EFFECTS OF THE ORBITAL ECCENTRICITY ON THE EQUIPOTENTIAL SURFACES IN BINARY SYSTEMS

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Abstract. In the frame of the elliptical restricted three-body problem, the differential equations for the motion of an infinitesimal body are established. In spite of the lack of Jacobi's integral, for fixed values of the true anomaly ($v = 0°$, $90°$, and $180°$), particular results were obtained. The pulsational character of the equipotential surfaces is evident.

1. Introduction

As was mentioned by Kopal and Lyttleton (1963), if the Keplerian orbit of the two finite masses m_1 and m_2 is eccentric, the geometrical properties of the equipotential surfaces are bound to depend on the time. In such a case '... in order to establish the instantaneous shape of the zero-velocity surface, we would need to know the value of $\mathrm{d}\Omega$ at each value of t (or v) all along any selected path' (Kopal, 1989, p. 35).

Such a subject is studied in the frame of the *Elliptical Restricted Three-Body Problem*, and the aim of the present note is to resume the effects of the orbital eccentricity on the equipotential surfaces. Here the two finite bodies will be considered as mass-points.

2. The Elliptical Three-Body Problem in an Inertial Coordinate System

Let us choose an inertial barycentric rectangular coordinate system (X', Y', Z'), where the X'-axis is directed towards the periastron, while $X'Y'$-plane coincides with orbital plane. In such a case, the differential equations for the motion of the infinitesimal body may be written in the form

$$\frac{\mathrm{d}^2X'}{\mathrm{d}t^2} = -G\frac{m_1}{r_1^3}(X' + R_1\cos v) - G\frac{m_2}{r_2^3}(X' - R_2\cos v),$$

$$\frac{\mathrm{d}^2Y'}{\mathrm{d}t^2} = -G\frac{m_1}{r_1^3}Y' - G\frac{m_2}{r_2^3}Y', \tag{1}$$

$$\frac{\mathrm{d}^2Z'}{\mathrm{d}t^2} = -G\frac{m_1}{r_1^3}Z' - G\frac{m_2}{r_2^3}Z';$$

with

$$r_1^2 = (X' + R_1\cos v)^2 + (Y')^2 + (Z')^2,$$
$$r_2^2 = (X' - R_2\cos v)^2 + (Y')^2 + (Z')^2; \tag{2}$$

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where the positions of the two finite bodies are:

$$S_1(-R_1 \cos v, -R_1 \sin v), \qquad S_2(R_2 \cos v, R_2 \sin v),$$

where v is the true anomaly; e, orbital eccentricity; A, semi-major axis,

$$R_1 = \frac{m_2}{m_1 + m_2} \frac{A(1 - e^2)}{1 + e \cos v}, \qquad R_2 = \frac{m_1}{m_1 + m_2} \frac{A(1 - e^2)}{1 + e \cos v},$$
$$R = R_1 + R_2 = \frac{A(1 - e^2)}{1 + e \cos v}. \tag{3}$$

3. The Use of a Non-uniform Rotating Coordinate System

Now the barycentric coordinate system (X, Y, Z) is considered in rotation with the binary system. Here the true anomaly v will be an 'angular' function, reckoned from the periastron passage. In such conditions the transformation equations between (X', Y', Z') and (X, Y, Z) are in the form

$$\left.\begin{aligned} X' &= X \cos v - Y \sin v, \\ Y &= X \sin v + Y \cos v, \\ Z' &= Z; \end{aligned}\right\} \tag{4}$$

and Equations (1) become

$$\frac{d^2X}{dt^2} - 2\left(\frac{dv}{dt}\right)\frac{dY}{dt} = \left(\frac{dv}{dt}\right)^2 X - G \frac{m_1}{r_1^3}(X + R_1) - G \frac{m_2}{r_2^3}(X - R_2) + Y \frac{d^2v}{dt^2},$$

$$\frac{d^2Y}{dt^2} + 2\left(\frac{dv}{dt}\right)\frac{dX}{dt} = \left(\frac{dv}{dt}\right)^2 Y - G \frac{m_1}{r_1^3} Y - G \frac{m_2}{r_2^3} Y - X \frac{d^2v}{dt^2},$$

$$\frac{d^2Z}{dt^2} = -G \frac{m_1}{r_1^3} Z - G \frac{m_2}{r_2^3} Z; \tag{5}$$

with

$$r_1^2 = (X + R_1)^2 + Y^2 + Z^2, \qquad r_2^2 = (X - R_2)^2 + Y^2 + Z^2. \tag{6}$$

In the first two Equations (5) the second terms on the left-hand side represent Coriolis accelerations. The first terms on the right-hand side are centrifugal effects. The next two terms are the gravitational effects, while the fourth terms represent the acceleration normal to the radius vector due to the non-uniform rotation of the coordinate system (X, Y, Z).

If we introduce now the potential function

$$U = G\,\frac{m_1}{r_1} + G\,\frac{m_2}{r_2} + \frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 (X^2 + Y^2)\,, \tag{7}$$

Equations (5) may be written in the form

$$\begin{aligned} \frac{\mathrm{d}^2X}{\mathrm{d}t^2} - 2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)\frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{\partial U}{\partial X} + Y\,\frac{\mathrm{d}^2v}{\mathrm{d}t^2}\,, \\ \frac{\mathrm{d}^2Y}{\mathrm{d}t^2} + 2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)\frac{\mathrm{d}X}{\mathrm{d}t} &= \frac{\partial U}{\partial Y} - X\,\frac{\mathrm{d}^2v}{\mathrm{d}t^2}\,, \\ \frac{\mathrm{d}^2Z}{\mathrm{d}t^2} &= \frac{\partial U}{\partial Z}\,. \end{aligned} \tag{8}$$

First, we multiply Equations (8) by $\mathrm{d}X/\mathrm{d}t$, $\mathrm{d}Y/\mathrm{d}t$, $\mathrm{d}Z/\mathrm{d}t$, respectively, and add the resulting equations together; then integrate with respect to time to obtain

$$\frac{1}{2}\,V^2 = U - \int_{t_0}^{t} \frac{\partial U}{\partial t}\,\mathrm{d}t - c_1 \int_{t_0}^{t} \frac{\mathrm{d}^2v}{\mathrm{d}t^2}\,\mathrm{d}t = \text{constant}\,, \tag{9}$$

V is the velocity, where

$$c_1 = X\,\frac{\mathrm{d}Y}{\mathrm{d}t} - Y\,\frac{\mathrm{d}X}{\mathrm{d}t} = \sqrt{G(m_1 + m_2)A(1 - e^2)} = R^2\,\frac{\mathrm{d}v}{\mathrm{d}t} \tag{10}$$

represents constant areas, while from Equation (7) we may write

$$\frac{\partial U}{\partial t} = f(X, Y, Z, v(t))\,.$$

For $V = 0$, Equation (9) becomes

$$U - c_1\,\frac{\mathrm{d}v}{\mathrm{d}t} - \int_{t_0}^{t} \frac{\partial U}{\partial t}\,\mathrm{d}t = \text{constant}\,. \tag{11}$$

Moreover, from Equation (10) we have at once

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\sqrt{G(m_1 + m_2)}}{A^{3/2}(1 - e^2)^{3/2}}\,(1 + e\cos v)^2$$

and

$$c_1\,\frac{\mathrm{d}v}{\mathrm{d}t} = R^2\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)^2 = \frac{G(m_1 + m_2)}{A(1 - e^2)}\,(1 + e\cos v)^2\,.$$

It is convenient to choose the following units: the semi-major axis ($A = 1$) as unit of length, the sum of the masses ($m_1 + m_2 = 1$) as unit of mass, and the reciprocal ω_k^{-1} of the mean angular velocity as unit of time. Therefore, we have $P = 2\pi$ for the orbital period and $G = 1$. In such conditions it follows that

$$m_1 = \frac{m_1}{m_1 + m_2} = \frac{1}{1+q}\,, \qquad m_2 = \frac{m_2}{m_1 + m_2} = \frac{q}{1+q}\,, \qquad q = m_2/m_1$$

and Equation (11) becomes

$$\frac{1}{2}\frac{(1 + e\cos v)^4}{(1 - e^2)^3}(X^2 + Y^2) + \frac{1}{1+q}\frac{1}{r_1} + \frac{q}{1+q}\frac{1}{r_2} - \\ - \frac{(1 + e\cos v)^2}{1 - e^2} - \int_{t_0}^{t} f(X, Y, Z, v(t))\,\mathrm{d}t = C\,. \tag{12}$$

Let us now return to Equations (5) which describe the motion of the infinitesimal body. Here we have in view the fact that, if we need the corresponding coordinates X, Y, Z, for the moment t, we must perform the integration of this system. But such an integration requires the adoption of six arbitrary constants, representing the initial conditions of the problem. Anyhow, for our task we can write

$$X = X(t, C_1, C_2)\,, \qquad Y = Y(t, C_3, C_4)\,, \qquad Z = Z(t, C_5, C_6)$$

and without performing the integration of Equations (5), a general solution is sufficient for this moment. Therefore, Equation (12) may be written as

$$\frac{1}{2}\frac{(1 + e\cos v)^4}{(1 - e^2)^3}(X^2 + Y^2) + \frac{1}{1+q}\frac{1}{r_1} + \frac{q}{1+q}\frac{1}{r_2} = C(v)\,, \tag{13}$$

where

$$r_1^2 = \left(X + \frac{q}{1+q}\frac{1 - e^2}{1 + e\cos v}\right)^2 + Y^2 + Z^2\,, \\ r_2^2 = \left(X - \frac{1}{1+q}\frac{1 - e^2}{1 + e\cos v}\right)^2 + Y^2 + Z^2\,; \tag{14}$$

and

$$C(v) = C + \frac{(1 + e\cos v)^2}{1 - e^2} + \int_{t_0}^{t} f(X(t), Y(t), Z(t), v(t))\,\mathrm{d}t$$

is a 'constant' for each peculiar value of the true anomaly $v(t)$.

4. The Location Determination of the Lagrangian Points

As it is known, at Lagrangian points all effective forces vanish. On the equipotential surfaces these are double points which are determined by the conditions

$$\frac{\partial U}{\partial X} = \frac{\partial U}{\partial Y} = \frac{\partial U}{\partial Z} = 0 . \tag{15}$$

Therefore, from Equations (13) and (14), for a fixed value of $v(t)$, we have at once

$$\frac{(1 + e\cos v)^4}{(1 - e^2)^3} X_i - \frac{1}{1 + q} \frac{X_i + \dfrac{q}{q + 1} \dfrac{1 - e^2}{1 + e\cos v}}{r_1^3} - \\ - \frac{1}{1 + q} \frac{X_i - \dfrac{1}{1 + q} \dfrac{1 - e^2}{1 + e\cos v}}{r_2^3} = 0 , \tag{16}$$

$$\frac{(1 + e\cos v)^4}{(1 - e^2)^3} Y_i - \frac{1}{1 + q} \frac{Y_i}{r_1^3} - \frac{q}{1 + q} \frac{Y_i}{r_2^3} = 0 , \tag{17}$$

$$- \frac{1}{1 + q} \frac{Z_i}{r_1^3} - \frac{q}{1 + q} \frac{Z_i}{r_2^3} = 0 , \tag{18}$$

where $i = \overline{1, 3}$ represents the succession of the first three Lagrangian points.

In Equation (18) we have always $Z = 0$. Now, the three abscissas of the collinear points will be given by Equation (16) with $Y = Z = 0$, while for the abscissas of the triangular points we have to resolve Equation (16) together with conditions $Y \neq 0$ and $Z = 0$.

For the inner Lagrangian point, for $q < 1$, we have to put

$$X_i = X_1 , \qquad r_1 = X_1 + R_1 , \qquad r_2 = R_2 - X_1$$

and Equation (16) becomes

$$\frac{(1 + e\cos v)^4}{(1 - e^2)^3} X_1 - \frac{1 + q}{\left[X_1(1 + q) + q \dfrac{1 - e^2}{1 + e\cos v}\right]^2} + \\ + \frac{q(1 + q)}{\left[X_1(1 + q) - \dfrac{1 - e^2}{1 + e\cos v}\right]^2} = 0 . \tag{19}$$

Therefore, Equation (19) may be solved for each peculiar value of the true anomaly v;

q and e being parameters of the corresponding binary system. Hence, X_1 may be determined.

Then, for the same first Lagrangian point L_1 ($Y = Z = 0$), and for the same value of the true anomaly, from Equation (13) we may write

$$C(v) = \frac{1}{2}\,\frac{(1 + e\cos v)^4}{(1 - e^2)^3}\,X_1^2 + \frac{1}{\left|X_1(1+q) + q\,\dfrac{1-e^2}{1+e\cos v}\right|} + \frac{q}{\left|X_1(1+q) - \dfrac{1-e^2}{1+e\cos v}\right|} = 0\,, \qquad (20)$$

whence it follows the value of the 'constant' $C(v)$.

Now, for each value of X, from Equation (13) we are able to determine the corresponding values of Y and to draw the equipotential curves of zero velocity in the orbital plane ($Z = 0$).

5. Practical Application

In order to illustrate the utility of the above-established formulas, we have drawn the Roche limit ($Z = 0$) for the binary system V380 Cygni, where $e = 0.22$, $q = 0.5703$ (Semeniuk, 1968). The obtained results, for $v = 0°$, $90°$, and $180°$, are displayed in Figure 1, whence the pulsational character of the equipotential surfaces is evident. The values of the stellar radii ($a_1 = 0.20$ and $a_2 = 0.14$) are taken from Svechnikov's (1969) catalogue.

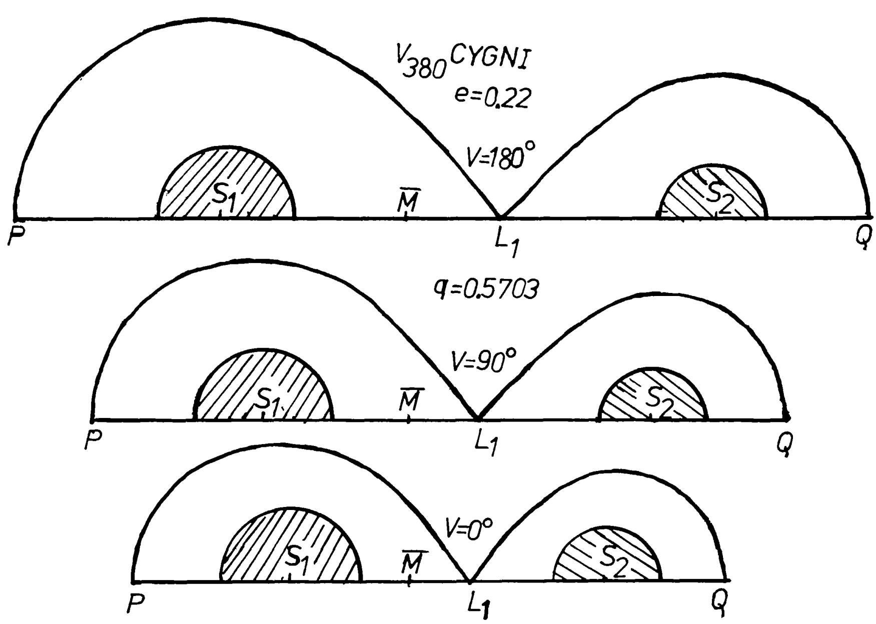

Fig. 1.

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