Simple Derivations of the Schwarzschild Metric

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Two derivations, one of the Schwarzschild line element, the other of the gravitational bending of light, both using only special relativity, the equivalence principle, and the Newtonian law of gravity, are examined in detail and shown to be incorrect. In both derivations equations of the correct form were derived, but the physical meanings of the symbols are not what they purport to be. The postulates, in addition to the above three that are necessary to derive the bending of light, are discussed briefly.

INTRODUCTION

SINCE 1944, there have been a number of attempts[1-3] to derive the Schwarzschild line element by using only the three postulates of special relativity, the equivalence principle, and the Newtonian law of gravity. Schiff[4] attempts to derive the gravitational bending of light using the same postulates. The amount of this bending involves the term of order $2GM/c^2r$ in the coefficient of dr^2 in the Schwarzschild line element, which is given by the expression

$$ds^2 = \frac{dr^2}{1-2GM/c^2r} + r^2\,(d\theta^2 + \sin^2\theta d\varphi^2) - c^2dt^2\left(1-\frac{2GM}{c^2r}\right). \quad (1)$$

In all these derivations an answer of the correct form is arrived at through a combination of invalid physical interpretations and coincidences.

Schild[5] has shown that the three postulates allow us to derive the first-order term in the coefficient of c^2dt^2 but not in the coefficient of dr^2. Yet many workers remain unsure of the impossibility of deriving the latter term, and attempts are still being made to refine earlier derivations. Eriksson's paper[3] is a refinement of Lenz's derivation,[1] and in their book Adler, Bazin, and Schiffer[6] offer a refinement of Schiff's derivation,[4] but mention explicitly that its validity is questionable.

We feel that a discussion of the errors in these derivations will help to resolve this disagreement. That discussion is the purpose of this paper.

I. DERIVATIONS OF LENZ, BALAZS, AND ERIKSSON

Balazs's paper is a defense of Lenz against a criticism of Kohler[7] to the effect that the Lorentz transformation between differentials is incorrect. We agree with Balazs that the particular criticism that Kohler makes does not invalidate Lenz's derivation, but Eriksson and Yngström[8] have correctly criticized it to the effect that Lenz makes no distinction between coordinate differentials and measured differentials of length and time. Eriksson[3] continues, however, to make another error that is common to both his work and Lenz's.

Eriksson's argument goes briefly as follows. Consider a spherically symmetric mass M surrounded by vacuum, and stationary with respect to the macroscopic frame S. Because of the spherical symmetry and stationary nature of the

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[1] W. Lenz, given in A. Sommerfeld, *Electrodynamics* (Academic Press Inc., New York, 1952), p. 313.
[2] N. L. Balazs, Z. Physik **154**, 264 (1959).
[3] K. E. Eriksson, Arkiv. Fysik **25**, 167 (1963).
[4] L. I. Schiff, Am. J. Phys. **28**, 340 (1960).
[5] A. Schild, Am. J. Phys. **28**, 778 (1960).
[6] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw–Hill Book Company, New York, 1965), pp. 194–197.
[7] M. Kohler, Z. Physik **130**, 139 (1951).
[8] K. E. Eriksson and S. Yngström, Phys. Letters (Neth.) **5**, No. 5, 327 (1963).

problem we can write a general expression for the metric as

$$ds^2 = a(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - b(r)c^2dt^2. \quad (2)$$

The coordinate r of a sphere can be seen to be proportional to the measured circumference of a great circle, or equivalently r^2 is proportional to the measured area of the sphere. These measurements are made by infinitesimal rods oriented transverse to the radial direction. Rods oriented along the radial direction do not directly measure the radius, as indicated by the metric.

Consider a local inertial frame L falling from rest at ∞ toward M. Consider another local inertial frame K momentarily at rest with respect to S just as L falls past it. Since these are both inertial frames locally they can be connected by an infinitesimal Lorentz transformation (with x_L and x_K chosen along the radial direction):

$$\left.\begin{aligned} dx_K &= [dx_L - v(r)dt_L]\,\Gamma(r), \\ dt_K &= [dt_L - v(r)dx_L/c^2]\,\Gamma(r), \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} dx_L &= [dx_K + v(r)dt_K]\,\Gamma(r), \\ dt_L &= [dt_K + v(r)dx_K/c^2]\,\Gamma(r), \end{aligned}\right\} \quad (4)$$

where $\Gamma(r) = [1 - v^2(r)/c^2]^{-1/2}$. Clearly, the transverse directions give

$$\left.\begin{aligned} dy_K &= dy_L = r d\theta, \\ dz_K &= dz_L = r\sin\theta d\varphi, \end{aligned}\right\} \quad (5)$$

and the line element is given by

$$ds^2 = dx_L{}^2 - c^2dt_L{}^2 = dx_K{}^2 - c^2dt_K{}^2, \quad (6)$$

where for simplicity we have dropped the transverse differentials, assuming them to be zero.

Let us consider briefly the meaning of Eq. (6). The line element expresses the relationship between the coordinate differentials of a pair of infinitesimally separated events or points in space–time. Indeed, the name "line element" refers to the line in space–time joining the two events. Only when the right-hand sides of Eq. (6) contain the infinitesimal differences in the corresponding coordinates of a *given pair* of events is the result an invariant. In other words, the sum of the squares of any arbitrary set of coordinate differentials will not result in a physically significant quantity, let alone an invariant.

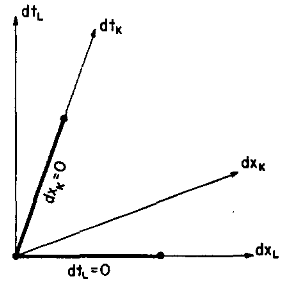

FIG. 1. Two different pairs of events for measurements of length contraction and time dilation.

While this seems trivial when pointed out, it is precisely in failing to realize this that Lenz, Balazs, and Eriksson make their fundamental error.

Eriksson proceeds after Eq. (6) by combining two steps, which masks this error. Let us separate the two steps and examine each of them. Consider a pair of events such that $dt_L = 0$ (see Fig. 1). The first Eq. (3) gives

$$dx_K = dx_L\,\Gamma(r). \quad (7)$$

Next, consider a pair of events such that $dx_K = 0$ (see Fig. 1). The second Eq. (4) gives, after rearrangement,

$$dt_K = dt_L/\Gamma(r). \quad (8)$$

If we insert Eqs. (7) and (8) into Eq. (6) (incorrectly for the reason given above), we get

$$ds^2 = dx_L{}^2\,\Gamma^2(r) - c^2dt_L{}^2/\Gamma^2(r). \quad (9)$$

Then using the Newtonian expression for energy conservation,

$$\tfrac{1}{2}\,v^2(r) = GM/r, \quad (10)$$

which at this stage we only know to be correct to first order, we get

$$ds^2 = \frac{dx_L{}^2}{1 - 2GM/c^2r} - c^2dt_L{}^2\left(1 - \frac{2GM}{c^2r}\right). \quad (11)$$

In his second step, Eriksson identifies what should more precisely be written as $dx_L(dt_L = 0)$ and $dt_L(dx_K = 0)$, with dr and dt through arguments that we shall examine below, thus arriving at what looks like the Schwarzschild line element

$$ds^2 = \frac{dr^2}{1 - 2GM/c^2r} - c^2dt^2\left(1 - \frac{2GM}{c^2r}\right). \quad (12)$$

Making these identifications before inserting them into Eq. (6) hides the obvious contradiction between Eqs. (6) and (11).

Thus, two different pairs of events, especially chosen, are used to arrive at an expression resembling the Schwarzschild line element. But then the right-hand side of Eq. (11) has no physical significance, and is not an invariant. Therefore, Eq. (12) cannot be the Schwarzschild line element even though it looks like it. Neither is it possible to find a single pair of events for which both Eqs. (7) and (8) hold, because, among other things, the first equation necessarily pertains to a spacelike pair and the second to a timelike pair. This use of two pairs of events is common to the derivations of Lenz, Balazs, and Eriksson.

Now let us examine Eriksson's identification of $dx_L(dt_L = 0)$ and $dt_L(dx_K = 0)$ with dr and dt. He unjustifiably defines $dt_L(dx_K = 0)$ to be equal to dt, thus passing over the fact that it does not have the same physical significance as the dt in the Schwarzschild line element. It turns out that they are numerically equal, and in Sec. II we show this. In identifying $dx_L(dt_L = 0)$ with dr, he essentially assumes in an inapparent way what he wishes to prove. Let us now see how.

Eriksson considers two radially oriented rods, of length Δr, falling together in from rest at ∞. There is an angle, $\Delta\theta$, between them which is maintained throughout the free fall. One of them, AB, is at rest in L; but since the other, $A'B'$, converges toward AB, it is not at rest in L (see Fig. 2). Since they start out together, B and B' are always on the same sphere about M, as are A and A'. According to the general form of the metric in Eq. (2) and because transverse rods are unaffected by their motion, rods at rest in L will measure the distance between B and B' to be $r\Delta\theta$, where r is the coordinate of the sphere that B and B' are on at the instant of measurement. Simultaneously, according to clocks in the locally inertial frame L, rods at rest in L will measure the distance from A to A', says Eriksson, to be $(r + \Delta r)\ \Delta\theta$, since the lengths of AB and $A'B'$ are picked to be Δr. Thus, he concludes that radially oriented rods in L measure the distance between spheres to be equal to the difference in the coordinates of the spheres.

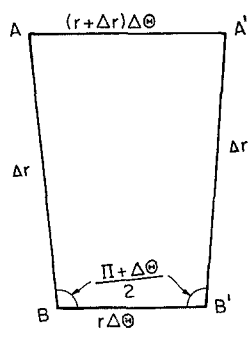

FIG. 2. Rods AB and $A'B'$ fall together toward mass M, with angle $\Delta\theta$ between them.

He arrives at the above result for the distance from A to A' solely by virtue of the fact that L has a locally Euclidean metric. But this fact alone does not guarantee such a result for the following reason. The local Euclidean nature of the metric in L will guarantee Euclidean results only to first order, but Eriksson gives the distance from A to A' to second order; i.e.—the product of Δr and $\Delta\theta$ is a second-order quantity. There is only one locally Euclidean frame that will give the correct result to second order, and it is the one that falls from rest at ∞, namely L. In other words, an equivalent measurement with AB and $A'B'$ at rest in, say, K, which is also locally Euclidean, will give the distance from A to A' to be

$$[r + \Delta r(1 - 2GM/c^2r)^{1/2}]\ \Delta\theta,$$

as can be calculated from the metric. This expression agrees to first order with the result obtained in L, as it must, but does not agree to second order. To get the correct second-order result, one must make the measurement in L, but as far as we know this fact cannot be shown without using the Schwarzschild line element to begin with.

A valid derivation of the Schwarzschild line element must show for independent reasons why L is to be used and we cannot simply call it a "natural" choice because of a certain unique property, namely zero Newtonian energy. The relevance of this property must be shown.

II. SCHIFF'S DERIVATION OF THE BENDING OF LIGHT

In deriving the behavior of rods and clocks in a spherically symmetric gravitational field,

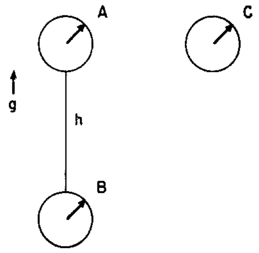

FIG. 3. Identical clocks A and B accelerate past C at rest in an inertial frame.

Schiff, unlike Eriksson, goes into detail about the clocks and simply assumes corresponding validity for rods. His argument goes as follows. Consider an inertial frame away from all gravitational fields with clock C at rest. Two other identical clocks A and B, connected by a rod of length h, accelerate past C along the direction of h with an acceleration g. Schiff compares the rates of A and B as observed by C. Using the time dilatation of special relativity, he finds that B is going slower than A, since by the time B passes C, A and B have undergone acceleration to a higher velocity (see Fig. 3).

The time-dilatation formula gives

$$\tau_c = \frac{\tau_A}{(1 - v_A^2/c^2)^{1/2}}. \tag{13}$$

(Schiff uses the periods of the clocks rather than the time measured by them.) Similarly, we have

$$\tau_c = \frac{\tau_B}{(1 - v_B^2/c^2)^{1/2}}, \tag{14}$$

where

$$v_B \cong v_A + gh/[\tfrac{1}{2}\,(v_A + v_B)], \tag{15}$$

or

$$v_B^2 - v_A^2 \cong 2gh. \tag{16}$$

Eliminating τ_c from Eqs. (13) and (14) gives

$$\begin{aligned} \tau_B &= \tau_A(1 - v_B^2/c^2)^{1/2}/(1 - v_A^2/c^2)^{1/2} \\ &\cong \tau_A[1 - (v_B^2 - v_A^2)/c^2]^{1/2} \\ &\cong \tau_A(1 - gh/c^2). \end{aligned} \tag{17}$$

Schiff then invokes the equivalence principle, placing A and B at rest in a gravitational field of local intensity g, and states that the above result is a derivation of the gravitational frequency shift. But this is incorrect, because the equivalence principle states only that an experiment done in an accelerating frame, away from gravitational fields, will give the same result as the *same* experiment done in a frame at rest in a gravitational field whose local intensity is equal to the acceleration of the first frame. This is not what Schiff has done, although in this case he could have done so and would have arrived at the same answer by coincidence, as we show below. The invalid extension to rods of a coincidentally correct (but invalid) result for clocks gives by further coincidence the correct answer for the bending of light and for the Schwarzschild line element.

Now let us see why the "comparison" of A and B is correct by coincidence only. First, Schiff's use of C to "compare" A and B is valid only within the context of this kind of comparison. That is, a proper use of the equivalence principle shows only that if C is in free fall past A and B in the gravitational field, observers on C, or more properly observers in C's frame with their own clocks synchronized by light signals, do, indeed, observe B to be going slower than A. The gravitational frequency shift, on the other hand, must be derived by letting A emit a photon of frequency ν_A, and letting B measure the frequency of that photon ν_B (or vice versa). The comparison of ν_B and ν_A in the case of an accelerating frame, away from gravitational fields, is then the same result as would be obtained with A and B at rest in a gravitational field of the proper intensity.

Let us carry out this calculation. When B receives the photon from A, because of the acceleration, he is going toward A with a velocity v_B' measured in the inertial frame in which A is at rest at time of emission. This is given (to first order) by

$$v_B' = gh/c. \tag{18}$$

The Doppler-shift formula of special relativity gives

$$\begin{aligned} \nu_B &= \nu_A(1 + v_B'/c)^{1/2}/(1 - v_B'/c)^{1/2} \\ &= \nu_A(1 + gh/c^2)^{1/2}/(1 - gh/c^2)^{1/2}. \end{aligned} \tag{19}$$

The periods of the photon as measured by A and B are related by

$$\begin{aligned} \tau_B' &= \tau_A\left(1 - \frac{gh}{c^2}\right)^{1/2} \Big/ \left(1 + \frac{gh}{c^2}\right)^{1/2} \\ &\cong \tau_A\left(1 - \frac{gh}{c^2}\right), \end{aligned} \tag{20}$$

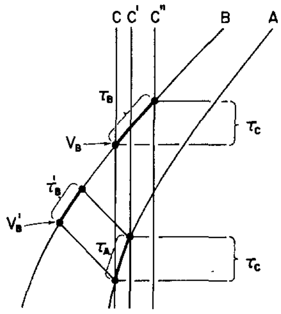

FIG. 4. A and B accelerate past C, C', and C''.

which is the same result as Eq. (17). But it has an entirely different physical meaning, hence the prime. Figure 4 shows the world lines of A, B, and C, and two other observers in C's frame, who are necessary to carry out the required measurements. Given that τ_c is the same above and below, it turns out that to first order $\tau_B' = \tau_B$ [see Eqs. (17) and (20)], but there appears to be no physical significance to this coincidence. Special relativity is a theory in which the expression $(1 - v^2/c^2)^{1/2}$ pops up everywhere, so it is perhaps not surprising that coincidences occur.

The extension of the "comparison" of clocks to the behavior of rods is similarly inapplicable to describing the behavior of light, and has not been shown to have any physical meaning outside of what C, in free fall in a gravitational field, observes about rods held by A and B. The $2GM/c^2r$ term in $a(r)$ [see Eq. (2)] describes a curvature of space in a section of space–time, $t =$ const; i.e.—$a(r)$ contains information about the geometry of a static mesh of infinitesimal rigid rods filling the space around M. In particular, the exact expression for $a(r)$, namely $(1 - 2GM/c^2r)^{-1}$, tells us that for equal intervals in r the measured distance along a radius becomes greater and greater as we get closer to M. Another way to look at it is that radially oriented rods shrink more and more relative to transverse rods as we get closer to M. No information about this shrinkage to any order is contained in the three postulates. All that they tell us is that observers in L will measure these radially oriented rods to be smaller and smaller as their velocity increases closer to M. But this is not the same as the shrinkage due to the curvature of space. These measurements by observers in L are above and beyond the shrinkage due to space curvature because their rods also participate in this shrinkage. The measured contractions are due solely to relative velocity and not to distance from M. The confusion arises because of the identical sizes of these two effects for observers starting at rest at ∞.

Since Eqs. (17) and (20) give the same result, we see that observers in the frame L falling in from rest at ∞ can make comparisons of clocks at rest in the gravitational field, in the fashion described in this section, and get, in addition, the correct answer for the gravitational frequency shift by coincidence. Without showing it in detail, we can see that since the gravitational frequency shift is implicitly contained in the Schwarzschild line element, the demonstration that Eqs. (17) and (20) give the same result is essentially a demonstration that the dt in the Schwarzschild line element is numerically equal to $dt_L(dx_K = 0)$. Thus, although Eriksson's method of equating the two quantities by definition is invalid, they are indeed numerically equal.

III. CONCLUSION

The question suggests itself: Is it possible to derive the $2GM/c^2r$ term in $a(r)$ with any additional postulates short of the field equations of general relativity? The answer is yes. Tangherlini[9] has derived it by adding the postulates of geodesic motion and what he calls the strong-equivalence principle. This principle states that the acceleration of a particle in purely radial motion must be independent of its speed and a function only of its distance from M. He goes beyond this and derives the exact expression for the Schwarzschild line element, but only by postulating an exact form of Newton's law of gravity. Before the exact form of the metric is known, there are several possible definitions of $v(r)$, differing from one another by factors of $(1 - 2GM/c^2r)$, any one of which might be the one that makes the Newtonian law, given by Eq. (10), exact. For example, $v(r)$ might be dr/dt, dx_K/dt_K, etc. To choose a particular one, along with the strong equivalence principle, is equivalent to choosing an exact form for both

[9] F. R. Tangherlini, Nuovo Cimento **25**, 1081 (1962).

$a(r)$ and $b(r)$. To choose a particular definition of $v(r)$ without postulating the strong equivalence principle, however, does not yield the $2GM/c^2r$ term in $a(r)$, nor does it yield the second-order term in $b(r)$.

In summary, the three postulates of special relativity, the equivalence principle, and the Newtonian law of gravity allow us to derive the first-order term in $b(r)$ but not in $a(r)$. If we further postulate the strong-equivalence principle, we can derive in addition the first-order term in $a(r)$, even with only first-order accuracy in the Newtonian law of gravity. If we further postulate the exact form of this law we can derive the Schwarzschild line element exact to all orders, as shown by Tangherlini.[9]

Fields Due to a Slow Charged Particle Moving Parallel to a Plane-Metal Surface*

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A solution is found for the fields due to a charged particle moving parallel to a plane-metal surface in the limit of low velocity. It is found that the current distribution in the metal is mathematically simple; the current density has a dipole pattern of flow. The solution offers interesting examples of charge conservation and the use of the uniqueness theorem in solving potential problems.

I. INTRODUCTION

WHEN a charged particle is moving near a metal surface, the fields in the metal are, in general, quite complicated, with the electric field exhibiting vorticity ("eddy currents"). If the charged particle is moving very fast, the skin-depth effect prevents the field from entering the metal beyond a thin surface layer. However, the thickness ("skin depth") of this surface layer increases if the velocity is less, and if the particle is going sufficiently slowly that the thickness of the penetration layer is much larger than the distance from the particle to the metal surface, a simple field pattern results.

In this paper we will find solutions for the electric and magnetic fields which are exact in the limit $(a/\delta)^2 << 1$, where a is the distance from the charged particle to the surface, and δ is the skin depth. It is found that the electric field (or current density) inside the metal has a simple dipole pattern.

In Sec. II, we derive a boundary condition for the electric field inside the metal, based on the well-known electrical-image solution for the electric field of a point charge near a plane-metal surface. We show in Sec. III that, if the condition $(V/c)^2 << (a/\delta) << 1$ is satisfied, we can set $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$ inside the metal, so that $\mathbf{E}$ is derivable from a potential Φ which obeys Laplace's equation. In Sec. IV we find a solution of Laplace's equation, using symmetry arguments, and employ the uniqueness theorem to verify that it is correct. The magnetic field also has a mathematically simple expression in the limit considered here, but it has no simple interpretation in terms of multipole fields and requires somewhat more elaborate techniques for solution. It is discussed in the Appendix.

II. BOUNDARY CONDITION ON THE ELECTRIC FIELD IN THE METAL

If a particle of charge Q is at rest a distance a from a plane-metal surface (see Fig. 1), the electric field vanishes inside the metal, and the field outside the metal is the superposed fields

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