GRAVITY AS A SPIN SYSTEM ☆

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General relativity is reformulated in a way which indicates gravity may be viewed as a three-dimensional spin system. This presents a new hamiltonian approach to quantum gravity.

The idea that gravitation may be viewed as a collective phenomenon is an attractive one and has been pursued by some [1], inspired by the BCS theory of superconductivity and the Nambu–Jona–Lasinio model. But there is little evidence supporting this view. In this note, we propose a new approach which favors the interpretation of gravity as an effective theory. We show that general relativity can be regarded as a spin system ‡1 in three dimensions, thus providing another approach to quantum gravity. We shall only present the main ideas here. Details will be published elsewhere.

In the hamiltonian description of general relativity, due to Dirac and Arnowitt–Deser–Misner (ADM) ‡2, the basic field is taken to be the riemannian metric on a spacelike three-manifold. Our motive is to replace this metric by some other "basic" variables from which the metric will be a derived concept. We start with a few crucial observations about the initial value formulation of general relativity.

Consider a spacetime with a metric satisfying the vacuum Einstein equations. The initial data on a spacelike hypersurface S is the pair of symmetric tensor fields (q_{ab}, π_{ab}) which are the induced metric and the second fundamental form respectively. And the pair must satisfy two constraints:

$$H^a = D_a(\pi^{ab} - \pi q^{ab}) = 0\ , \tag{1}$$

$$H = -R - \pi^{ab}\pi_{ab} + \pi^2 = 0\ , \tag{2}$$

where R is the scalar curvature of (S, q_{ab}) and $\pi = \pi^{mn} q_{mn}$; indices are raised and lowered with q_{ab} ‡3.

Since we have a riemannian metric q_{ab} on S, we can consider SU(2) spinor fields on S and, in particular, represent tensor fields by equivalent spinors. Thus, for example, we shall write $T^a \equiv T^{(AB)}$ obtained by replacing the tensor index by a pair of symmetric SU(2) spinor indices, such that $T^{\dagger(AB)} = -T^{(AB)}$ where † is the SU(2) adjoint. Spinor indices are raised and lowered with the skew spinor $\epsilon_{[AB]}$ ‡4.

Next we define two spinor connections $\nabla_a{}^{\pm}$ $(\equiv \nabla^{\pm}{}_{(AB)})$ by

$$\nabla_a{}^{\pm}\lambda_A = D_a\lambda_A \pm 2^{-1/2}\pi_{aA}{}^{B}\lambda_B\ , \tag{3}$$

where D_a is the derivative operator defined by q_{ab} and λ_A a smooth spinor field. Their action on spinors of higher valence is extended by Leibnitz rule. In particular, $\nabla_a{}^{\pm}\epsilon_{AB} = 0$. Furthermore,

$$(\nabla^{+}{}_{(MN)}\lambda_A)^{\dagger} = -\nabla^{-}{}_{(MN)}\lambda^{\dagger}{}_A\ , \tag{4}$$

i.e., they are SU(2) connections. The curvature $F^{\pm}{}_{[ab]M}{}^{N}$ of $\nabla_a{}^{\pm}$ can be regarded as a tensor field

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‡1 Spin systems are models of magnetism and phase transition well studied in statistical mechanics and field theory (e.g. ref. [2]).
‡2 See ref. [3] for recent reviews and ref. [4] for a critical discussion.
‡3 We choose signature (−, −, −).
‡4 See ref. [5] for notations. We use ( ) and [ ] for symmetric and skew symmetric indices respectively.

$F^{\pm}{}_{[ab]c}$ (lowering the spinor index N) or equivalently the tensor field $C^{\pm}{}_{ab} := \epsilon_a{}^{mn} F^{\pm}{}_{mnb}$. ($\epsilon_{abc}$ is the alternating tensor defined by q_{ab}.)

These connections have the following remarkable property. Constraints (1) and (2) respectively imply that $C^{\pm}{}_{ab}$ is trace free and symmetric. In spinor form $C^{\pm}{}_{ab} \equiv C^{\pm}{}_{(ABCD)}$, i.e., a pure spin 2 object. Thus the constraints are coded as symmetry conditions on the curvature, leaving only its spin 2 part. Moreover, $C^{+}{}_{(ABCD)}$ and $C^{-}{}_{(ABCD)}$ are respectively self-dual and anti self-dual. Finally, one can recover the metric [6] and π_{ab} from the pair $(\nabla_a{}^{+}, \nabla_a{}^{-})$.

Based on the above observations we propose the following framework. Fix a smooth three-manifold Σ (of arbitrary topology) and consider SU(2) spinor fields, denoted λ_A and $\lambda^{\dagger}{}_A$, on it and (complex) spinor connections $\nabla^{+}{}_{(AB)}$ and $\nabla^{-}{}_{(AB)}$ acting on them. Since we do not have a metric on Σ we cannot now freely replace tensor indices with equivalent spinor ones. We therefore work entirely with spinor fields.

Consider next the (complex, affine) space of all pairs of connections (∇^{+}, ∇^{-}), not necessarily satisfying eq. (4). The space of interest for quantum gravity is the subspace P consisting of only those pairs whose curvatures have only the spin 2 information. That is, if the curvature

$$F^{\pm}{}_{(AB)(CD)(MN)} \equiv C^{\pm}{}_{(MNAC)}\,\epsilon_{BD} + C^{\pm}{}_{(MNBD)}\,\epsilon_{AC}\,.$$

P can be viewed as the product of two spaces, $\{(\nabla^{+}, \underset{\circ}{\nabla}^{-})\}$ and $\{(\underset{\circ}{\nabla}^{+}, \nabla^{-})\}$ where $\underset{\circ}{\nabla}^{\pm}$ has zero curvature. These "left" and "right" flat spaces are precisely the spaces consisting of non-linear gravitons constructed by Penrose using twistor theory [‡5].

P is a complex space of connections while the phase space P* of general relativity is real. What is the relation between the two? P should be thought of as a complex phase space equipped with a real structure: the pair (∇^{+}, ∇^{-}) will be called real if they satisfy (4). P* is the real subspace of P.

We would now like to construct a hamiltonian using the spinor connections as the basic variables. From the index structure of the connections a candidate for a quadratic hamiltonian is

$$\mathrm{H} = \mathrm{i}/2 \int_{\Sigma} [(\nabla^{-}{}_{(MN)}\lambda_A)(\nabla^{+}{}_{(PQ)}\lambda^{\dagger}{}_B) + (\nabla^{+}{}_{(MN)}\lambda_A)(\nabla^{-}{}_{(PQ)}\lambda^{\dagger}{}_B)]\,\mathrm{d}S^{(AB)(MN)(PQ)}\,, \quad (5)$$

where λ_A is an arbitrary spinor field on Σ [‡6].

That (5) is indeed the correct hamiltonian for general relativity comes from an observation first made by Ashtekar and Horowitz [7] in a different context. In the ADM framework, the hamiltonian is of the form

$$H_{\mathrm{ADM}} = \int (NH - 2N_a H^a)\,\mathrm{d}S + M\,,$$

where N and N^a are smooth functions (called lapse and shift) on S and M is a boundary term giving the ADM mass [‡7]. It is easy to check that $\mathrm{H} = H_{\mathrm{ADM}}$ by integrating (5) by parts. The boundary term is precisely the form of ADM mass given by Nester [8]. The spinor field entering in H is to be interpreted as the "square root" of the lapse and shift functions defined by $N = 2^{-1/2}\lambda^{\dagger A}\lambda_A$, and $N^{(AB)} = \lambda^{\dagger(A}\lambda^{B)}$.

A more convenient form of eq. (5) is

$$\frac{-1}{2\sqrt{2}} \int_{\Sigma} [(\nabla^{-(MN}\lambda^{\dagger A)})(\nabla^{+}{}_{(MN}\lambda_{A)}) - \tfrac{2}{3}(\nabla^{-}{}_{AM}\lambda^{\dagger M})(\nabla^{+AM}\lambda_M) + (\nabla^{+(MN}\lambda^{\dagger A)})(\nabla^{-}{}_{(MN}\lambda_{A)}) - \tfrac{2}{3}(\nabla^{+}{}_{AM}\lambda^{\dagger M})(\nabla^{-AM}\lambda_M)]\,\mathrm{d}S\,, \quad (6)$$

which manifests the spin system structure with λ_A as a spin variable "gauge" coupled to the connections.

Observe that when λ_A satisfies the zero-mode equation [5]

$$\nabla^{\pm AB}\lambda_B = 0\,, \quad (7)$$

the hamiltonian is positive definite (noted also by Ashtekar and Horowitz) on P* and takes the non-linear sigma model form. Two of its features are peculiar to general relativity. One is the mixing of "internal" and "space" indices in $\nabla^{+}{}_{(AB}\lambda_{C)}$, giving a

‡5 See the article by Penrose and Ward in ref. [3].

‡6 In units $c = 1$ and $16\pi G = 1$.

‡7 We are considering asymptotically flat spaces here requiring N and N^a to have a certain fall off at infinity but otherwise arbitrary. We ignore these subtleties here. See ref. [4].

pure spin 3/2 field. The other is the nature of the constraint on λ_A imposed by (7). It should perhaps be viewed as a Gauss law constraint which generates spinor gauge transformations.

Several aspects of this formalism needs to be better understood, most important being the precise role of the spinor field λ_A and the choice of the SU(2) norm (given by †) both of which seem to have coded in them the notion of "time" and causal structure in quantum gravity.

From its similarities with the sigma model, together with the ideas suggested in [9], one might expect to probe the ultraviolet structure of the theory. Moreover, our formalism suggests a lattice version of the theory similar to the one discussed by Lewis [10] for three-dimensional gravity using the relation between Regge calculus and 6j symbols.

Some broad directions of enquiry are also suggested by the approach presented here. It gives a different "square root" of general relativity than that suggested by supergravity theory ‡8. Furthermore, it also seems to provide a natural bridge between quantum gravity and twistor theory which has hitherto remained obscure.

‡8 See the articles by Deser, and Teitelboim in ref. [3].

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