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The elliptical three-body problem. Triangular solution

I. Todoran and R. Roman, Cluj-Napoca, Romania
Astronomical Observatory

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The restricted three-body problem with eccentric orbit is reviewed and the positions of the triangular Lagrangian points (L_4, L_5) are determined. It is put in evidence the fact that L_4 and L_5 are situated at the corners of an isoscales triangle:

$$AB = BC = \frac{1-e^2}{(1+e\cos v)^{4/3}} \qquad \textit{and} \qquad AC = \frac{1-e^2}{(1+e\cos v)}$$

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1. Introduction

In a previous paper (Todoran 1992) we have analysed the effects of the orbital eccentricity on the shape of the equi-potential surfaces in a close binary system. So it was put in evidence the pulsational character and the variable positions of the linear Lagrangian points. In the frame of the same problem in the present note we shall discuss the positions of the triangular Lagrangian points.

2. Equations of the problem

Let us consider a non-uniform rotating barycentric coordinate system (X, Y, Z) assumed in rotation with the binary system. Here the true anomaly v will be an "angular" function, reckoned from the periastron passage. In such conditions for the double point positions the following equations are established (see Todoran 1992, eqs. (16) - (18)):

$$\frac{(1+e\cos v)^4}{(1-e^2)^3}X - \frac{1}{1+q}\frac{X+\frac{q}{1+q}\frac{1-e^2}{(1+e\cos v)}}{r_1^3} - \frac{q}{1+q}\frac{X-\frac{1}{1+q}\frac{1-e^2}{(1+e\cos v)}}{r_2^3} = 0\,, \tag{1}$$

$$\frac{(1+e\cos v)^4}{(1-e^2)^3}Y - \frac{1}{1+q}\frac{Y}{r_1^3} - \frac{q}{1+q}\frac{Y}{r_2^3} = 0\,, \tag{2}$$

$$\frac{1}{1+q}\frac{Z}{r_1^3} + \frac{q}{1+q}\frac{Z}{r_2^3} = 0\,, \tag{3}$$

where

$$\begin{aligned} r_1^2 &= \left[X + \frac{q}{1+q}\frac{1-e^2}{(1+e\cos v)}\right]^2 + Y^2 + Z^2\,, \\ r_2^2 &= \left[X - \frac{1}{1+q}\frac{1-e^2}{(1+e\cos v)}\right]^2 + Y^2 + Z^2\,. \end{aligned} \tag{4}$$

Equations (1)-(4) are written for a *fixed* value of the true anomaly v and the following system of units is considered: the semi-major axis ($A = 1$) as unit of length, the sum of the masses ($m_1 + m_2 = 1$) as unit of mass, and the reciprocal $1/\omega_k$ of the mean angular velocity as unit of time. Therefore, we have $P = 2\pi$ for the orbital period and $G = 1$. In such conditions we have

$$m_1 = \frac{m_1}{m_1 + m_2} = \frac{1}{1+q}, \qquad m_2 = \frac{m_2}{m_1 + m_2} = \frac{q}{1+q}, \qquad q = m_2/m_1$$

3. The location determination of the equilibrium triangular points

As it was mentioned in the previous paper (Todoran 1992) for $Y = 0$ and $Z = 0$, from eq. (1) we may determine the positions of the three linear Lagrangian points (L_1, L_2, L_3). On the other hand, for $Y \neq 0$ and $Z = 0$ we shall obtain the positions of the triangular points (L_4, L_5).

In order to do so, from eq. (2) we may write

$$\frac{(1+e\cos v)^4}{(1-e^2)^3} - \frac{1}{1+q}\frac{1}{r_1^3} - \frac{q}{1+q}\frac{1}{r_2^3} = 0 \; . \tag{5}$$

Moreover, eq. (1) may be written in the form

$$X\left[\frac{(1+e\cos v)^4}{(1-e^2)^3} - \frac{1}{1+q}\frac{1}{r_1^3} - \frac{q}{1+q}\frac{1}{r_2^3}\right] - \frac{q}{1+q}\frac{1-e^2}{(1+e\cos v)}\left[\frac{1}{r_1^3} - \frac{1}{r_2^3}\right] = 0 \; ,$$

whence, if we have in mind eq. (5), we have at once $r_1 = r_2 = r$.

Therefore, eq. (5) becomes

$$\frac{(1+e\cos v)^4}{(1-e^2)^3} - \frac{1}{1+q}\frac{1}{r^3} - \frac{q}{1+q}\frac{1}{r^3} = 0$$

$$\frac{1}{r_1} = \frac{1}{r_2} = \frac{1}{r} = \frac{(1+e\cos v)^{4/3}}{1-e^2}$$

that is

$$r_1^2 = r_2^2 = r^2 = \frac{(1-e^2)^2}{(1+e\cos v)^{8/3}} . \tag{6}$$

In such conditions, from eq. (4) follows that

$$\left[X + \frac{q}{1+q}\frac{1-e^2}{(1+e\cos v)}\right]^2 + Y^2 = \left[X - \frac{1}{1+q}\frac{1-e^2}{(1+e\cos v)}\right]^2 + Y^2 \; ,$$

whence we have

$$X_{4,5} = \frac{1}{2}\frac{1-q}{1+q}\frac{1-e^2}{(1+e\cos v)} \; . \tag{7}$$

In the same manner, from eqs. (4), (6), and (7) we may write

$$Y^2 = \frac{(1-e^2)^2}{(1+e\cos v)^{8/3}} - \left[\frac{1}{2}\frac{1-q}{1+q}\frac{1-e^2}{1+e\cos v} + \frac{q}{1+q}\frac{1-e^2}{1+e\cos v}\right]^2 = \left[\frac{1-e^2}{1+e\cos v}\right]^2\left[\frac{1}{(1+e\cos v)^{4/3}} - \frac{1}{4}\right]$$

that is

$$Y_{4,5} = \pm\frac{1-e^2}{(1+e\cos v)}\left[\frac{1}{(1+e\cos v)^{4/3}} - \frac{1}{4}\right]^{1/2} . \tag{8}$$

4. Concluding remark

In the elliptical three-body problem the triangular solution is referring to such a situation, when for a fixed value of the true anomaly v the three bodies are situated at the corners of an isoscales triangle:

$$S_1P = S_2P = \frac{1-e^2}{(1+e\cos v)^{4/3}}$$

and

$$S_1S_2 = \frac{1-e^2}{1+e\cos v)} ,$$

where S_1 and S_2 represent the two finite bodies, while P stands for the position of the infinitesimal body.

Now, it is evident that for $e = 0$ we obtain the well-known solution of an equilateral triangle

$$S_1P = S_2P = S_1S_2 = 1.$$

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References

Todoran, I. : 1992, Effects of the orbital eccentricity on the equipotential surfaces in binary systems, Astrophys. Space Sci., in press

Address of the authors:

Ioan Todoran, Rodica Roman
Astronomical Observatory
Strada Cireşilor 19
3400 Cluj-Napoca
Romania