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Why did the apple fall? A new model to explain Einstein's gravity

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Abstract

Newton described gravity as an attractive force between two masses but Einstein's General Theory of Relativity provides a very different explanation. Implicit in Einstein's theory is the idea that gravitational effects are the result of a distortion in the shape of space-time. Despite its elegance, Einstein's concept of gravity is rarely encountered outside of an advanced physics course as it is often considered to be too complex and too mathematical. This paper describes a new conceptual and quantitative model of gravity based on General Relativity at a level most science students should be able to understand. The model illustrates geodesics using analogies with paths of navigation on the surface of the Earth. This is extended to space and time maps incorporating the time warping effects of General Relativity. Using basic geometry, the geodesic path of a falling object near the surface of the Earth is found. From this the acceleration of an object in free fall is calculated. The model presented in this paper can answer the question, 'Why do things fall?' without resorting to Newton's gravitational force.

Keywords: relativity, gravity, Einstein, geodesic, education, spacetime, gravitational field

(Some figures may appear in colour only in the online journal)

Introduction

One of the most famous anecdotes in the history of science concerns a young Isaac Newton sitting under an apple tree when an apple falls. Newton, in a stroke of brilliant insight, creates

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his theory of gravity [1]. This story may be mythical but Newton's theory of gravity remains taught to this day. The falling apple epitomises Newton's gravity, a force exerted at a distance by planet Earth (and vice versa). However, encapsulated in Einstein's General Theory of Relativity is the postulate that gravitational effects are the result of a distortion in the shape of space-time. This concept is encapsulated in John Wheeler's expression:

> '*Matter tells space-time how to curve, and space-time tells matter how to move'* [2].

While the beauty of this idea is apparent to many who study it, it is seldom encountered in general science and physics courses where Newtonian concepts are predominantly taught. Explaining why an apple or any other object falls without resorting to Newton's gravity force can be a difficult task. This paper seeks to provide a simple and concise conceptual and quantitative model to help in the understanding of gravity as elucidated by General Relativity whilst avoiding the mathematical complexity often encountered in advanced university physics courses. Einstein's theory is now 100 years old but few science students have been exposed to this elegant theory. We advocate that high school teaching should embrace the modern concept of gravity and we propose a model which is suitable for students at this level.

In this paper we will use the terminology 'space and time map' to differentiate our approach from that of Minkowski space-time and we use the term 'geodesic' to describe the *shortest path* between two points on a curved surface. Our approach introduces the idea of a warped time axis on a space and time map. While this approach may not be consistent with a rigorous General Relativity methodology we believe it is good intermediate step between a Newtonian viewpoint and that of General Relativity. This simpler approach is designed to help and encourage students' progress in their understanding of General Relativity without them encountering the non-intuitive concepts inherent in relativity theory.

We start by introducing the idea of curved space and geodesics by investigating flight paths on a two-dimensional world map. We develop a method to find the shortest path (the geodesic) by means of scale lines and navigating a path perpendicular to these scale lines. Following a discussion of the time warping effects inherent in General Relativity this idea is then transferred to space and time maps where the geodesic is described as the path intersecting isochrones at right angles. Finally, the acceleration of a falling object from the viewpoint of an observer on the Earth is derived. We find that this derivation produces a result consistent with the equations of General Relativity.

The model

A fable of two pilots

Gould demonstrates that a common map of the world provides a good analogy to the effects of curved space, helping students visualise Einstein's concept of curved space [3]. He states that 'the map enables us to contrast Newton's and Einstein's models of gravity in a particularly visual and intuitive way'. We have expanded Gould's idea in the following allegory, using cities familiar to students in the Southern Hemisphere.

Perth in Western Australia and Durban in South Africa are located at approximately the same latitude. To fly a plane from Perth to Durban one needs to fly in a Westerly direction; at least that how it appears on a standard world map. Two pilots, allegorically named Newton and Einstein, embark on this journey. Their colleague, Euclid, plots the course and Newton points his plane in the direction indicated. He finds however that his plane has been pushed in a northerly direction. Newton exclaims, 'I have been pushed north-ward by a mysterious

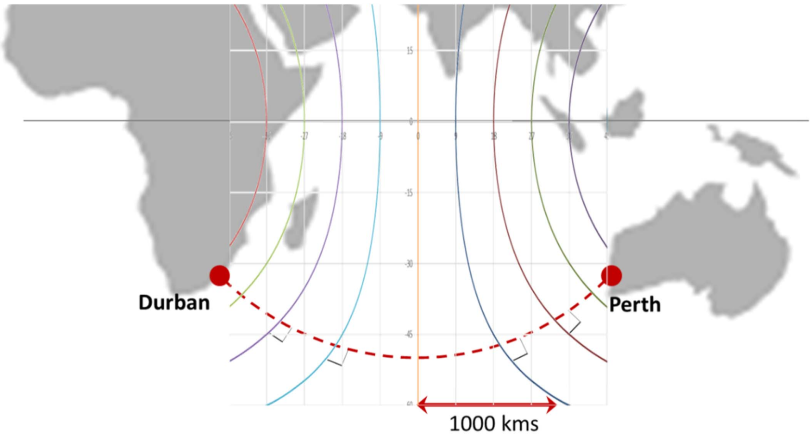

Figure 1. Scale-lines of equal east–west distance are shown. These lines diverge from lines of longitude as one approaches the poles. The east–west distance between each scale-line is 1000 km. By travelling in a direction perpendicular to each scale-line the shortest flight path is obtained.

force'. He counteracts this 'force' by banking to the south and arrives at his destination. Einstein however realises that the surface they are on is not flat but curved and by applying some clever geometry he plots his course along the great circle. On his journey he does not apply any lateral forces and arrives in the shortest time. Care must be taken to explain that this is a simple analogy. In reality planes move in three dimensions, not two, and the effect of prevailing winds and Coriolis forces are ignored.

Finding the geodesic

The geodesic is the shortest path between two points on a curved surface just as a straight line is the shortest path on a two-dimensional plane surface. In the story Einstein was able to determine the geodesic, his flight path, by understanding that he was on the spherical surface of the Earth. An equation could be constructed to describe his path but solving geodesic equations is often difficult and students at high school level seldom have the necessary skills to do this. We now introduce a simple method to determine this path based on drawing scale-lines of equal distance in an east–west direction as shown in figure 1. This popular map projection has lines of longitude running parallel to each other rather than converging at the poles. Distances in an east–west direction at the equator appear shorter on the map when compared to the same distance closer to the poles. By first drawing a vertical line joining points equidistant from each city, points at a distance of 1000 km (in an east–west direction) from this central line are joined to create the first curved scale line and so on for each successive scale line. Lines of longitude are also shown. At the equator nine degrees of separation in longitude is approximately 1000 km. Closer to the poles the scale-lines and lines of longitude diverge as shown.

The geodesic or shortest path between the two cities can be traversed by navigating in a direction perpendicular to each scale line. This path is shown in the figure by the curved dashed line. It is the same path that the Einstein in our story calculated. It is the shortest flight path between Perth and Durban and is coincidental with a great circle of the Earth.

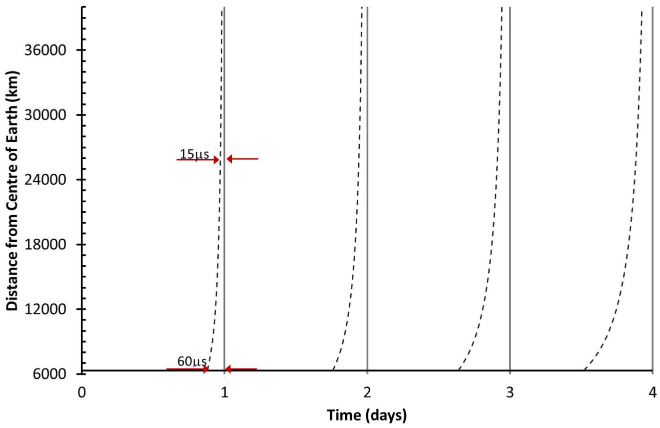

Figure 2. The effect of time warping in relation to height from the centre of the Earth. The solid vertical lines represent time in days as measured by a distant observer. The dashed curves (not to scale) show the distortion of these lines due to the proximity of the Earth (the Schwarzschild metric t and r coordinates correspond to the horizontal and vertical axes respectively).

The geodesic curve in General Relativity

In General Relativity a geodesic curve, or geodesic, is the path that a free particle (that is, a particle upon which no force acts) will follow in four-dimensional space-time. One of the central problems in General Relativity is the determination of these geodesic paths or curves which arise from the geometry described by Einstein's Field Equations. Finding solutions to these equations can be difficult and, unfortunately, the geometrical meaning is often obscured. The concept of time warping we believe can be introduced at a high school level bridging the conceptual gap between the world map analogy and the field equations. A large mass such as the Earth distorts the surrounding space-time—'mass tells space-time how to curve'. As emphasised by Gould [3] it is predominantly the time warping part of this distortion that leads to the observation of a falling object. We now introduce a model to find the geodesic on a space and time map in a manner similar to that used by the fictional Einstein in the story above.

Measuring the time warp

The time warp predicted by General Relativity can be measured with atomic clocks. Recent advances in these technologies suggest the possibility that these clocks will soon be able to detect relativistic changes in the rate of time over heights of just a few centimetres [4]. With these data we can construct a table of time-warp versus height above the Earth's surface. Furthermore, we can turn to a solution to Einstein's Field Equations published by Karl Schwarzschild in 1916 [5]. From this a convenient approximation can be derived showing the relationship between time warp and distance from the centre of the Earth (see appendix):

$$\frac{\Delta t_r}{T_\infty} = \frac{GM}{rc^2}. \tag{1}$$

The terms in the equations above can be understood as follows: T_∞ is the elapsed time measured by a clock in free space, far from all sources of gravity. A clock at a distance r from the centre of the Earth measures a corresponding time T_r. Δt_r is the difference between these

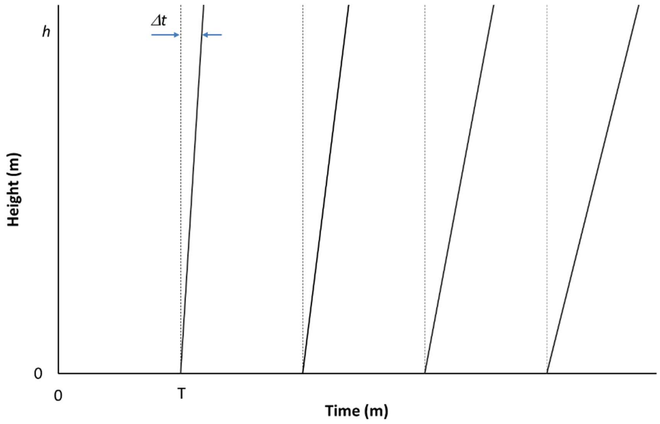

Figure 3. A space and time map showing the effect of time warping close to the surface of the Earth. The non-vertical isochrones represent a fixed proper time as read by a clock at some height from the surface. Δt is the time warp experienced at a height h over a time period T according to a reference clock on the surface.

two times, i.e. $\Delta t_r = T_\infty - T_r$. $\frac{\Delta t_r}{T_\infty}$ is therefore the fractional time difference due to the gravitational time warp.

A plot showing the effect of the gravitational time warping described by equation (1) is shown in figure 2. The solid vertical lines represent time in days as measured by a distant observer. The dashed lines (not to scale) show the result of time warping due to the proximity of a massive body, the Earth. The graph reveals a fractional time warp of $-60\ \mu\mathrm{s\ d^{-1}}$ at the surface of the Earth.

To understand the diagram students can consider two clocks synchronised at time $T = 0$ (the left most vertical line). One clock is on the surface of the Earth ($r = 6300$ km), the other is floating in space far from the Earth. After one day the distant clock records a greater elapsed time than the clock on the surface by an amount 60 μs. After two days the difference would be 120 μs and so on. Also shown is the time difference at a height of 20 000 km ($r \approx 26\,000$ km). The difference between these values (60 μs − 15 μs = 45 μs) is the time warp experienced after one day at a height of 20 000 km compared to a reference clock on the surface of the Earth. These values have been verified experimentally by, for example, Gravity Probe A [6, 7] and the GPS satellites [8].

Space and time maps

Students can comprehend three-dimensional space and are familiar with simple distance versus time graphs with time being displayed on the horizontal axis. To understand space-time, students need to add a fourth dimension, that of time, expressed in the same length unit as the other dimensions. By considering only one of the three space dimensions, four-dimensional space-time can be simplified by plotting distance in the direction of interest versus 'time' where the horizontal time axis is in units of length, that is, time multiplied by the speed of light. Making the time axis horizontal is nonstandard in relativity but may be more appropriate for high school students. This simplified depiction of space-time we have termed a 'space and time map'.

Simple Newtonian distance–time plots are usually drawn as a grid on a two-dimensional surface where time lines or isochrones represent a moment in time and are plotted vertically.

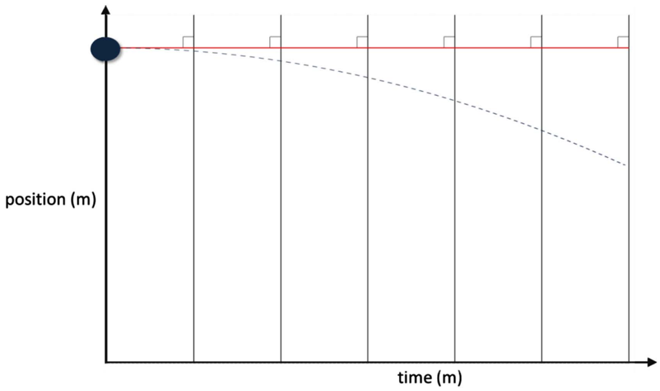

Figure 4. A space and time map showing an objects position in space versus time in units of length (time multiplied by the speed of light). A free particle in space follows a straight line as shown by the solid horizontal line. This path intersects the isochrones at right-angles. If it were observed to follow a parabolic path then it would necessarily be experiencing a force. The dashed line shows the path expected for a falling object in a reference frame on the surface of the Earth. This would suggest that the object experiences a force within the Earth's gravitational field.

As was demonstrated with the map of the Earth, vertical time-lines, like vertical lines of longitude, are a distortion of reality. On the world map, lines of latitude and longitude were drawn as horizontal and vertical lines but the surface is in fact curved. Similarly, on a space and time map, we are attempting to draw a curved surface on a two-dimensional page. Due to the time warping effects of General Relativity the isochrones can be considered to be slightly bent from the vertical. The effects of this time warp are strongest near a large mass such as the Earth. The effect diminishes as we travel further from the Earth's surface and the lines become closer to vertical as is evident in figure 2.

Students are more familiar with considering the motion of a falling object from the viewpoint of an observer on the surface of the Earth. With this in mind, equation (1) can be modified to find the fractional time dilation versus height from the surface of the Earth:

$$\frac{\Delta t_h}{T} = \frac{g}{c^2}h. \tag{2}$$

Here Δt_h is the time dilation at a height h expressed as a fraction of the time T as measured on the surface of the Earth rather than from the viewpoint of a distant observer. Close to the surface of the Earth or over short distances 'g' is approximately constant and the time warp is linear. The effect of this time warp is shown in the space-time map in figure 3.

In the figure the vertical grid lines (or isochrones) common to distance time plots are bent from the vertical. These lines represent a fixed proper time as read by a clock at some height from the surface. It is important to point out the extreme elongation of this map; 3×10^8 m along the horizontal axis is equivalent to approximately one second of time.

Einstein's 'first law'

Traditionally Newton's first law states that an object will remain at rest unless acted on by a net external force. A stationary object having no net external force acting on it will follow the path shown by the solid horizontal line in figure 4. Time progresses without the object changing its position in space, in this case its distance from the Earth. However, in a

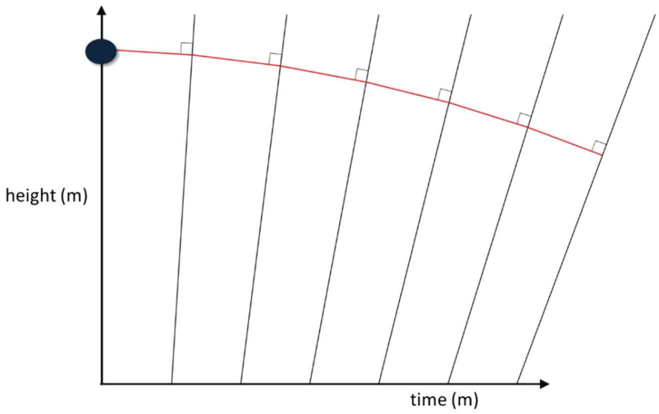

Figure 5. Space and time map for a frame at rest on the Earth's surface, showing the warping of time due the Earth's gravitational field. The solid line represents the path taken by an object in free-fall. It intersects each isochrone at a right angle.

gravitational field an observer on the Earth's surface will perceive that an object in 'free fall' will follow the path shown by the dashed parabolic curve. This observer would say that it accelerates downwards towards the surface of the Earth. In Newton's second law, the acceleration is accounted for by the existence of a downward force which Newton called gravity.

A closer look at the path of the object in figure 4 reveals a key feature of this model. When there are no forces on the object its path on a space and time map intersects the isochrones at 90°. In the case of an object floating in space, the path is a horizontal line. It could be said that the object follows a geodesic, that is, it takes the shortest path to the next moment in time. In a graphical sense this means that the path will intersect an isochrone at right angles. If there is a net external force acting on the object, it will deviate from this path depending on the direction of the force.

By analogy with Newton's first law, we introduce 'Einstein's first law' which states that an object will follow a geodesic on a space and time map when no external force acts on it (i.e. when it is in free fall). The importance of this generalisation of Newton's first law becomes evident when we consider the existence of curved space and time warping.

The diagram displayed in figure 5 is a key plank of this model. In the presence of a massive object such as the Earth the vertical isochrones displayed in figure 4 have become compressed near the large mass. 'Mass has told space-time how to curve'. This was shown previously in figure 3. The solid line shows what happens when an object obeys 'Einstein's first law'. It will follow a geodesic on a space and time map; it will intersect the isochrones at right angles. 'Space-time has told mass how to move'.

If there was no gravitational time warp the isochrones would be vertical and the shortest path would be described by a horizontal line indicating no change in the object's height. This would be true of an object floating in space far from any massive object. It can be seen that the object in free-fall near the surface of the Earth follows a path visually similar to the familiar result found from applying Newton's laws, but no force is involved. The shape of this trajectory is a result of the slowing of time near a large massive object. At this point students can now understand why an object falls; it is not pulled toward the Earth by an attractive force but simply follows a geodesic in space-time.

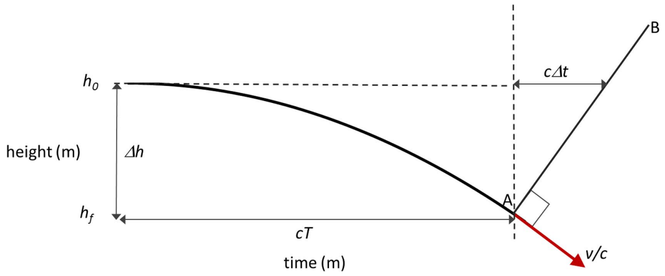

Figure 6. Analysis of a space and time map for an object near the Earth's surface. Initially at a height h_0 the object falls to a height h_f after an elapsed time T as measured by a clock on the surface. v/c is the space-time velocity vector of the object at time T. Δt is the time warp at height h_0 over the time period T. All values are relative to the observer on the surface.

But the question remains; is this graphical model able to produce a quantitative prediction of the motion of an object falling? Can it describe the position, velocity and acceleration of a falling object relative to an observer on the surface of the Earth, just as Newton's laws are able to do?

A calculation of the acceleration produced by time-dilation

The following derivation is included to validate the model and is a suitable exercise for students proficient in mathematics, but is not essential in presenting the model to younger students. We start by considering an object initially at rest at height h_0 above the surface of the Earth and falling a short distance.

The space and time map, figure 6, shows an object initially at rest at a height h_0 above the surface. At time $T = 0$ the object is released and after a time T it has fallen a short distance Δh and its final position is shown at position A where the falling object has velocity v.

The line passing through the points A and B is an isochrone as shown in figure 5. After an elapsed time T as measured by a clock at the surface (height = 0) a clock remaining at height h_0 will record a time $T + \Delta t$. The fractional time warp for a clock at a height h_0 compared to a reference clock on the surface is given by the expression $\frac{\Delta t}{T}$.

Following Einstein's first law we can say that the object in free fall will follow a geodesic on a space and time map. This path will intersect an isochrone perpendicularly requiring the velocity vector at any point to be at right angles to the isochrone at that point. The object's velocity vector v/c at point A in the diagram is shown to intersect the time-line AB perpendicularly. The diagram can now be analysed as follows.

(1) The velocity vector is perpendicular to the time-line AB. Students will note that velocity in space-time is dimensionless and is expressed as a fraction of the speed of light (v/c). When two lines are perpendicular most students would be aware that the product of their slopes (m_i) is equal to -1:

$$m_1 \cdot m_2 = -1.$$

The slope of line AB is:

$$\frac{\Delta h}{c\Delta t}$$

therefore:

$$\frac{v}{c} \cdot \frac{\Delta h}{c\Delta t} = -1. \tag{3}$$

(2) Velocity is related to acceleration by the familiar equation $v = v_0 + aT$. Here the initial velocity $v_0 = 0$ and therefore $v = aT$. Substituting this into equation (3) we get:

$$\frac{aT}{c} \cdot \frac{\Delta h}{c\Delta t} = -1. \tag{4}$$

(3) Rearranging to find $\Delta t/T$ and replacing a with $-g$, the gravitational acceleration near the Earth's surface, we get:

$$\frac{\Delta t}{T} = \frac{g}{c^2}\Delta h.$$

This is the same as equation (2). The model has successfully produced the time warping relationship derived from Schwarzschild's solution to Einstein's equations. Students can now use this result to predict the time warp at any height above the surface of the Earth. (For heights greater than a few metres the term 'g' can be replaced by $\frac{GM}{r^2}$). Alternatively, the equation can be re-arranged to find the acceleration of an object at a height h above the surface if the fractional time warp is known:

$$a = \frac{c^2}{h} \cdot \frac{\Delta t}{T}. \tag{5}$$

Over small changes in height the time warp is extremely small but it is this small change due to the curvature of space-time that explains why an object falls.

Conclusion

The graphical and visual model presented here is based on two simple principles: that time is warped in the vicinity of a large mass and that objects in free fall follow geodesics on a space and time map. The concept of curved space and geodesics was first demonstrated by investigating flight paths on a world map and realising that the separation of lines of longitude do not represent equal distances in an east–west direction. A method to find the shortest path (the geodesic) was described by navigating a path perpendicular to scale lines. This idea has then been applied to space and time maps where we describe the geodesic as the path intersecting isochrones at right angles. Finally, the acceleration of a falling object from the viewpoint of an observer on the Earth has been derived and found to be consistent with the equations of General Relativity. This model we believe is suitable for the introduction of Einstein's theory of General Relativity at a high school level. It provides an answer to the question purportedly asked by Newton, 'why did the apple fall?' without resorting to his gravitational force.

Acknowledgments

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Appendix. Simple derivation from the Schwarzschild metric

The metric for space and time near the Earth (ignoring its spin) can be written as follows:

$$\mathrm{d}s^2 = \left(1 - \frac{2MG}{rc^2}\right)(c\mathrm{d}t)^2 - \left(1 - \frac{2MG}{rc^2}\right)^{-1}\mathrm{d}r^2.$$

As argued by Gould [3], the space warp is negligible in the case of an object falling from rest near the Earth's surface. The equation can be reduced to find the fractional time dilation (or scale of time) as a function of the distance from the Earth's centre r as follows:

$$\frac{T_r}{T_\infty} = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}}.$$

The equation can be approximated using a binomial expansion and noting that for objects near the Earth's surface: $g = \dfrac{GM}{R^2}$

$$T_\infty = T_r\left(1 + \frac{gr}{c^2} + \frac{3g^2r^2}{2c^4}. + \ldots\ldots\right),$$

$$\frac{\Delta t_r}{T_\infty} = \frac{gr}{c^2} + \frac{3g^2r^2}{2c^4}\ldots\ldots\ldots,$$

where $\Delta t_r = T_\infty - T_r$.

The last terms are diminishingly small so the relationship becomes:

$$\frac{\Delta t_r}{T_\infty} = \frac{g}{c^2}r.$$

More generally, for objects further from the surface of the Earth the equation can be written:

$$\frac{\Delta t_r}{T_\infty} = \frac{GM}{rc^2}.$$

This is equation (1). In the above; T_∞ is the total time as measured by a clock at a large distance from the centre of the Earth. T_r is the total time measured at a distance r from the centre of the Earth. Δt_r is the difference between these two times. As $T_\infty > T_r$ Δt_r is expressed as $T_\infty - T_r$ to obtain a positive value.

For the investigation of the effects of gravity near to the surface of the Earth it would be useful to compare the time dilation Δt_h at a height h above the surface of the Earth with the time T_s as observed from the surface of the Earth. A simple derivation obtains the following relationship:

$$\frac{\Delta t_h}{T} = \frac{g}{c^2}h.$$

This is equation (2), the subscript 's' has been omitted.

$\frac{\Delta t_h}{T}$ is the fractional time difference between a clock at a height h above the surface of the Earth compared to a clock on the surface. All values as measured by an observer on the surface. Note that in this instance $\Delta t_h = T_h - T_s$ which gives a positive value.

References

[1] Stukeley W 2015 Memoirs of Sir Isaac Newton's life (viewed September at http://ttp.royalsociety.org/ttp/ttp.html?id=1807da00-909a-4abf-b9c1-0279a08e4bf2&type=book)

[2] Wheeler J A and Ford K W 2000 *Geons, Black Holes, and Quantum Foam: A Life in Physics* (New York: W W Norton & Company) p 153

[3] Gould R R 2016 Why does a ball fall?: a new visualization for Einstein's model of gravity *Am. J. Phys.* **84** 396

[4] Nicholson T L *et al* 2015 Systematic evaluation of an atomic clock at 2×10^{-18} total uncertainty *Nat. Commun.* **6** 6896

[5] Schwarzschild K 1916 Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie *Sitz.ber. Königlich Preussischen Akad. Wiss.* **7** 189–96 (Schwarzschild tragically died shortly after his paper was published while serving in World War I)

[6] Vessot R F C and Levine M W 2015 Gravitational Redshift Space-Probe Experiment (http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19800011717.pdf, retrieved November)

[7] Vessot R F C *et al* 1980 Test of relativistic gravitation with a space-borne hydrogen maser *Phys. Rev. Lett.* **45** 2081–4

[8] 'Real-World Relativity: The GPS Navigation System' (http://astronomy.ohio-state.edu/, retrieved 13 November 2015)