F_L(u)=-4\pi(-)^l l!\int_{-\infty}^u d\tau(\frac{u-\tau}{2})^l\times\int_{r_0}^{(u-\tau)/2}dx\frac{(\frac{u-\tau}{2}-x)^l}{l!}(\frac{2}{x})^{l-1}S_L(x,\tau+x) \tag{76}$$

Transforming the equation by $z = \frac{\tau - u + x}{x}$, it becomes

$$F_L(u) = \frac{4\pi}{2^{l+1}} \int_{-1}^{+1} dz(1-z^2)^l \int_0^{r_0} dx x^{l+2} S_L(x, u+zx) \tag{77}$$

Then using[4]

$$S_L(r,t) = \frac{(2l+1)!!}{(4\pi l!)} \int d\Omega \hat{n}_L S(\mathbf{x}, t), \tag{78}$$

finally we get

$$V = \sum_{l\geq 0} \frac{(-)^l}{l!} \hat{\partial}_L \left(\frac{F_L(t-r/c)}{r}\right) \tag{79}$$

where

$$F_L(u) = \int d^3\mathbf{x} \hat{x}_L S_l(\mathbf{x}, u) \tag{80}$$

with

$$S_l(\mathbf{x}, u) = \frac{(2l+1)!!}{2^{l+1} l!} \int_{-1}^{+1} dz(1-z^2)^l S(\mathbf{x}, u + z|\mathbf{x}|/c). \tag{81}$$

Therefore, we have

$$\mathcal{M}(\Delta^{\mu\nu}) = -\frac{4G}{c^4} \sum_{l=0}^{+\infty} \frac{(-)^l}{l!} \hat{\partial}_L \left(\frac{\mathcal{H}_L^{\mu\nu}(u)}{r}\right) \tag{82}$$

where

$$\mathcal{H}_L^{\mu\nu}(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B x_L [\tau^{\mu\nu} - \mathcal{M}(\tau^{\mu\nu})] \tag{83}$$

However, the moments are not match Eq.(67), as we need to apply the assumption of a post-Newtonian source here.

Because the sources are covered in their near zone, and the integral should have a compact support limited to the domain of the source, we can replace the terms in Eq.(83) with post-Newtonian expansion, which is valid in the near zone.

$$\mathcal{H}_L^{\mu\nu}(u) = \mathcal{FP}_{B=0} \int d^3\mathbf{x} |\mathbf{x}|^B x_L [\overline{\tau}^{\mu\nu} - \overline{\mathcal{M}(\tau^{\mu\nu})}] \tag{84}$$

The matching equation in the near zone, Eq.(65) and Eq.(60), gives us

$$\overline{\mathcal{M}(h)} = \sum \hat{n}_L r^m (\ln r)^p F_L, m, p(t) = \mathcal{M}(\overline{h}) \tag{85}$$

Thus, the second term in the integral is actually zero, proven in Ref.[2]. So, it will give Eq.(67).

3.2 Matching

We will use two equations shown as below, of which detailed proof can be found in Ref.[7].

$$\mathcal{M}(\Box_{inst}^{-1}(\tau)) = \Box_{inst}^{-1}(\mathcal{M}(\tau)) - \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{F_L(t-\frac{r}{c})+F_L(t+\frac{r}{c})}{2r}), \tag{86}$$

and

$$\overline{\Box_{ret}^{-1}[\mathcal{M}(\tau)]} = \Box_{inst}^{-1}\overline{\mathcal{M}(\tau)} - \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{R_L(t-\frac{r}{c})-R_L(t+\frac{r}{c})}{2r}) \tag{87}$$

where the two functions are defined as

$$F_L(t) = \mathbf{FP}_{B=0}\int d^3y\hat{y}_L|y|^B\int_{-1}^{+1}dz\delta_l(z)\overline{\tau}(t\pm z|y|/c, y), \tag{88}$$

and

$$R_L(t) = \int d^3y\hat{y}_L|y|^B\int_{+1}^{+\infty}dz[-2\delta_l(z)]\mathcal{M}(\tau)^{\mu\nu}(y, t-z|y). \tag{89}$$

Here

$$\delta_l(z) = \frac{(2l+1)!!}{2^{l+1}l!}(1-z^2)^l, \tag{90}$$

which is normalized by

$$\int_{-1}^{1}dz\delta_l(z) = 1 \tag{91}$$

Recall that

$$\overline{h}^{\mu\nu} = \frac{16\pi G}{c^4}\Box_{inst}^{-1}\overline{\tau}^{\mu\nu} - \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L[\frac{A_L^{\mu\nu}(t-\frac{r}{c})-A_L^{\mu\nu}(t+\frac{r}{c})}{2r}] \tag{92}$$

$$\mathcal{M}(h^{\mu\nu}) = \Box_{ret}^{-1}\mathcal{M}(\Lambda^{\mu\nu}) - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{U_L^{\mu\nu}(t-r/c)}{r}) \tag{93}$$

Then Eq.(65) becomes

$$\begin{aligned}
&\overline{\Box_{ret}^{-1}\mathcal{M}(\Lambda^{\mu\nu})} - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{U_L^{\mu\nu}(t-r/c)}{r}) \\
=&\frac{16\pi G}{c^4}\overline{\Box_{ret}^{-1}\mathcal{M}(\tau^{\mu\nu})} - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{U_L^{\mu\nu}(t-r/c)}{r}) \\
=&\frac{16\pi G}{c^4}\mathcal{M}(\Box_{inst}^{-1}\overline{\tau}^{\mu\nu}) - \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L[\frac{A_L^{\mu\nu}(t-\frac{r}{c})-A_L^{\mu\nu}(t+\frac{r}{c})}{2r}].
\end{aligned} \tag{94}$$

as $\frac{16\pi G}{c^4}\mathcal{M}(\overline{\tau}) = \mathcal{M}(\overline{\Lambda})$.

Applying Eq.(86) and (87), it becomes

$$
\begin{aligned}
&\Box^{-1}_{inst}\mathcal{M}(\overline{\Lambda}) - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{R_L(t-r/c)-R_L(t+r/c)}{2r})\\
&-\frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{U_L(t-r/c)}{r})\\
=&\frac{16\pi G}{c^4}\Box^{-1}_{inst}\mathcal{M}(\overline{\tau}) - \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L[\frac{F_L(t-\frac{r}{c})+F_L(t+\frac{r}{c})}{2r}]\\
&-\frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L[\frac{A_L(t-\frac{r}{c})-A_L(t+\frac{r}{c})}{2r}].
\end{aligned}
\tag{95}
$$

Then we finally have

$$
\begin{aligned}
U_L(t) &= F_L(t)\\
A_L(t) &= R_L(t) + F_L(t)
\end{aligned}
\tag{96}
$$

4 Summary

Post-Newtonian expansions are a good wasy to calculate an approximate solution of the Einstein field equations for the metric tensor. It expands the metrix tensor in small parameters, usually the ratio of the velocity of the matter in gravitational field to the speed of light, which indicates deviations from classical Newton's law. Thus, Post-Newtonian approximations are valid in the case of weak fields and slow motion.

Here, we rougly go through the main results of the post-Newtonian approximation with some proof. The post-Newtonian expansion have terms in power of $1/c$, which is valid in the near zone of the PN source (slow and weak self gravitation). In the exterior region, where the vaccum region doesn't have a compact-source support, we use the multiplole post-Minkowskian expansion. Due to the fact that there exists a "matching" region aroung post-Newtonian sources, we can use the "matching" equation. By matching the exterior solution to the metric of the post-Newtonian source in the near-zone we obtain the explicit expressions of the solutions.

This section will summarize and list those results.

(a) The post-Newtonian field in the near zone of a post-Newtonian source is in the form of

$$h^{\mu\nu} = \frac{16\pi G}{c^4}\Box^{-1}_{inst}\tau^{\mu\nu} - \frac{4G}{c^4}\sum_{l=0}^{\infty}\frac{(-)^l}{l!}\hat{\partial}_L[\frac{A_L^{\mu\nu}(t-\frac{r}{c}) - A_L^{\mu\nu}(t+\frac{r}{c})}{2r}] \quad (97)$$

The first term represents a particular solution of the post-Newtonian equations, while the second one is a homogeneous multipolar solution of the wave equation.

(b) The general solution of the linearized vacuum field equations (28, 29), which is stationary in the past is

$$h^{\alpha\beta}_{(1)} = k^{\alpha\beta}_{(1)} + \partial^\alpha\varphi^\beta_{(1)} + \partial^\beta\varphi^\alpha_{(1)} - \eta^{\alpha\beta}\partial_\mu\varphi^\mu_{(1)} \quad (98)$$

with details of the funtions in Section.(2).

(c) Under the hypothesis of matching, the multipole expansion of the solution of the Einstein field equation outside a post-Newtonian source is

$$\mathcal{M}(h^{\mu\nu}) = \Box^{-1}_{ret}\mathcal{M}(\Lambda^{\mu\nu}) - \frac{4G}{c^4}\sum_{l=0}^{+\infty}\frac{(-)^l}{l!}\hat{\partial}_L(\frac{\mathcal{H}_L^{\mu\nu}(t-r/c)}{r}), \quad (99)$$

where

$$\mathcal{H}_L^{\mu\nu}(u) = \mathcal{FP}_{B=0}\int d^3\mathbf{x}|\mathbf{x}|^B x_L\overline{\tau}^{\mu\nu}. \quad (100)$$

The inspiralling compact binaries are usually very relativistic, as the orbital velocities usually reach as high as $0.5c$ in the late rotations. Thus, they are ideal applications for post-Newtonian expansion. This article only mentions the metric tensor of the gravitational field around the Post-Newtonian sources. Further study and calculation can give an approxiamte expression of different orders for the waveform of the gravitational waves from the compact binaries. The mathematics are also complicated to reach those results, so I don't include them here.

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