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Classical Physics as Geometry

Gravitation, Electromagnetism, Unquantized Charge, and Mass as Properties of Curved Empty Space*

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If classical physics be regarded as comprising gravitation, source free electromagnetism, unquantized charge, and unquantized mass of concentrations of electromagnetic field energy (geons), then classical physics can be described in terms of curved empty space, and nothing more. No changes are made in existing theory. The electromagnetic field is given by the "Maxwell square root" of the contracted curvature tensor of Ricci and Einstein. Maxwell's equations then reduce, as shown thirty years ago by Rainich, to a simple statement connecting the Ricci curvature and its rate of change. In contrast to unified field theories, one then secures from the standard theory of Maxwell and Einstein an "*already* unified field theory." This purely geometrical description of electromagnetism is traced out in detail. Charge receives a natural interpretation in terms of source-free electromagnetic fields that (1) are everywhere subject to Maxwell's equations for free space but (2) are trapped in the "worm holes" of a space with a multiply-connected topology. Electromagnetism in such a space receives a detailed description in terms of the existing beautiful and highly developed mathematics of topology and harmonic vector fields. Elementary particles and "real masses" are completely excluded from discussion as belonging to the world of quantum physics.

"I transmit but I do not create; I am sincerely fond of the ancient."—Confucius.

I. IS THE SPACE-TIME CONTINUUM ONLY AN ARENA, OR IS IT ALL? CLASSICAL PHYSICS REGARDED AS COMPRISING GRAVITATION, ELECTROMAGNETISM, UNQUANTIZED CHARGE, AND UNQUANTIZED MASS; ALL FOUR CONCEPTS DESCRIBED IN TERMS OF EMPTY CURVED SPACE WITHOUT ANY ADDITION TO ACCEPTED THEORY; THE ELECTROMAGNETIC FIELD AS THE "MAXWELL SQUARE ROOT" OF THE CONTRACTED CURVATURE; UNQUANTIZED CHARGE DESCRIBED IN TERMS OF SOURCE FREE MAX-

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WELL FIELD IN A MULTIPLY-CONNECTED SPACE; UNQUANTIZED MASS ASSOCIATED WITH COLLECTION OF ELECTROMAGNETIC FIELD ENERGY HELD TOGETHER BY ITS OWN GRAVITATIONAL ATTRACTION; HISTORY OF IDEAS OF PHYSICS AS GEOMETRY; SUMMARY OF PAPER

Two views of the nature of physics stand in sharp contrast:

(1) The space time continuum serves only as *arena* for the struggles of fields and particles. These entities are foreign to geometry. They must be added to geometry to permit any physics.

(2) There is nothing in the world except empty curved space. Matter, charge, electromagnetism, and other fields are only manifestations of the bending of space. *Physics is geometry.*

To understand how far one can go in regarding classical physics as geometry is the object of this paper. Nothing will be said here about the fascinating issue[1] of quantizing this classical pure "geometrodynamics" (Table I).

In describing classical physics (in the sense of Table I) as geometry, we invent no new ideas. We accept Maxwell's 1864 electrodynamics of empty space, his formulation of the stress-momentum-energy tensor of the electromagnetic field, and Einstein's forty-one-year old description of gravitation in terms of curved space. Restricting attention to classical concepts (Table I) we take as source of metric fields, $g_{\mu\nu}$, *exclusively* electromagnetic fields, $F_{\alpha\beta} = (c^2/G^{1/2})f_{\alpha\beta}$—and electromagnetic fields that are themselves *free of all sources*[2]:

[1] See, however, Misner (*1*) and Wheeler (*2*) for a partial discussion of some features of this problem.

[2] We accept the following familiar conventions: Greek labels refer to four dimensional space; Latin labels refer to three dimensional space. The fourth coordinate, $x^0 (= T = ct =$ "cotime" in flat space) receives the label 0 to prevent confusion with the occasional use in special relativity of x^4 to designate ict. The proper distance, ds, or proper interval of cotime, $d\tau$, between two neighboring events is given by

$$(ds)^2 = -(d\tau)^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta.$$

In flat space and Euclidean coordinates, the metric tensor is diagonal with $-1, 1, 1, 1$ in the diagonal. Many of the considerations of this article deal with space like manifolds, on which it is a great convenience to have a positive definite metric, as given by the present convention (see also Pauli, Landau and Lifschitz, Jauch and Rohrlich). The determinant, $|g_{\alpha\beta}|$, of the metric in four space is designated by g, and the determinant, $|g_{ik}|$, of the metric on a spacelike manifold is designated by ${}^3g$. Other important quantities include the bending coefficients,

$$\Gamma_{\alpha\beta,\gamma} = \tfrac{1}{2}(\partial g_{\beta\gamma}/\partial x^\alpha + \partial g_{\alpha\gamma}/\partial x^\beta - \partial g_{\alpha\beta}/\partial x^\gamma);$$

the Riemann curvature tensor, with its twenty distinct components, $R_{\mu\nu\sigma\tau}$, where

$$R_\nu{}^\mu{}_{\sigma\tau} = \partial\Gamma_{\nu\tau}{}^\mu/\partial x^\sigma - \partial\Gamma_{\nu\sigma}{}^\mu/\partial x^\tau + \Gamma_{\sigma\eta}{}^\mu\Gamma_{\nu\tau}{}^\eta - \Gamma_{\tau\eta}{}^\mu\Gamma_{\nu\sigma}{}^\eta;$$

the symmetric Ricci tensor or contracted Riemann tensor,

$$R_{\mu\sigma} = R_\mu{}^\alpha{}_{\alpha\sigma};$$

$$(3!)^{-1}[\alpha\beta\gamma\delta](\partial f_{\beta\gamma}/\partial x^{\delta}) = 0 \text{ (half of Maxwell's equations)}, \quad (1)$$

$$(-g)^{-1/2}(\partial/\partial x^{\beta})(-g)^{1/2}f^{\alpha\beta} = 0 \text{ (the other half)}, \quad (2)$$

$$\text{“}\Box g_{\alpha\beta}\text{”} \equiv R_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}R = 2f_{\alpha\delta}f_{\beta}^{\ \delta} - \tfrac{1}{2}g_{\alpha\beta}(f_{\sigma\tau}f^{\sigma\tau})$$

(curvature of metric by Maxwell stress-momentum-energy density). (3)

These equations describe electromagnetism and gravitation as a coupled but closed dynamical system.

Solve Eqs. (3) for the reduced electromagnetic field, $f_{\sigma\tau}$, in terms of the contracted curvature tensor, $R_{\alpha\beta}$. Substitute the resulting expressions into Maxwell's equations. Thus re-express the content of the Maxwell-Einstein equations in a *purely geometrical form.* This program was carried out by Rainich in an important paper (*3*)[3] that has long lain neglected. The result is simple. (1) The symmetric

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the curvature invariant, $R = R_{\alpha}^{\ \alpha}$; the generalization of the notion of d'Alembertian of the gravitational potentials,

$$\text{“}\Box g_{\mu\nu}\text{”} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R;$$

and the electromagnetic potentials, A_{α}, such that

$$F_{\alpha\beta} = \partial A_{\beta}/\partial x^{\alpha} - \partial A_{\alpha}/\partial x^{\beta}.$$

The alternating quantity that is often written in the form $\epsilon_{\alpha\beta\gamma\delta}$ is not a tensor and is here written in the form $[\alpha\beta\gamma\delta]$. It changes sign on interchange of any two indices, and [0123] has the value unity. The dual, $(*F)_{\mu\nu}$, of an alternating tensor, $F_{\alpha\beta}$, is defined by the equation

$$(*F)_{\mu\nu} = \tfrac{1}{2}(-g)^{1/2}[\mu\nu\sigma\tau]g^{\sigma\alpha}g^{\tau\beta}F_{\alpha\beta}.$$

Associated with the geometrized electromagnetic field quantities, $f_{\alpha\beta} = (G^{1/2}/c^2)F_{\alpha\beta}$, are the geometrized electromagnetic potentials, $a_{\alpha} = (G^{1/2}/c^2)A_{\alpha}$, which are dimensionless. In flat space and Euclidean coordinates,

$dx^1 = dx_1$ = displacement in x-direction,

$dx^0 = -dx_0$ = interval of cotime,

$A^1 = A_1$ = x-component of usual vector potential,

$A^0 = -A_0$ = usual scalar potential, V(es volts),

$F_{23} = -F_{32}$ = x-component of magnetic field,

$F_{10} = -F_{01}$ = x-component of electric field.

[3] Even Rainich's later book, (*4*) does not summarize this paper, primarily because he was motivated by a different view of classical physics than that under investigation in the present article. We undertook the problem of expressing (1), (2), and (3) in "already unified form", and one of us (C. M.) independently derived Rainich's results, before becoming aware of his valuable contribution. The possibility of such an "already unified theory" was first suggested to us by Dr. Hugh Everett.

TABLE I

THE DISTINCTION BETWEEN CLASSICAL AND QUANTUM PHYSICS AS ENVISAGED IN THIS PAPER[a]

Classical physics as defined here	Description in terms of the geometry of empty curved space	Quantum physics; not discussed in this paper
Gravitation	Defined by curving of geodesics in a Riemannian space	Gravitons; photons; spin; neutrinos; *quantization* of charge; *quantization* of mass; electrons, mesons and other particles; characteristic fields that do not have zero rest mass, apparently associated with some of these particles; particle transformation processes; also all phenomena where quantum fluctuations in the metric are more important than any static gravitational fields.
Electromagnetism	Determined by curvature, and its rate of change, in this same Riemannian space (Fig. 2)	
Unquantized charge	Manifestation of lines of force trapped in a multiply connected topology (Fig. 3)	
Unquantized mass	Geons: semistable collection of electromagnetic or gravitational wave energy held together by its own gravitational attraction	
$G = 6.67 \times 10^{-8}$ cm^3/g sec^2 and c define no characteristic length, mass, or time. Electromagnetic field $F_{\mu\nu}$ (in gauss or electrostatic volts per cm or (g/cm sec^2)$^{1/2}$) translated into purely geometric quantities $f_{\mu\nu}$ (in units of cm^{-1}) by multiplication with $G^{1/2}/c^2 = 1/3.49 \times 10^{24}$ gauss cm.		G, c, and $\hbar$ define the characteristic units first introduced by Planck: $L^* = (\hbar G/c^3)^{1/2} = 1.63 \times 10^{-33}$ cm; $T^* = L^*/c$; and $M^* = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}$ g.

[a] The unquantized classical charge and mass in the table have no *direct* relation whatsoever with the elementary masses and charges that are seen in the quantum world of physics.

Ricci tensor, $R_{\alpha\beta}$, can be expressed as the "Maxwell square", as in Eq. (3), of an alternating tensor, $f_{\sigma\tau}$, if and only if this tensor (Fig. 1) (a) has zero trace and (b) has a square which is a multiple of the unit matrix:

$$R \equiv R_\alpha{}^\alpha = 0, \tag{4}$$

$$R_\alpha{}^\beta R_\beta{}^\gamma = \delta_\alpha{}^\gamma (R_{\sigma\tau} R^{\sigma\tau}/4). \tag{5}$$

We therefore demand these conditions of the Ricci tensor. (2) Then this contracted curvature tensor determines the local value of the reduced electromagnetic field tensor, $f_{\sigma\tau}$, uniquely up to an arbitrary angle, α, by way of an equation which we write symbolically in the form[4]

[4] We wish to express our appreciation to Professor V. Bargmann for bringing Eqs. (5) and (6) to our attention two years ago, noting that their gist had been independently discovered by several investigators, and expressing their content in essentially the above exceptionally simple form. An early proof is given by Rainich himself (*3*). The result is implicit in Theorem V of a study by Synge (*5*); see also Synge's book (*6*), a paper by Bonnor (*7*), and the thesis of Louis Mariot (*8*), for which we are indebted to M. Mariot.

$$f_{\sigma\tau} = (R_{\text{Maxwell root}})_{\sigma\tau} \cos\alpha + ({}^*R_{\text{Maxwell root}})_{\sigma\tau} \sin\alpha. \qquad (6)$$

(3) The expression for the electromagnetic field in terms of the Ricci curvature is substituted into Maxwell's equations. The laws of electrodynamics thereby take on the following purely geometrical character. First, out of the derivative

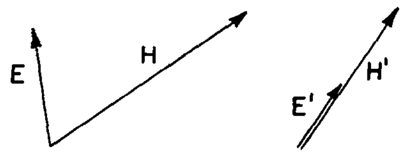

FIG. 1. Simplification of the analysis of the Maxwell stress-momentum-energy tensor by passage to a locally Lorentz frame of reference in which **E** and **H** are parallel. Left: Electric and magnetic vectors in the original reference system. Calculate the energy flux, $c(\mathbf{E} \times \mathbf{H})/4\pi$, and the energy density, $(\mathbf{E}^2 + \mathbf{H}^2)/8\pi$, and their ratio, the velocity, **v**. Right: View the fields in a frame of reference moving with this velocity. The energy flux must vanish. Therefore $\mathbf{E}'$ and $\mathbf{H}'$ must be parallel. Let their common direction be called the x' axis. There is a Maxwell tension $(\mathbf{E}'^2 + \mathbf{H}'^2)$ along this axis, and equally strong Maxwell pressures along the two perpendicular y' and z' axes. Therefore the stress-momentum-energy tensor has the form

$$(F'^2/8\pi)\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

where F'^2 is an abbreviation for the invariant

$$F'^2 = \mathbf{E}'^2 + \mathbf{H}'^2 = [(\mathbf{E}'^2 - \mathbf{H}'^2)^2 + 4(\mathbf{E}'\cdot\mathbf{H}')^2]^{1/2}$$
$$= [(\mathbf{E}^2 - \mathbf{H}^2)^2 + 4(\mathbf{E}\cdot\mathbf{H})^2]^{1/2} = [(\mathbf{E}^2 + \mathbf{H}^2)^2 - 4(\mathbf{E} \times \mathbf{H})^2]^{1/2}.$$

This tensor has two important properties (1) its trace is zero (2) its square is a multiple of the unit matrix. Both features are invariant to change of coordinate system. They hold whether the Maxwell tensor is diagonal or not.—Conversely, consider a real symmetric tensor which enjoys the properties (1) and (2). One can find a coordinate system with a favored direction, x', which puts it in the above diagonal form. In particular, one can find at once the invariant field magnitude,

$$F' = [(8\pi \text{ Maxwell tensor})^2/(\text{unit matrix})]^{1/4}.$$

Then pick any angle α, and *define* vectors $\mathbf{E}'$ and $\mathbf{H}'$ that point in the favored direction, x', with magnitudes

$$E' = F' \sin\alpha, \qquad H' = F' \cos\alpha.$$

The vectors $\mathbf{E}'$ and $\mathbf{H}'$ are determined uniquely apart from the single freely disposable parameter, α. Transform the electromagnetic field so defined back to the original reference system. Operating on this field, the Maxwell prescription for the stress energy tensor will produce the symmetric tensor with which one started.—These proofs break down when the electromagnetic field is a null field, with **E** perpendicular to **H** and equal in magnitude to **H**, but the statement in the text is still true.

of the Ricci tensor form the vector α_τ *defined* by the equation

$$\alpha_\tau = (-g)^{1/2}[\tau\lambda\mu\nu]R^{\lambda\beta;\mu}R_\beta{}^\nu/R_{\gamma\delta}R^{\gamma\delta}. \tag{7}$$

(The null-case where $R_{\gamma\delta}R^{\gamma\delta}$ vanishes requires special treatment). Second, demand that the curl of this vector shall vanish:

$$\alpha_{\tau;\eta} - \alpha_{\eta;\tau} = \alpha_{\tau,\eta} - \alpha_{\eta,\tau} = 0. \tag{8}$$

(4) *This differential equation* (8), *plus the algebraic equations* (4) *and* (5), *summarize in complete geometrical form both the whole of source-free Maxwell electromagnetism in curved space, and Einstein's laws for the production of curvature by this field.* These three equations, (4), (5), (8), comprise what we shall call "*already* unified field theory". Electric and magnetic fields are not signals to *invent* a unified field theory or to introduce one or another new kind of geometry. The "*already* unified field theory" of Maxwell, Einstein, and Rainich, summarized in this paper, describes electric and magnetic fields in terms of the rate of change of curvature of pure Riemannian geometry, and nothing more.

The nature of this unification can be stated in mathematical terms as follows: Maxwell's equations are of the second order, and so are Einstein's; the two sets of equations can be combined into one set of equations (8) of the fourth order. In more physical terms, the electromagnetic field leaves an *imprint*[5] upon the metric that is so characteristic (Fig. 2), that from that imprint one can read back to find out all that one needs to know about the electromagnetic field.

Given a purely metric field that satisfies Eqs. (4), (5), and (8) of already unified field theory, one finds the electromagnetic field as follows. First, calculate everywhere the vector field α_μ of Eq. (7). Second, from some standard point 0 calculate the integral

$$\alpha(x) = \int_0^x \alpha_\mu \, dx^\mu + \alpha_0 \tag{9}$$

Since α_μ is curl free, the integral is independent of path, so long as alternative paths are continuously deformable into one another. (When instead the space is multiply connected, new considerations will be needed.) We therefore have a dimensionless number or angle, α, defined as a function of position, up to an additive *constant*, α_0 . Finally, we substitute this angle into Eq. (6) to find the electric and magnetic field at every point in space.

We find that long established theory has a well defined means to describe gravitation and electromagnetism in terms of empty curved space. What about charge?

Einstein emphasized that the field equations of electromagnetism and general relativity have a purely local character. They relate conditions at one point to conditions at points an infinitesimal distance away. They tell nothing about the topology of space in the large. Einstein was led by Mach's principle (*9*) to consider

[5] We are indebted for this phrase to Professor Peter Bergmann.

a space not topologically equivalent to Euclidean space, a spherical or nearly spherical universe. But Einstein confesses his indebtedness to a thinker who had still more far reaching ideas. Riemann[6] in his famous inaugural lecture envisaged

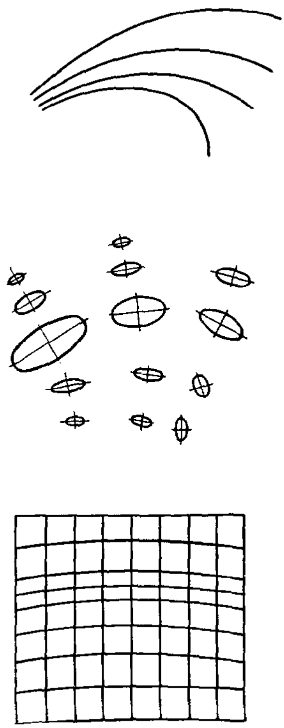

FIG. 2. Relation between the electromagnetic field and geometry, schematically represented. Above: lines of force. Middle: Maxwell stress tensor due to these lines of force. This stress tensor serves as source of the gravitational field and equals the contracted curvature tensor of the space-time continuum, up to a multiplicative constant, according to Einstein. Below: The metric of four-space as distorted by this curvature. In brief, the electromagnetic field leaves its footprints on space. Moreover, these footprints on the metric are so specific and characteristic that from them one can work back and find out all that needs to be known about the electromagnetic field. One has a purely geometrical description of electromagnetism.

a connection between physics and a curvature of space that would be sensible not only at very great distances, but also at very small distances: "... es kann dann in jedem Punkte das Krümmungsmass in drei Richtungen einen beliebigen Werth haben, wenn nur die ganze Krümmung jedes messbaren Raumtheils

[6] In the opening passage of this lecture (*10*) Riemann declares that "the properties which distinguish space from other conceivable triply extended magnitudes are only to be deduced from experience." (translation of Clifford (*11*)).

nicht merklich von Null verschieden ist; . . ." Dying of tuberculosis twelve years later, occupied with an attempt at a unified explanation of gravity and electromagnetism, Riemann communicated to Betti his system of characterization of multiply-connected topologies[7]. What is the character of charge-free electromagnetism in a space endowed with such a multiply connected topology?

One can outline a complete classification of the everywhere regular initial conditions for Maxwell's equations in a closed space. This analysis *forces* one to consider situations—such as described by one of us (*14*) (Fig. 3)—where there is a net flux of lines of force through what topologists would call a handle of the multiply-connected space and what physicists might perhaps be excused for more vividly terming a "wormhole". The flux of lines of force that emerge from the mouth of a small wormhole appears to an observer endowed with poor resolving power to come from an elementary electric charge. But there is nowhere that one can put his finger and say, "This is where some charge is located[8]." Lines of force never end. This freedom from divergence by no means prevents changes in field strengths. Lines of forces which are not trapped into the topology can be continuously shrunk to extinction, as in familiar examples of electromagnetic induction and electromagnetic waves. However, lines of force which are trapped in wormholes cannot diminish in number. The flux out of the mouth of a wormhole cannot change with time, no matter how violent the disturbances in the electromagnetic field, no matter how roughly the metric changes, no matter how rapidly corresponding wormholes recede or approach, up to the moment when they actually coalesce and change the topology. Either Maxwell's equa-

[7] Weyl (*12*) emphasizes that the field equations provide no means whatever to rule out either multiply-connected spaces, or spaces which are nonorientable, such as a Klein bottle. He notes (original German in 1927; translation and revision in 1949) "that a more detailed scrutiny of a surface might disclose that, what we had considered an elementary piece, in reality has tiny handles attached to it which change the connectivity character of the piece, and that a microscope of ever greater magnification would reveal ever new topological complications of this type, *ad infinitum*. The Riemann point of view allows, also for real space, topological conditions entirely different from those realized by Euclidean space. I believe that only on the basis of the freer and more general conception of geometry which had been brought out by the development of mathematics during the last century, and with an open mind for the imaginative possibilities which it has revealed, can a philosophically fruitful attack upon the space problem be undertaken." Einstein and Rosen (*13*) proposed in 1935 to regard ordinary space as connected with a duplicated "mirror" space by short tubes. This topology is much more particular than anything contemplated here or in the following paper (*2*). Einstein and Rosen also took the electromagnetic field to have a *negative*-definite energy density, in contradiction to experience. We learn that Professor J. L. Synge also once mentioned in a lecture at Dublin in 1947 the idea of multiply connected space.

[8] In 1895 the great physicist Henry A. Rowland said, ". . . electricity no longer exists, for the name electricity as used up to the present time signifies at once that a substance is meant, and there is nothing more certain than that electricity is not a substance." (Quoted by Darrow (*15*).) His words are apropos here!

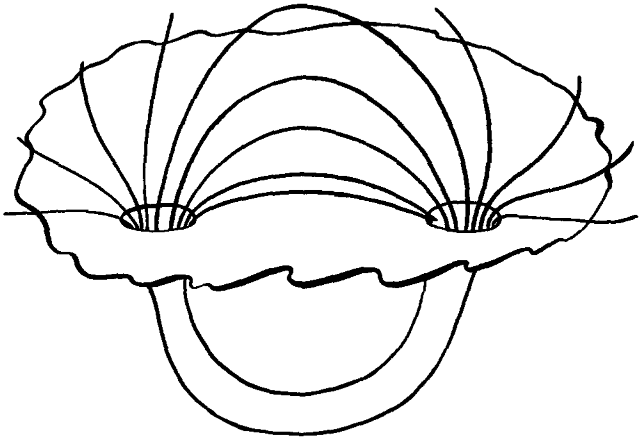

FIG. 3. Symbolic representation of the unquantized charge of classical theory. For ease of visualization the number of space dimensions is reduced from three to two. However, the two dimensional curved and multiply-connected space is pictured as imbedded in a three dimensional Euclidean space. The third dimension, measured off the surface, has no physical meaning. Of course the topology and geometry of the 2-space receives its best mathematical formulation in intrinsic terms, without this imbedding of the manifold in a space of higher dimensionality. The 2-space is multiply-connected, but free of all singularities. An imaginary ant crawling over the surface and entering the tunnel or handle or "wormhole" finds there the same two dimensional space he experienced everywhere else. Electric lines of force that converge on the right-hand mouth of the tunnel continue to obey at each point Maxwell's equation, div $\mathbf{E} = 0$. The field is everywhere free of singularity. The lines of force have no escape but to continue through the tunnel. They emerge from the left-hand mouth. Outside the tunnel mouths the pattern of lines of force is identical with that due to equal positive and negative charges. An observer endowed with poor vision sees evidence for two point charges. He may even construct a boundary around the right-hand charge, determine the flux through this boundary, incorrectly apply the theorem of Gauss, and "prove" that there is a charge inside the boundary. He does not recognize that he has been making tacit and unjustified assumptions about the topology of space. He is not aware that his "boundary" does not bound any region interior to it. He assumes, either that Maxwell's equations fail in the vicinity of the charge, or that there exists there some magic substance at which lines of force end and to which he gives the name "electricity". But a closer inspection discloses that the lines of force do not end. Neither is there any violation of Maxwell's equations for charge-free space. Nowhere can one place his finger and say, "Here there is some charge". Such is the purely topological picture of unquantized electric charge which is adopted in this paper. This classical charge has no *direct* relation whatsoever to quantized electric charge. At this classical level there is a freedom of choice in the strength of the charge, and an individuality about the connection between one charge and another, that must be entirely changed in any proper *quantum* theory of electricity.—The distance along the wormhole from one mouth to the other need have no correspondence whatever with the distance in the open space between the same two mouths. The connection can be as short as the radius of the wormhole itself, for example, even when the openings are very far apart in the upper space, as one sees by bending the upper space to bring the backs of the two holes into coincidence (diagram reproduced from Ref. *14*).

tions, or Faraday's equivalent physical picture of lines of force, plus the conception of multiply connected space, force one to the conclusion that the wormhole flux remains invariant. *This constant of the motion represents the charge.*

The charge or wormhole flux is unquantized. It can have one value as well as another. It has nothing whatsoever directly to do with the quantized charge observed on the elementary particles of quantum physics. This circumstance is not an objection to the concept of a classical unquantized charge. It is a warning that quantized charge is quite another concept. This distinction is not unacceptable at a time when one has learned how great a difference there is between the "undressed" and "dressed" charge of quantum electrodynamics (*16*). To limit attention to purely classical unquantized charge will therefore not appear unreasonable in an article that restricts itself to classical physics (Table I).

Around the mouth of a wormhole lies a concentration of electromagnetic energy that gives *mass* to this region of space. Mass arises even in singly connected space, where there is no charge connected with the source-free Maxwell field. The equations of Maxwell and Einstein predict the possibility of a long-lived concentration of electromagnetic energy, or "geon", held together by its own attraction. Both in the multiply-connected space and in the singly-connected continuum, the mass with which one has to do is classical, nonlocalized, and unquantized. It has nothing whatsoever directly to do with the quantized mass of elementary particles.

Summarized in paradoxical form, the existing well-established already unified classical theory [Eqs. (4), (5), (8)] allows one to describe in terms of empty curved space

1. gravitation without gravitation
2. electromagnetism without electromagnetism
3. charge without charge
4. mass without mass.

It has *nothing at all* to contribute directly to an understanding of

5. spin without spin
6. elementary particles without elementary particles,

or any other issues of quantum physics. Nevertheless, we would hardly have taken up the analysis of classical geometrodynamics if we did not hope ultimately to find out what, if anything, *quantum* geometrodynamics has to do with elementary particle physics. It is our long range objective to discover if quantum physics, like classical physics (Table I), can be expressed in terms of pure geometry, and nothing more.

It is not customary today to adopt either extreme view, either that space time is only an arena, or that it is everything. One analyzes the states of particles and fields into plane waves that move as foreign elements in a preassigned flat space. At the same time one thinks of the curvature of space, not as exactly zero, but only as very small over distances short compared to the extension of the universe. Einstein's geometrical description of gravitation is taken seriously. His attempts at an equally geometric description of electromagnetism—by *modifying* Riemannian geometry—are recognized to be incompatible (*17*) with the well-tested Lorentz law of force and are rejected. Particles, and fields other than gravitation, are considered to be added to geometry, not as derived from geometry. Nature can be said to be described today in a mixed fashion, partly in terms of pure geometry, partly in terms of foreign entities.

To go to the logical extreme, however, and think of a purely geometrical description of nature, was not a new idea even before one knew enough to distinguish between classical and quantum physics. The distinguished mathematician Clifford delivered a paper to the Cambridge Philosophical Society on February 21, 1870 "On the Space-Theory of Matter," in which he proposed that "in the physical world nothing else takes place but this variation [of the curvature of space], subject (possibly) to the law of continuity;" and later he spoke of considerations "which indicate that distance or quantity may come to be expressed in terms of *position* in the wide sense of the analysis situs;" and again about the finite volume of a uniformly curved space, but with the explicit statement that "The assumptions here made about the Zusammenhang of space are [merely] the simplest" (*18*). Before Einstein, Clifford, and Riemann—and Riemannian geometry—was there ever current anything like the concept of physics as geometry? What were Newton's views of field theory and the idea that empty space is the universal building material? His letter to Bentley has long been known: "That one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent manner of thinking, can ever fall into it"[9]. Maxwell says, "We find in his 'Optical Queries' and in his letters to Boyle, that Newton had very early made the attempt to account for gravitation by means of the pressure of a medium, and that the reason he did not publish these investigations 'proceeded from hence only, that he found he was not able, from experiment and observation, to give a satisfactory account of this medium, and the manner of its operation in producing the chief phenomena of Nature.' "

New insight into Newton's ideas and their origins comes from the recent scholarly and most interesting analysis by Fierz (*20*). Fierz cites especially Patrizzi (*21*), who writes of space as a substance, "Spacium ergo hoc, quod ante mundum fuit, et post quod mundus est, et quod mundum at capit, et excedit, quidnam tandem est. . . . Quid ergo substantia ne est? Si substantia est, id quod per substat, spacium maxime omnium substantia est." Also some of the Vedas

[9] This and the following quotation come from notes of Cajori (*19*).

of old India (*22*) suggest[9a] that the idea is very old, that nature derives its whole structure and way of action from properties of space.

Can space be regarded as a marvellous creation of all-encompassing properties? Independent of the origins of this idea, both ancient and modern, let us now proceed to analyze it.

In Sec. II we recapitulate in present day notation the derivation of the equations of Rainich for already unified field theory. The starting point, the theory of Einstein and Maxwell, deals entirely with local properties and so does Rainich's final system of equations.

To pass from local to global or topological properties, and still to keep the discussion simple, we return in Sec. III to the more familiar dual language of metric *plus* field. We sketch out the necessary topological background, and introduce the theorem of Gauss and the theory of harmonic vector fields in the necessary generality. Much of the required mathematics is most readily expressed in terms of Cartan's calculus of exterior differential forms. Most results we give both in this notation and in the familiar tensor form. A few conclusions would appear so complicated in the tensor formalism that we omit their transcription to the conventional notation. We prove that Maxwell's equations demand the

[9a] For very early ideas related to "physics is geometry" we have been referred by Professor G. L. Chandratreya to the Indian Vedas. In this connection we wish to thank Swami Nikhilananda who explains to us the relevant writings: "According to the Vedas, *akasa* (often translated as 'space' or 'ether') is the rudimentary first element from which the other elements, namely, air, fire, water, and earth, have evolved. These five are the only material elements spoken of by the Vedas. Hindu philosophers populated five elements because a man reacts to the outside world in five ways: through his hearing, touch, sight, taste, and smell. . . . [quoting in this connection from Taittirîya Upanishad II. i. 3.]" Thus we find *akasa* as the primary element in this early (c. 700 B. C.) sketch of physics. (Brahman, although preceding *akasa*, is pure spirit, outside the realm of physics.) We can say that in this physics space was the primary element out of which all else was to have come, only if we can satisfy ourselves that *akasa* meant something like the current word space. In this connection we quote the most authoritative early commentators on the Upanishad just mentioned (*22*). Sankarâchârya (A. D. 788–820) tells us "*Akasa* is that thing which has sound for its property and which affords space to all corporeal substances." Then Sâya*n*a elucidates: " . . . the power of *akasa* to afford space to all (corporeal) things constitutes its own peculiar nature . . . And it has sound for its property. The echo heard in mountain-caves etc., is supposed to be inherent in *akasa* and is therefore said to be the property of *akasa*." Except for the curious references to sound, these explanations seem to corroborate a tentative identification of *akasa* with space: a physicist might write 'space provides room for all things' where the translator wrote '*akasa* affords space to all things'. The reference to sound is understood when we recall that the five elements were chosen to correspond to the five senses. *S*ankarâchârya writes "Thence, i.e., from *akasa*, comes into being V*a*yu, the air, with two properties, the property of touch which is its own, and the property of sound belonging to *akasa* already evolved." It is perhaps too much to expect that at such an early date men would know that sound is transmitted through the air, and not through empty space.

conservation of flux through each wormhole independently, thus justifying the identification of this flux with charge.

Section IV deals with specific examples of nonsingular multiply-connected metrics that manifest both charge and mass. First the Schwarzschild metric is written in nonsingular form, to put into evidence the special case of a space free of either charge or mass—in the conventional sense of those words—which nevertheless exhibits mass. Next a nonsingular form of the Reissner Nordström metric is exhibited. It describes a spherically symmetric space free of all "real" charge and mass which nevertheless exhibits both properties. Finally, a closed mathematical form is given for a class of metrics endowed with a plurality of wormhole mouths, each with its own charge and mass. The initial conditions being thus specified, the future evolution of the space with time is of course determined by the field equations. In other words, the arguments of Einstein, Infeld, and Hoffman apply to this situation. The entire dynamics of the system of singularity-free charges and masses becomes a matter of pure geometrodynamics.

Section V outlines points that require further investigation to complete classical geometrodynamics and to extend its domain of application.

II. RAINICH'S ALREADY UNIFIED FIELD THEORY: THE MAXWELL TENSOR; DUALITY; DUALITY ROTATIONS; INVARIANCE OF MAXWELL TENSOR TO DUALITY ROTATION; COMPLEXION OF FIELD DEFINED; SQUARE OF MAXWELL TENSOR; THE ALGEBRAIC RELATIONS ON THE CURVATURE; THE REVERSE PROBLEM—FROM THE CURVATURE TO FIND THE FIELD; RESULTING DIFFERENTIAL EQUATIONS OF THE CURVATURE

The Maxwell Tensor

We begin by recalling the relation between the electromagnetic field, $F_{\mu\nu}$ and the Maxwell stress-momentum-energy tensor $T_{\mu\nu}$. This relation is purely algebraic. Hence we may concentrate our attention on a single point of space time and, when convenient, use coordinates that give the metric components, $g_{\mu\nu}$, their Minkowski values at that point. To keep geometry to the fore, we shall use instead of **F** the "geometrized" or "reduced" field strength, $\mathbf{f} = (G^{1/2}/c^2)\,\mathbf{F}$, and instead of the electric and magnetic field strengths, **E** and **H**, the reduced field strengths, $\mathbf{e} = (G^{1/2}/c^2)\mathbf{E}$ and $\mathbf{h} = (G^{1/2}/c^2)\mathbf{H}$ (dimensions cm^{-1}). When the metric is Minkowskian, the reduced field tensor has the form

$$f_{\mu\nu} = \begin{bmatrix} 0 & -e_x & -e_y & -e_z \\ e_x & 0 & h_z & -h_y \\ e_y & -h_z & 0 & h_x \\ e_z & h_y & -h_x & 0 \end{bmatrix} (\text{cm}^{-1}); \tag{10}$$

and the dual tensor,

$$*f_{\mu\nu} = \tfrac{1}{2}(-g)^{1/2}[\mu\nu\alpha\beta]f^{\alpha\beta}, \tag{11}$$

differs from **f** only by the interchange **e** → **h**, **h** → −**e**. Two familiar invariants form themselves out of the field tensor:

$$\begin{aligned} \mathbf{f}^2 &\equiv \tfrac{1}{2}f_{\mu\nu}f^{\mu\nu} \ (= \mathbf{h}^2 - \mathbf{e}^2 \text{ in a Minkowski frame}), \\ \mathbf{f} \times \mathbf{f} &\equiv \tfrac{1}{2}f_{\mu\nu}{*f^{\mu\nu}} \ (= 2\mathbf{e}\cdot\mathbf{h} \text{ in a Minkowski frame}). \end{aligned} \tag{12}$$

The equations connecting the contracted curvature tensor with the stress tensor, and that in turn with the field,

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu} &= (8\pi G/c^4)[(1/4\pi)(F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})], \\ &= 2(f_{\mu\alpha}f_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}f_{\alpha\beta}f^{\alpha\beta}) \equiv \mathfrak{T}(\mathbf{f}) \end{aligned} \tag{13}$$

justify one in using interchangeably for the quantity on the right the terms "reduced" or "geometrized" stress energy tensor, or contracted curvature tensor, insofar as one deals with pure geometrodynamics. In the Minkowski frame of reference, typical components of (13) are

$R_{00} = R^{00} = -R_0{}^0 = (\mathbf{e}^2 + \mathbf{h}^2) = (8\pi G/c^4)$(energy density);

$-R_{10} = R^{10} = R_1{}^0 = 2\mathbf{e} \times \mathbf{h} = (8\pi G/c^4)$(density of c times x-component of momentum) $= (8\pi G/c^4)$(flow of electromagnetic energy per cm^2 of area normal to x and per cm of elapsed cotime);

$R_{11} = R^{11} = R_1{}^1 = (-e_x^2 + e_y^2 + e_z^2 - h_x^2 + h_y^2 + h_z^2) = (8\pi G/c^4)$(pressure) $= (8\pi G/c^4)$(force exerted in x direction, per unit area normal to x, by electromagnetic fields in medium at $x - \epsilon$, acting on medium at $x + \epsilon$);

$R_{12} = R^{12} = R_1{}^2 = -2e_xe_y - 2h_xh_y = (8\pi G/c^4)$(shear) $= (8\pi G/c^4)$(force exerted in x direction, per unit area normal to y, by electromagnetic forces due to medium at $y - \epsilon$ acting upon medium at $y + \epsilon$). (14)

The invariance of the stress energy tensor under the interchange **e** → **h**, **h** → −**e**, is apparent from (14), but does not show itself clearly in (13). Therefore it is preferable to rewrite the reduced stress tensor, or the "Maxwell square of **f**", on the right-hand side of (13) in the more symmetrical form

$$\mathfrak{T}(\mathbf{f}) = f_{\mu\alpha}f_\nu{}^\alpha + {*f_{\mu\alpha}}{*f_\nu{}^\alpha}, \tag{14a}$$

as follows from the identity[10]

$$f_{\mu\alpha}f^{\nu\alpha} - {*f_{\mu\alpha}}{*f^{\nu\alpha}} = \tfrac{1}{2}\delta_\mu{}^\nu f_{\alpha\beta}f^{\alpha\beta} \equiv \delta_\mu{}^\nu \mathbf{f}^2. \tag{15}$$

Duality Rotations[4]

The stress tensor, or Maxwell square of **f**, shows a further symmetry. Note that the operation of taking the dual of **f**, twice repeated, leads back to −**f**, so that the square of the operation $*$ is the negative of the identity operation. Consider therefore an angle α and define the operation $e^{*\alpha}$ by the equation

$$e^{*\alpha}\mathbf{f} = \mathbf{f}\cos\alpha + {*\mathbf{f}}\sin\alpha. \tag{16}$$

In a Minkowski coordinate system this operation takes the form

$$\left.\begin{aligned} h_{x\,\text{new}} &= h_x \cos\alpha + e_x \sin\alpha \\ e_{x\,\text{new}} &= -h_x \sin\alpha + e_x \cos\alpha \end{aligned}\right\} \text{(same for } x, y, z) \tag{17}$$

This operation appears at first sight to be an ordinary rotation: applied to any linearly polarized monochromatic wave, with $|\mathbf{e}| = |\mathbf{h}|$ and $\mathbf{e} \perp \mathbf{h}$, it turns the direction of polarization through the angle α around the direction of propagation. However, when this operation is applied to less special fields, it produces no such simply describable result. Moreover, it treats all three space axes alike. It is not an ordinary rotation in 3-space. We shall therefore call it a *duality rotation.* It has the additivity property

$$e^{*\alpha}e^{*\beta} = e^{*\beta}e^{*\alpha} = e^{*(\alpha+\beta)} \tag{18}$$

and the special value

$$e^{*\pi/2} = *. \tag{19}$$

The dual of the duality rotation yields the field tensor

$$*(e^{*\alpha}\mathbf{f}) = -\mathbf{f}\sin\alpha + {*\mathbf{f}}\cos\alpha. \tag{20}$$

The duality rotation has the following important property as a consequence of (14), (16), and (20): *The Maxwell square of a duality-rotated field is identical with the Maxwell square of the original field:*

$$\mathfrak{T}(e^{*\alpha}\mathbf{f}) = \mathfrak{T}(\mathbf{f}). \tag{21}$$

The electric and magnetic fields individually are changed, but every component of the stress energy tensor is unaltered. In contrast (Table II) the duality rota-

TABLE II

CONTRAST BETWEEN PROPER LORENTZ TRANSFORMATIONS AND DUALITY ROTATIONS

Quantity	General proper Lorentz transformation	Duality rotation
Components of the Maxwell tensor or Maxwell square of $\mathbf{f}$	Transformed	Unchanged
The invariants, $\mathbf{f}^2$ and $\mathbf{f} \times \mathbf{f}$	Unchanged	Transformed

tion alters the invariants of the field. Make the definition

$$\xi = e^{-*\alpha}\mathbf{f} \tag{22}$$

for a field which has undergone a duality rotation by the angle $-\alpha$. Then the invariants transform as by a rotation through the angle -2α:

$$\begin{aligned} \xi^2 &= (\mathbf{f} \cos \alpha - *\mathbf{f} \sin \alpha)^2 \\ &= \mathbf{f}^2 \cos 2\alpha - \mathbf{f} \times \mathbf{f} \sin 2\alpha, \\ \xi \times \xi &= \mathbf{f}^2 \sin 2\alpha + \mathbf{f} \times \mathbf{f} \cos 2\alpha. \end{aligned} \tag{23}$$

Assume that the invariants, $\mathbf{f}^2$ and $\mathbf{f} \times \mathbf{f}$, of the original field do not both vanish. Then choose the angle α so that the one invariant quantity, $\xi \times \xi$, is zero:

$$\tan 2\alpha = -(\mathbf{f} \times \mathbf{f})/\mathbf{f}^2. \tag{24}$$

Then solve for sin 2α and cos 2α up to a $\pm$ ambiguity and evaluate the other invariant, finding

$$\xi^2 = \pm[(\mathbf{f}^2)^2 + (\mathbf{f} \times \mathbf{f})^2]^{1/2}. \tag{25}$$

Demand that the *minus* sign shall appear on the right, thus determining the angle 2α uniquely up to a positive or negative additive integral multiple of 2π. Then the field tensor ξ represents a pure *electric* field along the x-axis, or a Lorentz transformation thereof (Table III). We say that the original field has received a duality rotation into an *extremal* field, or into an essentially *electric field.* In a preferred Lorentz system where this field has no magnetic components, and points along the x-axis, the field magnitude is

$$\begin{aligned} e_x''' &= [(\mathbf{h}^2 - \mathbf{e}^2)^2 + (2\mathbf{e}\cdot\mathbf{h})^2]^{1/4} \\ &= [(\mathbf{h}^2 + \mathbf{e}^2)^2 - 2(\mathbf{e} \times \mathbf{h})^2]^{1/4} \end{aligned} \tag{26}$$

THE COMPLEXION OF THE FIELD

Referred to an extremal, or essentially electric field, ξ, as standard of reference, the actual field, $\mathbf{f}$, evidently arises by a duality rotation through the angle α:

$$\mathbf{f} = e^{*\alpha}\xi.$$

TABLE III

TRANSFORMATIONS OF THE GENERAL (NON-NULL) ELECTROMAGNETIC FIELD TENSOR $\mathbf{f} = (\mathbf{e}, \mathbf{h})$ IN A LOCALLY MINKOWSKIAN REFERENCE SYSTEM

Field values	At start	After canonical duality rotation
At start	$\mathbf{e}, \mathbf{h}$	$\mathbf{e}''$ and $\mathbf{h}''$ perpendicular, and $\mathbf{e}''$ greater than $\mathbf{h}''$
After canonical Lorentz transformation	$\mathbf{e}'$ and $\mathbf{h}'$ parallel to each other and to the x-axis	$\mathbf{e}$ parallel to x axis; $\mathbf{h} = 0$

Under a Lorentz transformation the components of the three tensors, **f** and $\boldsymbol{\xi}$, and $*\boldsymbol{\xi}$, transform alike. *The angle* α therefore remains unchanged. *It is a significant and Lorentz invariant scalar property of the field,* **f**. We shall call the angle α the *complexion* of the electromagnetic field.

When the field **f** is a *null* field, with

$$(\mathbf{e}\cdot\mathbf{h}) = 0 \quad \text{and} \quad \mathbf{h}^2 - \mathbf{e}^2 = 0$$

or

$$\mathbf{f} \times \mathbf{f} = 0 \quad \text{and} \quad \mathbf{f}^2 = 0$$

then Eq. (24) for the angle α becomes indeterminate. Then the complexion is not definable on a purely local basis.

The Square of the Maxwell Tensor and the Algebraic Relations on the Curvature

Now return to the case where the field is not a null field. Evaluate the Maxwell square—or stress energy tensor—of the original field by using its equality to the Maxwell square of the extremal field [see Eq. (21)] or the Maxwell square of the dual of the extremal field

$$\begin{aligned} \mathfrak{T}_\mu{}^\kappa(\mathbf{f}) &= \mathfrak{T}_\mu{}^\kappa(\boldsymbol{\xi}) = 2\xi_{\mu\alpha}\xi^{\kappa\alpha} - \delta_\mu{}^\kappa(\boldsymbol{\xi}^2), \\ \mathfrak{T}_\kappa{}^\nu(\mathbf{f}) &= \mathfrak{T}_\kappa{}^\nu(*\boldsymbol{\xi}) = 2{*\xi_{\kappa\sigma}}{*\xi^{\nu\sigma}} + \delta_\kappa{}^\nu(\boldsymbol{\xi}^2). \end{aligned} \tag{27}$$

Now *square* the Maxwell square of **f** by multiplying the first tensor by the second. The cross terms between $\boldsymbol{\xi}$ and $*\boldsymbol{\xi}$ that arise in the calculation reduce to zero by reason of the identity[10]

$$2\xi_{\mu\alpha}{*\xi^{\kappa\alpha}} = \tfrac{1}{2}\delta_\mu{}^\kappa\xi_{\sigma\tau}{*\xi^{\sigma\tau}} = \delta_\mu{}^\kappa(\boldsymbol{\xi}\times\boldsymbol{\xi}) \tag{28}$$

and the extremal property, $\boldsymbol{\xi}\times\boldsymbol{\xi} = 0$. We also used in the evaluation the identity (15). We find for the square the result

$$\begin{aligned} \mathfrak{T}_\mu{}^\kappa\mathfrak{T}_\kappa{}^\nu &= 2(\boldsymbol{\xi}^2)\,(\xi_{\mu\alpha}\xi^{\nu\alpha} - {*\xi_{\mu\alpha}}{*\xi^{\nu\alpha}}) - \delta_\mu{}^\nu(\boldsymbol{\xi}^2)^2 \\ &= \delta_\mu{}^\nu(\boldsymbol{\xi}^2)^2 \\ &= \delta_\mu{}^\nu[(\mathbf{f}^2)^2 + (\mathbf{f}\times\mathbf{f})^2] \\ &= \delta_\mu{}^\nu[(\mathbf{h}^2 - \mathbf{e}^2)^2 + (2\mathbf{e}\cdot\mathbf{h})^2] \\ &= \delta_\mu{}^\nu[(\mathbf{h}^2 + \mathbf{e}^2)^2 - (2\mathbf{e}\times\mathbf{h})^2]. \end{aligned} \tag{29}$$

The proof of the same result in the case of null a field is even simpler,

[10] A special case of the relation

$$A_{\mu\alpha}B^{\nu\alpha} - {*A_{\mu\alpha}}{*B^{\nu\alpha}} = \tfrac{1}{2}\delta_\mu{}^\nu A_{\alpha\beta}B^{\alpha\beta},$$

which is valid for every pair of antisymmetric tensors, **A**, **B**, in 4-space.

$\mathbf{f} \times \mathbf{f} = 0$ and $\mathbf{f}^2 = 0$ and the right-hand side of Eq. (29) vanishes. In summary, *the square of the Maxwell stress-energy-momentum tensor is a multiple of the unit matrix*. This beautiful and interesting relation is central in Rainich's already unified field theory. In terms of curvature components, it has the form

$$\mathfrak{T}_\mu{}^\alpha \mathfrak{T}_\alpha{}^\nu = R_\mu{}^\alpha R_\alpha{}^\nu = \delta_\mu{}^\nu(\tfrac{1}{4} R_{\alpha\beta} R^{\alpha\beta}). \tag{30}$$

Here the value of the constant is obtained by comparing the traces of the two sides of the equation. To this relation he adds the vanishing of the trace of the reduced Maxwell tensor.

$$\mathfrak{T}_\alpha{}^\alpha = R_\alpha{}^\alpha \equiv R = 0, \tag{31}$$

[Eq. (4)], that follows directly from the equation of definition, (13), and the statement [Eq. (14)] that the electromagnetic energy density is positive definite:

$$\mathfrak{T}_{00} = R_{00} \geqslant 0. \tag{32}$$

In other words, for any time like vector $\mathbf{v}$ the quantity

$$v^\alpha \mathfrak{T}_{\alpha\beta}(\mathbf{f}) v^\beta \tag{33}$$

is non-negative. Equations (30), (31), and (32) summarize the algebraic relations on the curvature in already unified field theory.

It is *necessary* that the contracted curvature tensor $R_{\mu\nu}$ satisfy (30), (31), and (32) if it is to be representable as the Maxwell square of some antisymmetrical field tensor, $\mathbf{f}$. Figure 1 derives the same conditions in a slightly different way.

It is evident from Fig. 1 that a suitable Lorentz transformation puts the tensor $R_{\mu\nu} = \mathfrak{T}_\mu{}^\nu$ associated with a non-null field, $\mathbf{f}$, into a diagonal form,

$$[(\mathbf{f}^2)^2 + (\mathbf{f} \times \mathbf{f})^2]^{1/2} \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{34}$$

The diagonal form necessarily has this appearance for any symmetric tensor with a zero trace and a square which is a nonzero multiple of the unit matrix. The tensor $R_\mu{}^\nu$ defines what in the language of Schouten is a *two-bladed structure* ($\mathfrak{B}$) in space-time at the point in question.[11] A rotation in the yz plane about the x-axis leaves the tensor (34) unchanged; electric and magnetic fields remain parallel to the x-axis (Fig. 1). That picture of parallel field vectors is also left unchanged by any Lorentz transformation in the x, T plane. The yz plane and the xT plane are the two *blades* defined by the Maxwell tensor $\mathfrak{T}_\mu{}^\nu(\mathbf{f})$, and therefore in turn defined by $\mathbf{f}$ itself.

In diagonal form the Maxwell tensor is characterized by a single parameter. To this parameter there are added only four additional parameters by the

[11] We are indebted to Professor Schouten for several illuminating discussions.

TABLE IV

PARTIAL CLASSIFICATION OF RIEMANNIAN GEOMETRY

Feature	Name	Remarks
$R_{\alpha\beta\gamma\delta} = 0$	Uncurved space	All such 3-spaces recently classified by L. Markus, in publication.
$R_{\mu\nu} = 0$	Pure gravitation field	The 10 "local" components of $R_{\alpha\beta\gamma\delta}$ are zero, but the other 10 "remote action" components of $R_{\alpha\beta\gamma\delta}$ have to be found by solving the differential equations $R_{\mu\nu} = 0$ for the metric. No free components for $R_{\mu\nu}$.
$R_\mu{}^\mu = 0$; $R_{00} \geqslant 0$; $R_\mu{}^\alpha R_\alpha{}^\nu = \delta_\mu{}^\nu(\frac{1}{4}R_{\alpha\beta}R^{\alpha\beta}) = (\mathbf{h}^2 - \mathbf{e}^2)^2 + (2\mathbf{e}\cdot\mathbf{h})^2$	"Electromagnetic Riemannian geometry" or "geometrodynamics"	Five free components for $R_{\mu\nu}$. Metric must be found by solving field equations of already unified theory.
$R_\mu{}^\mu = 0$; $R_{00} \geqslant 0$; $R_\mu{}^\alpha R_\alpha{}^\nu = 0$	Null field; $\mathbf{e}\cdot\mathbf{h} = 0$ and $\mathbf{h}^2 - \mathbf{e}^2 = 0$; a special case of electromagnetism	$R_{\mu\nu} = \mathfrak{T}_{\mu\nu}(\mathrm{f}) = k_\mu k_\nu$, where $\mathbf{k}$ is a null vector; only three free parameters in $R_{\mu\nu}$.
$R_\mu{}^\mu = 0$; $R_{00} \geqslant 0$; $R_\mu{}^\alpha R_\alpha{}^\nu \neq 0$; α = constant for all space and time	Static field; by a change of names (duality rotation) can be translated into a condition where there is an electric field but never any magnetic field; another special case of electromagnetism	Extra non local (differential) requirements imposed on $R_{\mu\nu}$ in addition to the standard field equation of already unified field thoery
$R_{\mu\nu}$ arbitrary	Unrestricted Riemannian geometry	No physical laws

general 6-parameter proper Lorentz transformation, because rotations in the yz and xT planes have no effect. Therefore a total of five parameters characterize the contracted curvature tensor, $R_{\mu\nu}$, of "electromagnetic Riemannian geometry" or "geometrodynamics" (Table IV)—this despite the fact that the general $R_{\mu\nu}$ has 10 distinct components, and despite the fact that (29) and (31) can be said to constitute 10 conditions on these 10 components. Evidently these nonlinear algebraic equations are not all independent.

It is not only *necessary*—as previously shown—but also *sufficient* that the contracted curvature tensor $R_{\mu\nu}$ *satisfy the Rainich conditions* (30), (31), and (32) in order that one be *then able to represent $R_{\mu\nu}$ as the Maxwell square of an electromagnetic field* tensor, $f_{\mu\nu}$. Moreover, this field tensor is *unique* up to a duality rotation. We call this field tensor the *Maxwell root* of the Ricci curvature tensor, $R_{\mu\nu}$. We give separately the proofs for the cases where $R_{\mu\nu}$ is not a null tensor and where it *is* a null tensor ($R_{\mu\nu}R^{\mu\nu} = 0$; only 3 free parameters left in $R_{\nu\mu}$).

(1) Form the "Ricci part" $E_{\tau\sigma}{}^{\mu\nu}$ of the Riemann curvature tensor

$$E_{\tau\sigma}{}^{\mu\nu} \equiv \tfrac{1}{2}(-\delta_\tau{}^\mu R_\sigma{}^\nu + \delta_\sigma{}^\mu R_\tau{}^\nu - \delta_\sigma{}^\nu R_\tau{}^\mu + \delta_\tau{}^\nu R_\sigma{}^\mu). \tag{35}$$

It has the same symmetrics as the Riemann curvature tensor, contracts to the same Ricci curvature tensor,

$$E_{\tau\alpha}{}^{\alpha\nu} = -\tfrac{1}{2}R_\tau{}^\nu + 2R_\tau{}^\nu - \tfrac{1}{2}R_\tau{}^\nu + 2R_\alpha{}^\alpha\,\delta_\tau = R_\tau{}^\nu \equiv R_{\tau\alpha}{}^{\alpha\nu}, \tag{36}$$

by virtue of the condition $R_\alpha{}^\alpha = 0$, and introduces the antisymmetry we need for taking the Maxwell root. (2) Define the *extremal Maxwell root* $\xi_{\mu\nu}$ of the Ricci tensor—a pure electric field—up to a single $\pm$ sign by the equation

$$\xi_{\mu\nu}\xi_{\sigma\tau} = -\tfrac{1}{2}E_{\mu\nu\sigma\tau} - \tfrac{1}{2}(R_{\alpha\beta}R^{\alpha\beta})^{-1/2}E_{\mu\nu\gamma\delta}E_{\sigma\tau}{}^{\gamma\delta}. \tag{37}$$

Find any given component, $\xi_{\mu\nu}$, up to a $\pm$ sign by setting $(\sigma, \tau) = (\mu, \nu)$ and taking the root of (37). Then use (37) to determine the relative sign of different components, $\xi_{\mu\nu}$ and $\xi_{\sigma\tau}$. The consistency of the magnitudes is guaranteed by the Rainich conditions.

The prescription just given for the Maxwell square root is checked most easily in a Minkowskian reference system where the Ricci curvature tensor has the diagonal form

$$R_\mu{}^\nu = \begin{pmatrix} -(\xi_{01})^2 & & & \\ & -(\xi_{01})^2 & & \\ & & +(\xi_{01})^2 & \\ & & & +(\xi_{01})^2 \end{pmatrix}, \tag{38}$$

where ξ_{01} is a real positive number, known as soon as $R_\mu{}^\nu$ is known. Use this number to *define* an antisymmetrical extremal field tensor, ξ, of which all the components vanish in the present Lorentz system except ξ_{01} and $\xi_{10} = -\xi_{01}$. Then the dual of this tensor has all components zero except

$$({*\xi})_{23} = -({*\xi})_{32} = -\xi_{01}.$$

Thus the extremal field has the properties

$$\begin{aligned}\xi^2 \equiv \frac{1}{2}\,\xi_{\alpha\beta}\xi^{\alpha\beta} &= -(\xi_{01})^2 \\ &= \begin{pmatrix}\text{magnetic component of} \\ \text{reduced extremal field}\end{pmatrix}^2 - \begin{pmatrix}\text{electric component of} \\ \text{reduced extremal field}\end{pmatrix}^2 \\ &= -\frac{1}{2}\,(R_{\mu\nu}R^{\mu\nu})^{1/2};\end{aligned}$$

$$\xi \times \xi = 0 = \text{(magnetic component)} \cdot \text{(electric component)};$$

$$(*\xi)^2 = +(\xi_{01})^2 = -\xi^2; \qquad *\xi \times *\xi = 0. \tag{39}$$

In this same Minkowski frame the only nonvanishing distinct components of the tensor E of (35) are

$$E_{01}{}^{01} = -\tfrac{1}{2}(R_1{}^1 + R_0{}^0) = (\xi_{01})^2$$

and

$$E_{23}{}^{23} = -\tfrac{1}{2}(R_3{}^3 + R_2{}^2) = -(\xi_{01})^2 = -(*\xi_{23})^2. \tag{40}$$

Therefore the general tensor component of **E** can be written in the covariant form

$$E_{\alpha\beta\gamma\delta} = -\xi_{\alpha\beta}\xi_{\gamma\delta} - *\xi_{\alpha\beta}*\xi_{\gamma\delta}\,. \tag{41}$$

The product of this tensor by itself has the value

$$\begin{aligned} E_{\alpha\beta\gamma\delta}E^{\gamma\delta}{}_{\mu\nu} &= 2(\xi_{01})^2(-\ \xi_{\alpha\beta}\xi_{\mu\nu} + *\xi_{\alpha\beta}*\xi_{\mu\nu}) \\ &= (R_{\sigma\tau}R^{\sigma\tau})^{1/2}(-\ \xi_{\alpha\beta}\xi_{\mu\nu} + *\xi_{\alpha\beta}*\xi_{\mu\nu}) \end{aligned} \tag{42}$$

by virtue of the properties (39) of the extremal field. Multiply (41) by $-\tfrac{1}{2}$ and (42) by $-\tfrac{1}{2}(R^{\sigma\tau}R_{\sigma\tau})^{-1/2}$ and add, to cancel out the terms in the dual field. There results Eq. (37) for the components of the reduced extremal electromagnetic field tensor. Being a tensor equation true in the simple coordinate system, it must be true in all coordinate systems. So much for the machinery for taking the Lorentz square root when the Ricci tensor is non-null; that is, when its square is a nonzero multiple of the unit matrix.

Consider now the other case where the contracted curvature tensor is a null tensor,

$$R_{\mu\nu}R^{\mu\nu} = 0, \tag{43}$$

but a tensor which is not identically zero. As before, discuss the tensor in a Minkowski reference system, where the components, $g_{\mu\nu}$, of the metric tensor have their Lorentz values. The frame of reference is still free to the extent of a 6-parameter Lorentz transformation. Make such a rotation in 3-space (3 parameters) as will diagonalize the 3×3, space-space part of $R_{\mu\nu}$; thus, $R_{12} = R_{23} = R_{31} = 0$. The vanishing of the square of $R_\mu{}^\nu$ (Table IV),

$$(g^{\alpha\beta})_{\text{Lorentz}}R_{\mu\alpha}R_{\beta\nu} = 0, \tag{44}$$

then makes conditions of the form

$$\text{time-time:} \qquad -R_{00}{}^2 + R_{10}{}^2 + R_{20}{}^2 + R_{30}{}^2 = 0, \tag{45}$$

$$\text{time-space:} \qquad -R_{00}R_{01} + R_{01}R_{11} + 0 + 0 = 0, \tag{46}$$

$$\text{space}_1\text{-space}_1: \qquad -R_{01}^{\ \ 2} + R_{11}^{\ \ 2} + 0 + 0 = 0, \tag{47}$$

$$\text{space}_1\text{-space}_2: \qquad -R_{10}R_{02} + 0 + 0 + 0 = 0. \tag{48}$$

We conclude from equations of the type (48) that only one of the components, R_{01}, R_{02}, R_{03}, can differ from zero. Let this nonzero component be R_{01}, and let the sense of the x axis be so chosen that R_{01} is a negative quantity, $-2\kappa^2$. Then R^{01} is positive, corresponding to a Poynting flux in the plus x direction. Then from Eqs. (45)–(48) plus the requirement, $R_{00} \geqslant 0$, of positive definite energy density it follows that the Ricci curvature tensor has the form

$$R_{\mu\nu} = \begin{pmatrix} 2\kappa^2 & -2\kappa^2 & 0 & 0 \\ -2\kappa^2 & 2\kappa^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{row } 0 \\ 1 \\ 2 \\ 3 \end{matrix}. \tag{49}$$

This tensor may be written

$$R_{\mu\nu} = 2k_\mu k_\nu, \tag{50}$$

where $\mathbf{k}$ is the null vector

$$\begin{aligned} k_\mu &= (-\kappa, \kappa, 0, 0); \\ \mathbf{k}^2 &= k_\alpha k^\alpha = 0. \end{aligned} \tag{51}$$

Being covariant and true in one reference system, the decomposition (50) is valid in any reference system.

There is no Lorentz transformation that will diagonalize a null Ricci tensor any more than there is a Lorentz transformation that will make a null vector time-like, or parallelize field vectors $\mathbf{e}$ and $\mathbf{h}$ that satisfy the null condition, $(\mathbf{e}\cdot\mathbf{h}) = 0$, $\mathbf{h}^2 - \mathbf{e}^2 = 0$. The theorem that every symmetric tensor can be reduced to diagonal form does not hold when the metric is indefinite, a circumstance for the elucidation of which we are indebted to our colleague, Professor V. Bargmann. This feature in no way prevents taking the Maxwell root of (49):

$$f_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\kappa & 0 \\ 0 & 0 & \kappa & 0 \\ \kappa & -\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{52}$$

or

$$\mathbf{e} = (0, \kappa, 0); \qquad \mathbf{h} = (0, 0, \kappa) \tag{53}$$

as one checks by direct substitution in formula (13) or (14) for the Maxwell square. The tensor (49) describes a flow of energy in the x-direction at the speed of light, and (52) or (53) decompose the Poynting flux into factors, $\mathbf{e}$ and $\mathbf{h}$.

The polarization direction alone is free in the Maxwell square root. Application of a duality rotation of parameter α to the field (52) rotates the polarization vector by the angle α about the direction of propagation, leaving unaltered its Maxwell square, $R_{\mu\nu}$.

The prescription for the Maxwell square of a null Ricci tensor can be summarized in covariant form: (1) Take the "ordinary" square root, k_μ, according to Eq. (50). (2) Take a four vector $\mathbf{v}$ which (a) has unit magnitude,

$$\mathbf{v}^2 = v_\alpha v^\alpha = 1 \tag{54}$$

and (b) stands normal to $\mathbf{k}$,

$$k_\alpha v^\alpha = 0 \tag{55}$$

and (c) in the special Minkowski frame of (49) and (52)—for example—has the components

$$\mathbf{v} = (0, 0, 1, 0). \tag{56}$$

(3) Form the antisymmetrical product

$$f_{\mu\nu} = k_\mu v_\nu - k_\nu v_\mu \tag{57T}$$

or—in the so-called intrinsic notation—

$$\mathbf{f} = \mathbf{k} \wedge \mathbf{v}. \tag{57I}$$

(4) Then *the reduced field tensor* (57) *is a Maxwell square root of the null tensor* $R_{\mu\nu}$, *and apart from a duality rotation, is the only Maxwell square root of* $R_{\mu\nu}$.

The reduced field $\mathbf{f}$ is a null field in the Minkowski frame which we have used by preference, as one easily shows from (51), (54), and (55). Therefore it is a null field in any other frame of reference. The same is true of the field after it has experienced a duality rotation.

The effect of a duality rotation on $\mathbf{f}$ can be stated in terms of the constituent factors of $\mathbf{f}$; thus, $\mathbf{k}$ is left unchanged, and $\mathbf{v}$ is rotated about $\mathbf{k}$. However, $\mathbf{v}$ is never uniquely determined by $\mathbf{f}$: the new vector, $\mathbf{v}' = \mathbf{v} +$ (constant) $\mathbf{k}$, gives the same field, $\mathbf{f}$, and satisfies the conditions (54) and (55) as well as does $\mathbf{v}$ itself. Moreover, the descriptive word "polarization" should be taken cautiously, as indicating in the present context only the orientation of the mutually perpendicular pair of 3-vectors, $\mathbf{e}$ and $\mathbf{h}$. It does not stand for the polarization of a monochromatic directed wave train, which could not be determined without a knowledge of the field at nearby points—even if one had a monochromatic directed wave to talk about!

The Two-Way Connection between Field and Curvature.

In summary, any reduced electromagnetic field tensor, $f_{\mu\nu}$, null or not, produces a Ricci curvature that satisfies the Rainich conditions. Conversely, any

Ricci curvature tensor, null or not, that satisfies the Rainich conditions, has a Maxwell square root, f_{ik}, that is unique up to a duality rotation. In addition, in the non-null case the Ricci curvature—or the field—determines a structure with two blades, A and B, at each point in space time.

The Blades Do Not Mesh.

How is the geometric structure at one point in space-time related to that at a neighboring point? A geometrical description of the electromagnetic field has to answer this question. It therefore appeared useful to us to raise the issue, what happens when one moves in the local two dimensional surface, or tangent plane, of blade A to neighboring points where blade A is tilted at slightly different orientations. Will one arrive in this way at a well defined two dimensional surface,

$$x^\mu = x_A{}^\mu(u, v)? \tag{58}$$

Or will the pattern of the blades lead to structures like those in a spiral dislocation in a crystal, so that one can get from one point to any other point by moving about on blades A via a suitably selected route? In this case no surface of the type (58) will exist. Does the *demand* that blades fit into a surface of type (58) lead to Maxwell's equations? We investigated and found that this demand is too restrictive to be satisfied by the general solution of the equations of electromagnetism. Therefore we decided to let Maxwell's equations speak for themselves. We expressed the electromagnetic field as the Maxwell root of the Ricci tensor, where

$$\begin{gathered} f_{\mu\nu} = e^{*\alpha}\xi_{\mu\nu}, \quad \text{or} \quad \mathbf{f} = e^{*\alpha}\boldsymbol{\xi}, \\ \boldsymbol{\xi} = \text{extremal Maxwell root of } R_{\mu\nu}, \end{gathered} \tag{59}$$

inserted into Maxwell's equations,

$$\begin{aligned} f^{\mu\nu}{}_{;\nu} &= 0, \\ *f^{\mu\nu}{}_{;\nu} &= 0, \end{aligned} \tag{60}$$

and learned at first hand what Rainich had already learned before us about the true geometrical content of electromagnetism.

Four of Maxwell's Equations as Consequences of the Identities Satisfied by the Curvature or Stress Energy Tensor.

The eight equations (60) of electrodynamics take the form

$$\begin{aligned} 0 = f^{\mu\nu}{}_{;\nu} &= (\xi^{\mu\nu}{}_{;\nu} + *\xi^{\mu\nu}\partial\alpha/\partial x^\nu)\cos\alpha + (*\xi^{\mu\nu}{}_{;\nu} - \xi^{\mu\nu}\partial\alpha/\partial x^\nu)\sin\alpha, \\ 0 = *f^{\mu\nu}{}_{;\nu} &= (-\xi^{\mu\nu}{}_{;\nu} - *\xi^{\mu\nu}\partial\alpha/\partial x^\nu)\sin\alpha + (*\xi^{\mu\nu}{}_{;\nu} - \xi^{\mu\nu}\partial\alpha/\partial x^\nu)\cos\alpha, \end{aligned} \tag{61}$$

or, by simple combination

$$\xi^{\mu\nu}{}_{;\nu} + *\xi^{\mu\nu}\partial\alpha/\partial x^{\nu} = 0, \tag{62}$$

$$*\xi^{\mu\nu}{}_{;\nu} - \xi^{\mu\nu}\partial\alpha/\partial x^{\nu} = 0. \tag{63}$$

We now combine these equations in such a way as to separate the information they give about $\boldsymbol{\xi}$ and about α. Multiply (62) by $\xi_{\alpha\mu}$ and sum over μ. Use the properties (39) of $\boldsymbol{\xi}$ and the identity

$$\xi_{\alpha\mu}*\xi^{\mu\nu} = 0, \tag{64}$$

which, being true in a simple Minkowski frame, and being covariant, is true in any frame. We find the result

$$\xi_{\alpha\mu}\xi^{\mu\nu}{}_{;\nu} = 0, \tag{65a}$$

and, by similar reasoning starting from (63),

$$*\xi_{\alpha\mu}*\xi^{\mu\nu}{}_{;\nu} = 0. \tag{65b}$$

Only four of the eight Eqs. (65a), (65b) are independent, as may be seen by going to the simple coordinate system where ξ_{01} and $*\xi_{23} = -\xi_{01}$ are the only nonvanishing components of $\boldsymbol{\xi}$. In these coordinates we see that the eight Eqs. (65a), (65b) are equivalent to the four independent equations

$$\xi_{\alpha\mu}\xi^{\mu\nu}{}_{;\nu} + *\xi_{\alpha\mu}*\xi^{\mu\nu}{}_{;\nu} = 0. \tag{66}$$

The identity[12]

$$\begin{aligned}\xi_{\alpha\mu;\nu}\xi^{\mu\nu} &= \tfrac{1}{2}(\xi_{\alpha\mu;\nu} + \xi_{\nu\alpha;\mu} + \xi_{\mu\nu;\alpha})\xi^{\mu\nu} - \tfrac{1}{4}(\xi_{\mu\nu}\xi^{\mu\nu})_{;\alpha} \\ &= *\xi_{\alpha\mu}*\xi^{\mu\nu}{}_{;\nu} - \tfrac{1}{2}(\boldsymbol{\xi}^2)_{;\alpha}\end{aligned}$$

and a similar equation for $*\boldsymbol{\xi}$ in place of $\boldsymbol{\xi}$ puts Eq. (66) in the form

$$\tfrac{1}{2}(\xi_{\alpha\mu}\xi^{\mu\nu} + *\xi_{\alpha\mu}*\xi^{\mu\nu})_{;\nu} = 0. \tag{67}$$

The quantity in parentheses is the Maxwell stress tensor. This quantity is to be identified with the Ricci curvature, whose trace, R, is zero so that (67) can be written in the form

$$[R_{\mu}{}^{\nu} - \tfrac{1}{2}\delta_{\mu}{}^{\nu}R]_{;\nu} = 0. \tag{68}$$

However, this condition is no requirement at all. Bianchi proved that the Ricci tensor calculated from an *arbitrary* metric tensor will satisfy (68) *identically*. It

[12] The second line of this identity is a special case of the identity

$$\tfrac{1}{2}A^{\alpha\beta}(B_{\mu\alpha;\beta} + B_{\alpha\beta;\mu} + B_{\beta\mu;\alpha}) = *A_{\mu\alpha}*B^{\alpha\beta}{}_{;\beta}$$

which is valid for any two antisymmetrical tensors in 4-space.

has long been known that half of Maxwell's equations are given by these Bianchi identities. Now what have the other half of Maxwell's equations to say about the curvature of space?

Maxwell's Other Four Equations Demand That a Certain Vector Combination of the Curvature and Its First Derivative Shall Have Zero Curl.

Multiply (62) by $*\xi_{\beta\mu}$ and (63) by $\xi_{\beta\mu}$, sum over μ, use (15), and find the result

$$*\xi_{\beta\mu}\xi^{\mu\nu}{}_{;\nu} + \xi_{\beta\mu}*\xi^{\mu\nu}{}_{;\nu} + \delta_\beta{}^\nu\xi^2\partial\alpha/\partial x^\nu = 0$$

or

$$\partial\alpha/\partial x^\beta = \alpha_\beta, \tag{69}$$

where we make the *definition*

$$\alpha_\beta = -(*\xi_{\beta\mu}\xi^{\mu\nu}{}_{;\nu} + \xi_{\beta\mu}*\xi^{\mu\nu}{}_{;\nu})/\xi^2. \tag{70}$$

This vector expresses itself in terms of the Ricci curvature in the form

$$\alpha_\beta = (-g)^{1/2}[\beta\lambda\mu\nu]R^{\lambda\gamma;\mu}R_\gamma{}^\nu/R_{\sigma\tau}R^{\sigma\tau}. \tag{71}$$

For proof of (71), recall the expression (41) for the Ricci part (35) of the Riemann curvature tensor in terms of the extremal field, ξ; and form the first contracted covariant derivative of this tensor:

$$\begin{aligned} E^{\gamma\delta\beta\tau}{}_{;\tau} &= -\xi^{\gamma\delta}\xi^{\beta\tau}{}_{;\tau} - *\xi^{\gamma\delta}*\xi^{\beta\tau}{}_{;\tau} - \xi^{\gamma\delta}{}_{;\tau}\xi^{\beta\tau} - *\xi^{\gamma\delta}{}_{;\tau}*\xi^{\beta\tau} \\ &= \tfrac{1}{2}(-g^{\gamma\beta}R^{\delta\tau} + g^{\gamma\tau}R^{\delta\beta} - g^{\delta\tau}R^{\gamma\beta} + g^{\delta\beta}R^{\gamma\tau})_{;\tau} \\ &= \tfrac{1}{2}(R^{\delta\beta;\gamma} - R^{\gamma\beta;\delta}). \end{aligned} \tag{72}$$

Here we use the fact that every component of the covariant derivative of $g^{\gamma\beta}$ vanishes, and employ the Bianci identities, $R^{\delta\tau}{}_{;\tau} = 0$, to annul all but two terms among the eight that arise from the differentiation. Define

$$\begin{aligned} F_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}(-g)^{1/2}[\gamma\delta\mu\nu]E_{\alpha\beta}{}^{\mu\nu} \\ &= \tfrac{1}{2}(-g)^{1/2}[\gamma\delta\mu\nu]\,(-\xi_{\alpha\beta}\xi^{\mu\nu} - *\xi_{\alpha\beta}*\xi^{\mu\nu}) \\ &= -\xi_{\alpha\beta}*\xi_{\gamma\delta} + *\xi_{\alpha\beta}\xi_{\gamma\delta} \\ &= \tfrac{1}{2}(-g)^{1/2}[\gamma\delta\mu\nu]\tfrac{1}{2}(-\delta_\alpha{}^\mu R_\beta{}^\nu + \delta_\alpha{}^\nu R_\beta{}^\mu - \delta_\beta{}^\nu R_\alpha{}^\mu + \delta_\beta{}^\mu R_\alpha{}^\nu) \\ &= \tfrac{1}{2}(-g)^{1/2}[\gamma\delta\mu\nu]\,(\delta_\alpha{}^\nu R_\beta{}^\mu - \delta_\beta{}^\nu R_\alpha{}^\mu). \end{aligned} \tag{73}$$

Finally, form the product

$$\begin{aligned} F_{\alpha\beta\gamma\delta}E^{\gamma\delta\beta\tau}{}_{;\tau} &= -2\xi^2(*\xi_{\alpha\beta}\xi^{\beta\tau}{}_{;\tau} + \xi_{\alpha\beta}*\xi^{\beta\tau}{}_{;\tau}) \\ &= \tfrac{1}{2}(-g)^{1/2}[\gamma\delta\mu\nu]\,(\delta_\alpha{}^\nu R_\beta{}^\mu - \delta_\beta{}^\nu R_\alpha{}^\mu)R^{\delta\beta;\gamma} \\ &= \tfrac{1}{2}(-g)^{1/2}[\alpha\delta\gamma\mu]R^{\delta\beta;\gamma}R_\beta{}^\mu. \end{aligned} \tag{74}$$

The equation

$$\xi_{\gamma\delta} \times \xi^{\gamma\delta}{}_{;\tau} \equiv \tfrac{1}{2}(\xi_{\gamma\delta} * \xi^{\gamma\delta})_{;\tau} = 0,$$

which follows from $\boldsymbol{\xi} \times \boldsymbol{\xi} = 0$ by differentiation, as well as the identity (64), were used in computing (74) from (72) and (73). Now divide (74) by $2(\boldsymbol{\xi}^2)^2 = \tfrac{1}{2}R_{\sigma\tau}R^{\sigma\tau}$ to obtain (71), as was to be proven.

The vector α_β of (71) has a well defined existence in *any* Riemannian space where the Ricci curvature tensor $R_{\mu\nu}$ is non-null and differentiable. From such general Riemannian spaces the geometry of the Einstein-Maxwell theory is distinguished by the circumstance that *this vector*—as shown by Rainich—is not arbitrary, but *is the gradient of the complexion*, α, *of the electromagnetic field.* Consequently it follows that the curl of α_β must vanish:

$$\alpha_{\beta;\gamma} - \alpha_{\gamma;\beta} = \alpha_{\beta,\gamma} - \alpha_{\gamma,\beta} = 0. \tag{75}$$

Conversely, when the curl (75) vanishes, then the line integral of the vector α_β of (71) from some selected point 0 to any arbitrary point x^μ *defines* a scalar complexion,

$$\alpha = \int_0^x \alpha_\beta \, dx^\beta + \alpha_0 \tag{76}$$

up to an additive constant, α_0, provided that the region of space under consideration is simply connected, and provided that the line of integration does not include any point where $R_{\sigma\tau}R^{\sigma\tau}$ vanishes. For a multiply-connected space it is necessary to replace (75) by the demand that the line integral of α_β around any closed path shall be an integral multiple of 2π,

$$\oint \alpha_\beta \, dx^\beta = 2\pi n \tag{77}$$

provided the line of integration does not touch any null points. This condition, plus the algebraic requirements of Rainich

$$R_\alpha{}^\alpha = 0; \qquad R_{00} \geqslant 0; \qquad R_\alpha{}^\mu R_\mu{}^\beta = \delta_\alpha{}^\beta(\tfrac{1}{4}R_{\sigma\tau}R^{\sigma\tau}), \tag{78}$$

gives the *necessary and sufficient* conditions that a Riemannian geometry shall reproduce the physics of Einstein and Maxwell, provided that the curvature $R_{\mu\nu}$ is non-null. It may well be that trivial changes in the statement of the theorem will cover the case of null fields, but this point remains to be investigated.

Why Rainich-Riemannian Geometry?

We shall give the name Rainich-Riemannian geometry to any 4-space with signature $- + + +$ which satisfies (77) and (78). The question poses itself insistently to find a point of view which will make Rainich-Riemannian geometry seem a particularly natural kind of geometry to consider. Presumably a variational principle based on an appropriate scalar Lagrange density will prove the most natural starting point to discuss this question. Whatever the deeper simplicities may be, it is extraordinarily beautiful that the Rainich-Riemannian geometry (77), (78) of empty curved space reproduces all the standard machinery of Maxwell stresses, electromagnetic waves, and generation of gravitational forces by field energy.

III. CHARGE AS FLUX IN MULTIPLY-CONNECTED SPACE: ELECTROMAGNETISM WITHIN THE ARENA OF A PRESCRIBED METRIC; PLAN OF THIS SECTION; TOPOLOGY; DIFFERENTIAL GEOMETRY; MAXWELL'S EQUATIONS AND CHARGE

A. Electromagnetism Within the Arena of a Prescribed Metric

"Already unified field theory" or geometrodynamics appears to consist entirely of differential equations and algebraic relations on the curvature (Eqs. 75 and 78)—in other words, appears to have an exclusively local character—so long as space is assumed to be simply connected. As soon as the possibility is admitted of two or more topologically distinct routes to pass from one point to another (Fig. 3), then in addition one has to impose a *periodicity condition*

$$\oint_{C_i} \alpha_\beta \, dx^\beta = 2\pi n_i \tag{79}$$

for each topologically distinct closed circuit C_i that is free of null points. Only when this condition is satisfied will the electromagnetic field, $\mathbf{f} = e^{*\alpha}\xi$, be a single valued function of position. The condition (79) has a nonlocal character. This circumstance forces us to ask, what *additional* features of a nonlocal character appear when Einstein-Maxwell physics goes on in a multiply-connected space?

To *recognize charge as a nonlocal manifestation of charge free electrodynamics in a multiply-connected space*, we will find it helpful to revert from the ideas of the more familiar geometrodynamics to language of Maxwell theory in a preexisting space time continuum. In keeping close to the most easily visualized terminology, we will not deny that electrodynamics is intrinsically nonlinear. A field has an energy density that curves space. This curvature affects the propagation of the field. A field of twice the strength therefore propagates differently from the original field. This nonlinearity will remain hidden behind the scenes in the following analysis. Attention will be limited to Maxwell's equations and their consequences. However, these consequences will remain valid when Maxwell's equations for the field are supplemented by Einstein's equations for the metric.

The concept of multiple connectedness is topological in character and logically precedes any idea of metric. Section B therefore summarizes the necessary topological preliminaries, including the concepts of continuity, manifold, boundary, homology class, Betti number, differentiable manifold, and coordinate patches. No space topologically inequivalent to an open subset of Euclidean space can be covered without singularity by a single nonsingular coordinate system. This circumstance makes the analysis of vectors, tensors, and other quantities by way of components—in the traditional spirit of tensor analysis—less appropriate than an *intrinsic* type of calculus, such as is familiar from vector analysis in 3-space. Section C uses this intrinsic notation of Cartan side by side with the familiar notation to define the needed concepts of differential geometry in a space not yet endowed with any metric: vectors, alternating tensors, cross product, curl, Stokes theorem, the metric-free half of Maxwell's equations, conservation of flux, charge, vector potential, de Rham's theorem, and the issue of electric *versus* magnetic charges. The rest of Sec. C traces some of the more important additional consequences that flow from existence of a metric: duality, divergence, differential operators, and relations and integral formulas involving these operators. Section D analyzes electric and magnetic fields on one or two space like surfaces as initial value data for Maxwell's equations; the new feature of charge brought in by multiple connectedness; and the description of charge by way of the theory of harmonic vector fields.

B. Topology

Topology and Point Sets.

Topology[13] is the study of a nonquantitative idea of "nearby". This idea itself is not ordinarily axiomatized; instead, topology axiomatically defines what is meant by a neighborhood, an open set, or a closed subset of space. The relationship of these ideas to "nearby" may be suggested as follows: (1) A neighborhood of a point is a subset containing all points sufficiently near x. (2) An open subset is one which contains all points sufficiently near any of its points. (3) A closed subset C is one which contains every point that is arbitrarily close to C.

Let two sets X_1, X_2 be equivalent as sets, so that they also contain the same number of points. Then the two sets are equivalent as topological spaces, that is *homeomorphic,* if there is a 1-1 transformation h of X_1 onto X_2, called a *homeomorphism,* under which the open sets of X_1 are in 1-1 correspondence with the open sets of X_2. The open sets define the notion of "nearby". Therefore we may say that a *homeomorphism h* is a 1-1 onto correspondence of X_1 and X_2 which always makes nearby points in one space correspond to nearby points in the other, i.e., it is continuous, and so is its inverse.

We have found outselves forced to consider spaces that are topologically more general than those usually treated in physics. Nevertheless, we shall restrict ourselves to a very special class of spaces, called manifolds: *An n*-manifold (*27*) *is a topological space which* (1) *is locally Euclidean of dimension n,* (2) *is Hausdorff,*

[13] A standard text on point set topology is one by Kelley (*24*). For a brief development of the theory see a book by Pontrjagin (*25*). Statements of definitions and theorems from point set topology which are most pertinent for a study of manifolds can be found in a book by Aleksandrov (*26*).

and (3) *has a countable basis.* (1) *A space is locally Euclidean of dimension n* if each point has a neighborhood homeomorphic to Euclidean n-space. (2) A space is *Hausdorff* if every two distinct points have disjoint neighborhoods. Effectively Hausdorff means that no two distinct points are arbitrarily close to each other, but this intuitive notion may also be made precise in other ways. (3) A space has a *countable basis* if there is a countable collection of open sets such that every open set is a union of open sets in this collection. This condition is imposed so that a manifold will not be uncomfortably large.

Examples of Manifolds.

The prime examples of manifolds are the Euclidean spaces. Euclidean n-space, R^n, is the space of all n-tuples of real numbers, $\mathbf{x} = (x^1, x^2, \cdots x^n)$, with the topology customary in analysis. The open subset of R^n defined by

$$\mathbf{x}\cdot\mathbf{x} = \Sigma(x^i)^2 < 1 \tag{80}$$

is called the n-ball. It is also an n-manifold, and is in fact homeomorphic to R^n by the transformation $\mathbf{x} \rightarrow \mathbf{x}(1 - \mathbf{x}\cdot\mathbf{x})^{-1}$. Another familiar example of a manifold is the n-sphere S^n which is a subspace of R^{n+1} consisting of all points $\mathbf{x}$ satisfying the condition

$$\mathbf{x}\cdot\mathbf{x} = \sum_{x=1}^{n+1} (x^i)^2 = 1. \tag{81}$$

From S^n we may obtain projective n-space P^n by identifying antipodal points on S^n; i.e., a point x of P^n is an unordered pair $(\mathbf{x}, -\mathbf{x})$ of points $\mathbf{x}$ of R^{n+1} satisfying $\mathbf{x}\cdot\mathbf{x} = 1$. In general relativity, S^3 and P^3 have been used in cosmological models (*28*). Another manifold, the 3-torus T^3, is encountered in theoretical physics in the frequent instances where one imposes periodic boundary conditions. The n-torus T^n is a space whose points are families of points in R^n, $x = \{\mathbf{x} + 2\pi\mathbf{m}\}_{\mathbf{m}}$ where $\mathbf{m}$ ranges over all n-tuples of integers $\mathbf{m} = (m_1, m_2, \cdots, m_n)$. As a final example we construct a space W_m which illustrates in a simple way some of the topological possibilities which we are interested in investigating in relation to the idea of charge. Starting from the 3-sphere S^3, we consider only those points $\mathbf{x} = (x^1, x^2, x^3, x^4)$ which satisfy both $\mathbf{x}\cdot\mathbf{x} = 1$ and $|x^4| \leqslant 1 - \epsilon$. We then identify each point $(x^1, x^2, x^3, 1 - \epsilon)$ with $(x^1, x^2, x^3, -1 + \epsilon)$. This may be called the pierced sphere, or W_1. By cutting out k pairs of antipodal balls (such as $x^4 > 1 - \epsilon$ and $x^4 < 1 + \epsilon$ above) around k pairs of points of S^3 (such as $\mathbf{x} = (0, 0, 0, \pm 1)$ above) and making similar identifications of the boundaries, we construct the k-pierced sphere W_k. (See Fig. 4.)

Product Manifolds.

When $\mathfrak{M}^p$ is a p-manifold and $\mathfrak{M}^q$ a q-manifold, then we can construct from them a $(p + q)$-manifold $\mathfrak{M}^p \times \mathfrak{M}^q$ whose points are pairs (x, y) where x is any

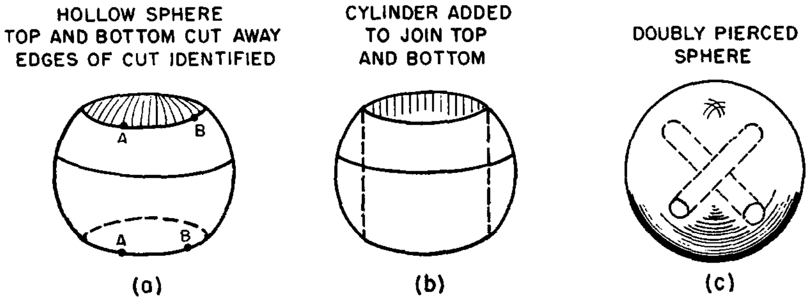

FIG. 4. The pierced spheres. When the polar caps are excluded from a sphere and the resulting boundaries identified as indicated by the corresponding points A, B, in (a), there results a space which may be called a pierced sphere, W_1. The identification may be visualized by pulling the edges together through the center of the sphere as in (b), so that the resulting manifold is the surface of a ball with a hole drilled through the center. Drilling k such holes in a sphere, in such a way that the drill holes never intersect, gives the k-pierced sphere W_k. In (c) a view of W_2 is shown. The figures show two dimensional manifolds, while the W_k in the text are 3-manifolds.

point of $\mathfrak{M}^p$ and y any point of $\mathfrak{M}^q$. Euclidean 2-space R^2 is, for example, the product $R^1 \times R^1$, and similarly $R^n = R^1 \times R^1 \times \cdots \times R^1$ (n factors). We note that S^1 and T^1 are homeomorphic,

$$\{x + 2\pi n\}_n \rightarrow (x^1, x^2) \text{ with } x^1 + ix^2 = e^{ix}. \tag{82}$$

(We write $T^1 = S^1$ since we usually do not need to distinguish homeomorphic manifolds, unless the names of the points, *i.e.*, $\{x + 2\pi n\}_n \in T^1$ *vs* $(x^1, x^2) \in S^1$, are needed in a construction.) Then T^n is a product of n circles, $T^n = S^1 \times S^1 \times \cdots \times S^1$. It may also be seen that $W_1 = S^2 \times S^1$.

In the examples above we have not defined the topology, or in other words, we have not made the choice of open sets, since the appropriate choice is fairly obvious.

Differentiable Manifolds.

To discuss differential geometry we need both a manifold, and a concept of *differentiable function* on this manifold. Let a criterion be supplied which decides which functions defined on a manifold $\mathfrak{M}$ are to be called differentiable. Then this criterion is called a *differentiable structure* on $\mathfrak{M}$, and $\mathfrak{M}$ is called a *differentiable manifold.* The class of differentiable functions is however subject to certain axioms (*27*):

For every point x of $\mathfrak{M}$ there exists an open neighborhood U of x, and n real valued functions $x_1(x), x_2(x), \cdots x_n(x)$ defined on U such that:

(a) The transformation $x \rightarrow (x_1(x), x_2(x), \cdots x_n(x))$ of U into R^n is a homeomorphism of U onto an open set of R^n, so that every function f defined in U can be expressed in terms of the x_i,

$$f(x) = f(x_1, x_2 \cdots x_n). \tag{83}$$

(b) A function $f(x)$ is differentiable at a point of U if and only if it is defined in an open neighborhood W of that point, with W contained in U, and $f(x_1, x_2, \cdots x_n)$ has continuous derivatives of all orders with respect to the x_i for values of the x_i corresponding to points in W.

The neighborhood U with the functions x_i is called a *coordinate patch,* and the set of functions $x_1(x)$, $x_2(x)$, $\cdots x_n(x)$ is called a system of *local coordinates* in U. It follows from the definition that a coordinate $x_i(x)$ is a differentiable function.

We apply the definition of differentiable function by way of illustration to some of the manifolds already mentioned. On the n-sphere we may define $x_i(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \cdots x_{n+1})$ to be a differentiable function of the points x in S^n. Then some n of these $n + 1$ functions will serve as coordinates about any point in part (a) of the axioms, while the remaining differentiable functions are given by (b). For the projective n-space P^n whose points x are pairs of points, $(\mathbf{x}, -\mathbf{x})$, of S^n, we say $f(x)$ is a differentiable function on P^n if $f((\mathbf{x}, -\mathbf{x}))$ is a differentiable function of $\mathbf{x} \in S^n$. For the space W_1, the pierced sphere, we consider x_1, x_2 and x_3 as differentiable functions, as they were on the sphere. The quantity x^4 is to be differentiable only in the region $|x^4| < 1 - \epsilon$. This function is not defined at the rims that have been identified with each other. We need another function which is defined and which can be called differentiable across this region of identification. For this purpose we define $f(x)$ by

$$\begin{aligned} f(x) &= 1 - \epsilon - x^4(x) \quad \text{for} \quad 0 < x^4 \leq 1 - \epsilon, \\ f(x) &= -1 + \epsilon - x^4(x) \quad \text{for} \quad -1 + \epsilon \leq x^4 < 0, \end{aligned} \tag{84}$$

and insist that $f(x)$ is differentiable for $x^4 \neq 0$. From among the functions x_1, x_2, x_3, x_4, and f we can choose local coordinates about any point.

Although a topological manifold may be given more than one differentiable structure (*29*), we shall abbreviate differentiable manifold to manifold in what follows, and the word *topology* will include differentiable structure as well as topology.

C. Differential Geometry and the Formulation of Maxwell's Equations

Differentiable Functions and Fields

To state physical laws in quantitative form, we are compelled to deal with vectors and tensors in a curved space. These quantities are not most conveniently described in the familiar language of tensor analysis when the space is multiply connected. A new or *intrinsic* formulation is required.

Tensor analysis is not adequate. It demands a nonsingular coordinate system with respect to which one can give the components of vectors and tensors. However, according to the definition of a differentiable manifold, a single nonsingular coordinate system is not enough to cover a manifold that is topologically inequivalent to an open set in Euclidean space. It is essential for our physical applications that we be able to distinguish singular tensor fields from nonsingular ones. A singularity in a field will ordinarily imply a localized *source term* in the differential equations. Such a source term will represent a nongeometric charge or mass, which has not been eliminated from the theory, but merely idealized to a point charge or a point mass. To investigate the content of pure geometrodynamics we therefore *exclude all singularities in the fields*. Consider for example the two dimensional surface of the unit sphere. The polar angles, θ, φ, are ordinarily called coordinates, but they do not cover the surface without singularity: (1) The metric,

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \tag{85}$$

has a component, $g^{\varphi\varphi} = \sin^{-2} \theta$, which goes to infinity at the two poles, $\theta = 0$ and $\theta = \pi$. (2) A vector field with apparently nonsingular components, $v_\theta = 1$, $v_\varphi = 0$, is not well defined with respect to direction at either pole. This singularity in the vector $\mathbf{v}$ = grad θ raises a question about calling θ a coordinate. (3) Along the Greenwich meridian the angle φ suffers a discontinuous change from 0 to 2π, in this region making this quantity also inappropriate for a coord nate. (4) Only in the region $0 < \theta < \pi$, $0 < \varphi < 2\pi$ do the polar angles supply a well defined "coordinate patch."

Another "coordinate patch" is needed. Let a small circular ink pad be centered at the north pole and let the pad be moved down the Greenwich meridian to the south pole. We ask for a new coordinate system that is regular everywhere in the blackened region. Of course the new coordinate system may be regular over a wider region. For example, consider the point $\theta = \pi/2$, $\varphi = \pi/2$. Let this be north pole of a new set of polar angles, θ', φ', in terms of which (1) the previous north pole is $\theta' = \pi/2$, $\varphi' = \pi/2$ (2) the previous Greenwich meridian runs from $\theta' = \pi/2$, $\varphi' = \pi/2$ to $\theta' = \pi/2$, $\varphi' = 3\pi/2$ (3) the previous south pole lies at $\theta' = \pi/2$, $\varphi' = 3\pi/2$. There is evidently a wide region where the old and new coordinate patches overlap. Moreover, at any fixed point on the sphere at least *one* non-singular pair of coordinates θ, φ or θ', φ', is well behaved. If we do not know what nonsingular coordinates are, then we have not yet defined the differentiable structure of the manifold. However, as soon as we have said what we mean by a differentiable function, then the idea of a *differentiable tensor field* is well defined: A tensor is differentiable at a point x if its components with respect to a (nonsingular) coordinate system around x are differentiable functions at x.

Intrinsic Representation of Tensors.

In the general differentiable manifold it will be impossible to describe a field, such as the Maxwell field, by giving its components $F_{\alpha\beta}$ with respect to a single set of coordinates. If components are to be used at all, they have to be given with respect to the coordinates of the several distinct coordinate patches. In each patch there is great freedom of choice about the coordinate system. Consequently the components, $F_{\alpha\beta}$, of the Maxwell field in the several coordinate systems are not individually so important as the concept that can be abstracted from them: the intrinsic value, **F**, of the Maxwell field. This procedure of abstraction is familiar from the analysis of Cartesian vectors. To say that the velocity vector **v** is known is to say that on demand one can give the components of **v** in any nonsingular coordinate system. We propose to generalize this abstract formalism to the case of curved space and tensors of order higher than the first.

In vector analysis it is convenient to express the connection between a vector **v** and its components by the formula

$$\mathbf{v} = v^m \mathbf{e}_m , \tag{86}$$

where the components v^m depend upon the choice of base vectors $\mathbf{e}_m$. "Intrinsic differential geometry" expresses the same type of relation more explicitly in the form

$$\mathbf{v} = v_\alpha \operatorname{grad} x^\alpha, \tag{87}$$

and more briefly in the notation,

$$\mathbf{v} = v_\alpha \mathbf{d}x^\alpha, \tag{88}$$

where the letter **d** stands for gradient. Modern text books on differential geometry (*30*) even replace the boldface **d** by lightface *d*—as we also shall do later—but for the time being we keep the boldface notation to emphasize the vectorial character of the gradient operation.

The gradient operation, applied to any function of position, $f = f(P)$, expresses itself in intrinsic notation in the familiar form

$$\operatorname{grad} f = (\partial f/\partial x^\alpha) \operatorname{grad} x^\alpha$$

or

$$\mathbf{d}f = (\partial f/\partial x^\alpha)\, \mathbf{d}x^\alpha. \tag{89}$$

Covariant vectors, such as grad f, or $\mathbf{d}f$, and **v** are linear combinations of the gradients of the coordinates and are called differential forms of the first order, or 1-forms.

Instead of using the gradients of a particular set of coordinates as base vectors in a certain coordinate patch, one can use there alternatively as base vectors any other set of n linearly independent vectors, $\boldsymbol{\omega}^\alpha$. Here the superscript α tells which vector, not which component of one vector. Relative to this set of basic vectors the components of the vector v are again read out of the expansion

$$\mathbf{v} = v_\alpha \boldsymbol{\omega}^\alpha. \tag{90}$$

The components of grad f or df in such a frame are called its Pfaffian derivatives $f_{,\alpha}$ with respect to the $\boldsymbol{\omega}^\alpha$:

$$\text{grad}\, f = \mathbf{d}f = f_{,\alpha}\boldsymbol{\omega}^\alpha \tag{91}$$

Exterior Differential Forms.

To describe the Maxwell field requires one to deal with a higher order tensor, or 2-form. The simplest 2-form, $\boldsymbol{\alpha}$, is the cross product of two 1-forms, $\mathbf{u}$ and $\mathbf{v}$, thus,

$$\boldsymbol{\alpha} = \mathbf{u} \wedge \mathbf{v}$$

in intrinsic notation; or

$$\alpha_{\mu\nu} = u_\mu v_\nu - v_\mu u_\nu \tag{92}$$

in the familiar language of tensor analysis, or

$$\boldsymbol{\alpha} = \tfrac{1}{2}\alpha_{\mu\nu}\mathbf{d}x^\mu \wedge \mathbf{d}x = \tfrac{1}{2}\alpha_{\mu\nu}(\text{grad}\, x^\mu) \times (\text{grad}\, x^\nu)$$

in a formulation that connects the intrinsic form with the component representation.

The *exterior product* operation $\wedge$ ("hook") generalizes the cross product into a means for multiplying vectors and alternating tensors in such a way as to obtain tensors that are also alternating. This operation does not apply to symmetric tensors. The $\wedge$ multiplication is defined by the requirements (1) it is associative (2) it is distributive (3) exterior multiplication of two vectors is anticommutative; that is, we ask

$$\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u} \tag{93}$$

for any two 1-forms or vectors $\mathbf{u}$, $\mathbf{v}$. In particular, $\mathbf{u} \wedge \mathbf{u} = 0$.

The exterior product of p vectors, or a linear combination of such vectors, gives a *p-form*. Let each vector be expressed in terms of base vectors $\boldsymbol{\omega}^\alpha$. The product of the p vectors, or the linear combination of such products, can be expressed in the form

$$\begin{aligned} a &= \sum_{\alpha_1 < \alpha_2 \cdots < \alpha_p} a_{\alpha_1\alpha_2\cdots\alpha_p}\boldsymbol{\omega}^{\alpha_1} \wedge \boldsymbol{\omega}^{\alpha_2} \wedge \cdots \boldsymbol{\omega}^{\alpha_p} \\ &= (p!)^{-1} a_{\alpha_1\alpha_2\cdots\alpha_p}\boldsymbol{\omega}^{\alpha_1} \wedge \boldsymbol{\omega}^{\alpha_2} \wedge \cdots \wedge \boldsymbol{\omega}^{\alpha_p}. \end{aligned} \tag{94}$$

Here the coefficients or *tensor components* in the first sum are well defined. In terms of these components those that appear in the second sum are *defined* by the requirement that they be alternating functions of the indices.

The exterior product $\mathbf{d}x^1 \wedge \mathbf{d}x^2 = (\text{grad}\, x^1) \times (\text{grad}\, x^2)$ signifies geometrically an area spanned by base vectors along the x^1 and x^2 coordinate axes. Similarly the product $\mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3$ represents the parallelepipedal volume spanned by three base vectors.

The commutation rule for a p-form $\mathbf{a}$ and a q-form $\mathbf{b}$ is easily found:

$$\mathbf{a} \wedge \mathbf{b} = (-1)^{pq}\mathbf{b} \wedge \mathbf{a}. \tag{95}$$

Functions and vectors are included in this scheme of exterior differential forms as 0-forms and 1-forms, respectively.

Exterior Differentiation.

The operator $\mathbf{d}$ which produces the 1-form $\mathbf{d}f = \text{grad } f$ from the 0-form f can be extended to a form of higher degree,

$$\mathbf{a} = (p!)^{-1}a_{\alpha_1\cdots\alpha_p}\mathbf{d}x^{\alpha_1} \wedge \cdots \wedge \mathbf{d}x^{\alpha_p}, \tag{96}$$

by way of the definition

$$\mathbf{da} = (p!)^{-1}\mathbf{d}a_{\alpha_1\cdots\alpha_p} \wedge \mathbf{d}x^{\alpha_1} \wedge \cdots \wedge \mathbf{d}x^{\alpha_p}. \tag{97}$$

In the language of tensor analysis this equation takes the form

$$(\mathbf{da})_{\alpha_1\cdots\alpha_{p+1}} = \Sigma(-1)^P \partial a_{\beta_2\cdots\beta_{p+1}}/\partial x^{\beta_1}, \tag{98}$$

where $P = 0$ or $+1$ according as the indices $\beta_1 \cdots \beta_{p+1}$ form an even or an odd permutation of the indices $\alpha_1 \cdots \alpha_{p+1}$. The operation $\mathbf{d}$, the *exterior derivative*, is an alternating differentiation which generalizes the familiar operations of gradient and curl. It is linear:

$$\mathbf{d}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{da}_1 + \mathbf{da}_2. \tag{99}$$

Applied to a product where the first factor is a p-form, it gives the result

$$\mathbf{d}(\mathbf{a} \wedge \mathbf{b}) = (\mathbf{da}) \wedge \mathbf{b} + (-1)^p\mathbf{a} \wedge \mathbf{db}. \tag{100}$$

Applied twice it gives zero

$$\mathbf{d}(\mathbf{da}) = 0. \tag{101}$$

This circumstance was employed in passing from expression (96) for an exterior differential form $\mathbf{a}$ to expression (97) for its exterior derivative, $\mathbf{da}$: the base vectors $\mathbf{d}x^\alpha = \text{grad } x^\alpha$ give no contribution when the operation of exterior differentiation is applied to them:

$$\mathbf{dd}x^\alpha = \text{curl grad } x^\alpha = 0. \tag{102}$$

No such simplicity results in the case where the same exterior form $\mathbf{a}$ is expressed in terms of a more general system of base vectors $\boldsymbol{\omega}^\alpha$; thus $\mathbf{d}\boldsymbol{\omega}^\alpha = \text{curl } \boldsymbol{\omega}^\alpha$ will not ordinarily vanish.

Maxwell's equations deal with two antisymmetrical tensors, $f_{\alpha\beta}$ and $*f_{\alpha\beta}$; or in invariant terminology, with two 2-forms,

$$\mathbf{f} = \tfrac{1}{2}f_{\alpha\beta}\mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$$

and

$$*\mathbf{f} = \tfrac{1}{2}{*f_{\alpha\beta}}\, dx^{\alpha} \wedge \mathbf{d}x^{\beta}. \tag{103}$$

Consider the exterior derivative of $\mathbf{f}$; that is, the 3-form

$$\mathbf{df} = \tfrac{1}{2}(\partial f_{\alpha\beta}/\partial x^{\gamma})\mathbf{d}x^{\gamma} \wedge \mathbf{d}x^{\alpha} \wedge \mathbf{d}x^{\beta}. \tag{104}$$

Here the coefficient of a typical basic 3-form, such as $\mathbf{d}x^{\lambda} \wedge \mathbf{d}x^{\mu} \wedge \mathbf{d}x^{\nu}$, with $\lambda < \mu < \nu$, has the value

$$\partial f_{\mu\nu}/\partial x^{\lambda} + \partial f_{\lambda\mu}/\partial x^{\nu} + \partial f_{\nu\lambda}/\partial x^{\mu}, \tag{105}$$

or in a locally Minkowskian frame,

$$\begin{aligned} &\operatorname{div} \mathbf{h} && (\lambda, \mu, \nu = 1, 2, 3), \\ &\operatorname{curl} \mathbf{e} + \partial \mathbf{h}/\partial T && (\lambda = 0; \mu, \nu = \text{space}). \end{aligned} \tag{106}$$

These expressions vanish, according to Maxwell, so one half of his equations take the form

$$\mathbf{df} = 0. \tag{107}$$

Similarly, in charge free space the other half of Maxwell's equations take the form

$$\mathbf{d}{*\mathbf{f}} = 0. \tag{108}$$

To satisfy Maxwell's first set of equations it is *sufficient* to express the field $\mathbf{f}$ as the curl of a four potential

$$\mathbf{f} = \mathbf{da};$$

or

$$\begin{aligned} \tfrac{1}{2}f_{\alpha\beta}\mathbf{d}x^{\alpha} \wedge \mathbf{d}x^{\beta} &= \mathbf{d}(a_{\gamma}\mathbf{d}x^{\gamma}) = \mathbf{d}a_{\gamma} \wedge \mathbf{d}x^{\gamma} \\ &= (\partial a_{\gamma}/\partial x^{\delta})\mathbf{d}x^{\delta} \wedge \mathbf{d}x^{\gamma} = \tfrac{1}{2}(\partial a_{\beta}/\partial x^{\alpha} - \partial a_{\alpha}/\partial x^{\beta})\mathbf{d}x^{\alpha} \wedge \mathbf{d}x^{\beta} \end{aligned}$$

or

$$f_{\alpha\beta} = \partial a_{\beta}/\partial x^{\alpha} - \partial a_{\alpha}/\partial x^{\beta}. \tag{109}$$

Then the exterior derivative of $\mathbf{f}$ vanishes automatically:

$$\mathbf{df} = \mathbf{dda} \equiv 0. \tag{110}$$

Integration: Motivation and Method.

From the differential properties of the 2-form $\mathbf{f}$ what integral properties can be deduced? How do these consequences differ (1) when we know merely that $\mathbf{df} = 0$ and (2) when we know in addition $\mathbf{f} = \mathbf{da}$? More generally, how does one define integrals of forms?

A p-form can be integrated over a p-dimensional surface to give a scalar or coordinate-independent quantity. To combine a local q-form ($q \neq p$) with a local p-dimensional surface element would give a result with local directional properties; but there is no invariant way to add together vectors or other directed quantities at different points in curved space to obtain a well-defined total. Therefore integrals of p-forms on p-dimensional surfaces alone need be considered.

The surface of integration will require for its coverage one or more p-dimensional coordinate patches. Break the surface up into curvilinear p-dimensional parallelepipeds at least equal in number to the number of required coordinate patches. Consider the *standard unit p-dimensional* cube, I^p, with coordinates λ^α:

$$0 \leq \lambda^\alpha \leq 1 \ (\alpha = 1, 2, \cdots p). \tag{111}$$

Map the points of I^p into $\mathfrak{M}$,

$$C: I^p \to \mathfrak{M}. \tag{112}$$

This map C may be thought of as a curvilinear parallelepiped in $\mathfrak{M}$, but we shall call it a *cube in* $\mathfrak{M}$ because we have no metric with which to distinguish "curved" from "straight". In terms of coordinates the cube takes the form

$$x^\alpha = C^\alpha(\lambda^1, \lambda^2, \cdots \lambda^p), \tag{113}$$

where the C^α are differentiable functions. A great number of different cubes cover the same set of points in $\mathfrak{M}$. For instance

$$x^\alpha = C^\alpha(1 - \lambda^1, \lambda^2, \cdots \lambda^p) \tag{114a}$$

and

$$x^\alpha = C(\lambda^2, \lambda^1, \cdots \lambda^p) \tag{114b}$$

are cubes in $\mathfrak{M}$ different from C, and with the opposite *orientation* to C, while

$$x^\alpha = C^\alpha(1 - \lambda^1, 1 - \lambda^2, \lambda^3, \cdots \lambda^p) \tag{115}$$

is a cube in $\mathfrak{M}$ different from C, but with the same orientation. The orientation of the cube governs the sign of the integral we are evaluating, but other differences in cubes that cover the same points of $\mathfrak{M}$ will not influence the value of the integral. If $f: I^p \to I^p$ is an orientation preserving homeomorphism of I^p onto itself, if in other words it has a positive Jacobian, we will not distinguish the cube $x = C(\lambda^1, \lambda^2, \cdots \lambda^n)$ from the cube $x = C(f(\lambda^1, \lambda^2, \cdots \lambda^n))$. We should therefore speak of an "*oriented cube in* $\mathfrak{M}$." Substitute the parametric description (113) of the surface into the expression (96) for a p-form, $\mathbf{a}$, to get the mapping

$$C: \mathbf{a} \to \mathbf{a}^C \tag{116}$$

of the form $\mathbf{a}$ on $\mathfrak{M}$ into the form $\mathbf{a}^C$ on I^p:

$$\begin{aligned}\mathbf{a}^C &= (p!)^{-1}a_{\alpha_1\cdots\alpha_p}\mathbf{d}x^{\alpha_1}(\lambda) \wedge \cdots \wedge \mathbf{d}x^{\alpha_p}(\lambda) \\ &= (p!)^{-1}a_{\alpha_1\cdots\alpha_p}(\partial x^{\alpha_1}/\partial\lambda^{\beta_1})\mathbf{d}\lambda^{\beta_1} \wedge \cdots \wedge (\partial x^{\alpha_p}/\partial\lambda^{\beta_p})\mathbf{d}\lambda^{\beta_p} \\ &= \sum_{\alpha_1<\cdots<\alpha_p} a_{\alpha_1\cdots\alpha_p} \frac{\partial(x^{\alpha_1}, \cdots, x^{\alpha_p})}{\partial(\lambda^1, \cdots, \lambda^p)} \mathbf{d}\lambda^1 \wedge \cdots \wedge \mathbf{d}\lambda^p.\end{aligned} \tag{117}$$

Replace the volume element $\mathbf{d}\lambda^1 \wedge \cdots \wedge \mathbf{d}\lambda^p$ by the volume element of the standard unit cube, $d\lambda^1\, d\lambda^2 \cdots d\lambda^p$ and *define* the integral of $\mathbf{a}$ over the oriented cube C as the quantity

$$\int_C \mathbf{a} = \int_{I^p} \mathbf{a}^C = \int \sum_{\alpha_1<\cdots<\alpha_p} a_{\alpha_1\cdots\alpha_p} \frac{\partial(x^{\alpha_1}, \cdots, x^{\alpha_p})}{\partial(\lambda^1, \cdots, \lambda^p)}\, d\lambda^1 \cdots d\lambda^p. \tag{118}$$

The appearance of the Jacobian in the integral gives assurance what the value of the integral is independent of (a) the choice of coordinates x^α in $\mathfrak{M}$ and (b) the choice of the map $I^p \to \mathfrak{M}$ used to represent the oriented cube in $\mathfrak{M}$.

Having defined an integral over one oriented curvilinear p-cube, we have now to piece together the entire p-dimensional surface in $\mathfrak{M}$ as the sum of a finite number of such oriented p-cubes,

$$c = C_1 + C_2 + \cdots + C_N \tag{119}$$

Instead of concentrating our attention on such surfaces, it turns out to be more convenient and more significant (see Homology Theory, below) to include as well a slightly more general object, a *p-chain*, c, which is simply a formal linear combination of a *finite* number of oriented p-cubes:

$$C = \sum_{i=1}^{N} s^i C_i\,. \tag{120}$$

It might appear that the only algebraic structure necessary is $\pm$ signs, so that if C is the oriented cube defined by the map (113), we may write $-C$ for the oriented cube defined by (114). However it may happen that the same cube C appears several times on the boundary of a surface and we therefore wish to write nC, where n is a positive or negative integer or zero. If the coefficients s^i in equation (120) were required to be integers, we would call c an integral chain. In connection with integration, it is more appropriate to let the coefficients s^i be any real numbers, so that we have in Eq. (120) a *real p-chain*. The integral of a p-form $\mathbf{a}$ over the p-dimensional surface, or p-chain, c, then is defined as the sum of elementary contributions of the form (118):

$$\int_c \mathbf{a} = \sum_{i=1}^{N} s^i \int_{C_i} \mathbf{a} = \sum_{i=1}^{N} s^i \int_{I^p} \mathbf{a}^{C_i}. \tag{121}$$

Surfaces or p-Chains, and Their Boundaries: Homology Theory.

Particularly interesting is the relation between a p-chain or surface c and its *boundary*, ∂c. This relation is very simple in the case of a cube, C. Use the symbol C^{j+} to stand for the jth upper face of C; that is, for the $(p-1)$ dimensional cube,

$$x^\alpha = C^\alpha(\lambda^1, \cdots \lambda^{j-1}, 1, \lambda^{j+1}, \cdots \lambda^p). \tag{122}$$

Likewise let C^{j-} stand for the lower face,

$$x^\alpha = C^\alpha(\lambda^1, \cdots \lambda^{j-1}, 0, \lambda^{j+1}, \cdots \lambda^p). \tag{123}$$

Then define the boundary of the oriented cube, C as the chain

$$\partial C = \sum_{j=1}^{p} (-1)^{j-1}(C^{j+} - C^{j-}). \tag{124}$$

The p-dimensional cube has a $(p-1)$ dimensional boundary. It might be thought that this $(p-1)$ dimensional surface has in turn a $(p-2)$ dimensional boundary. However, being closed, the surface evidently has no boundary at all. One comes to the same conclusion by applying Eq. (124) twice in succession. The result derived in this way for cubes applies to volumes and surfaces and more generally to any p-chain built up out of cubes.

Let a p-chain be expressed in the form

$$c = \sum_i s^i C_i . \tag{125}$$

Then we *define* the boundary of the p-chain in terms of the boundaries of the constituent cubes, C_i, in $\mathfrak{M}$, through the equation

$$\partial c = \sum_i s^i \partial C_i . \tag{126}$$

When the boundary of the p-chain vanishes,

$$\partial c = 0, \tag{127}$$

then c is called a *cycle* or *closed p-chain*. For example, a closed orientable surface which is broken up into cubes and represented by a chain c will be represented by a closed chain. Whether the chain is closed or not, whether ∂c vanishes or not, it follows from the formula $\partial\partial C_i = 0$ and from the Definition (125) that the boundary of any chain is a closed chain; i.e., has *no* boundary:

$$\partial\partial c = 0. \tag{128}$$

Table V gives a few illustrations.

The notion of boundary as now defined is adequate for deriving the theorem of Stokes in a general form, and from this as a special case the theorem of Gauss with its electromagnetic applications. A fuller analysis of boundaries and their

TABLE V

EXAMPLES OF p-CHAINS, BOTH CLOSED (=CYCLES) AND OPEN[a]

Dimensionality, p	Description of p-chain, c	Is it a closed chain (= cycle)?	Its boundary, ∂c
1	open line	no	2 end points
1	closed line in a Euclidean space	yes	0
1	closed line about minor circumference of a torus	yes	0
1	closed line about major circumference	yes	0
2	upper face of a disc	no	circle
2	surface of a sphere in flat 3-space	yes[b]	0
2	surface of a sphere inscribed about mouth of one wormhole	yes[c]	0
2	surface of a torus	yes	0
3	volume of a sphere in flat 3-space	no	sphere
3	3-dimensional closed space	yes	0

[a] There is no need to include in the table a last column to say that the $(p - 1)$ dimensional boundary, ∂c, of the p-cycle, c, has itself in every case no boundary: $\partial(\partial c) = 0$.

[b] Yes; and it also *bounds* a certain volume interior to the sphere.

[c] Yes; but there is no limited volume interior to the sphere which the surface can be said to bound.

relationships is given by that branch of algebraic topology which is known as homology theory[14]. Some of the ideas of this theory are needed to analyze charge-free electrodynamics in a multiply-connected space. Consider by way of example a sphere drawn in 3-space around one mouth of a wormhole—or, in the lower dimensional illustration of Fig. 3, consider a circle drawn on the indicated 2-space around one mouth of the handle. As a consequence of the existence of the wormhole, there is *no well-defined interior* to the sphere or the circle. An imaginary being supposedly locked up on the inside can enter the wormhole, pass freely through it, emerge from the other mouth, and come around outside and look at the exterior of his prison. The same topological information can be stated in another form by thinking of the spherical boundary as a bubble. Let the bubble be shrunk but not be allowed to break. It contracts into the interior of the wormhole. Let it be pushed through the tunnel and out the other side and expanded about the other wormhole mouth. Let the positive sides of the original sphere and of the new sphere be defined by the direction of a line of force that goes through the tunnel and crosses both spheres. Let the two *oriented* surfaces or

[14] Basic theory in extenso is given by Eilenberg and Steenrod (*31*); the forthcoming Vol. II is to consider practical methods of computation. Presentation of the theory in a less extensive form, with applications, is made by Seifert and Threlfall (*32*). For details and additional intuitive insight see Lefschetz (*33*) and Seifert and Threlfall (*34*). Rigorous treatment of *cubical* singular homology theory is given by Serre (*35*).

2-chains so defined be called c_1 and c_2, or more explicitly, $c_1^{(2)}$ and $c_2^{(2)}$ to indicate the dimensionality, $p = 2$, of the chains. Then c_1 and c_2 are said to be *homologous* to each other. More generally, when two p-chains, c_1^p and c_2^p together form the boundary of a finite $(p + 1)$ dimensional chain, in the sense

$$\partial c_a^{p+1} = c_1^p - c_2^p, \tag{129}$$

then we say that the *two p-chains are homologous*. In the example of Fig. 3, the two 1-chains or circles about the two mouths together completely bound the two-dimensional region of the tunnel, or 2-chain, c_a^2. In contrast, a circle, c_g^1, drawn in the 2-space of Fig. 3 in a place where it does *not* enclose the mouth of a wormhole *will* by itself completely bound the 2-space, or 2-chain, c_b^2, interior to it:

$$\partial c_b^2 = c_g^1. \tag{130}$$

If in this case the bounding line c_g^1 is considered subject to pressure from one side, and considered to deform accordingly, it will not have any wormhole to pass through nor any chance to reexpand into a new circle. Instead, it will collapse to nothingness. More generally, when a p-cycle c^p all by itself is a boundary,

$$c^p = \partial c^{p+1}, \tag{131}$$

it is said to be *homologous zero.*

Two surfaces—or more generally, two p-chains—that are homologous to each are said to belong to the same homology class. A homology class is defined to consist of all the closed p-chains or p-cycles that are homologous to each other. We denote the homology class containing a p-cycle c_i^p by $\{c_i^p\}$. In space that is topologically Euclidean all the p-cycles are shrinkable to nothingness; are therefore all homologous zero; and therefore all belong to the zero homology class, $\{c_0^p\} = \{0\}$. In a multiply connected space the p-cycles are not in general all homologous to each other. The number of *linearly independent* homology classes of order p,

$$\{c_1^p\}, \{c_2^p\}, \cdots, \{c_{R_p}^p\} \tag{132}$$

is called the pth *Betti number* R_p of the space (Table VI). The number

$$\chi = \sum_{p=0}^{n} (-1)^p R_p \tag{133}$$

is called the Euler characteristic of the space.

Consider two surfaces of the same homology class, one of which is identical with the other except for having a wart or incipient bubble on its surface. Combine the two surfaces with opposite signs. Then they cancel out except at the bubble. The resultant surface, like a small sphere, is homologous zero. These considerations generalize readily from surfaces to p-cycles of any order. In count-

TABLE VI

HOMOLOGY CLASSES $\{c_1^p\}$, $\{c_2^p\}$, $\cdots$ $\{c_{R_p}^p\}$ AND BETTI NUMBERS, R_p, FOR TWO SAMPLE SPACES. A p-CYCLE IS A p-DIMENSIONAL REGION, OR p-CHAIN, WITH THE SPECIAL PROPERTY THAT IT HAS NO BOUNDARY

Space: The two dimensional surface of a torus imbedded in Euclidean 3-space; more briefly, a 2-torus, T^2. Its Betti numbers: $R_0 = 1$; $R_1 = 2$; $R_2 = 1$. Euler characteristic $\chi = 0$

Homology classes	Description of typical p-cycle in this homology class
$\{c^0\}$	A point; all points are homologous
$\{c_1^1\}$	A closed curve that goes around minor circumference[a]
$\{c_2^1\}$	A closed curve around major circumference[a]
$\{c^2\}$	The 2-torus itself

Space: Start with a 3-dimensional closed space—visualizable as the 3-surface of a sphere imbedded in Euclidean 4-space—and modify this space topologically to the extent needed to install one wormhole with mouths M_1 and M_2. This is the pierced sphere $W_1 = S^2 \times S^1$. Its Betti numbers: $R_0 = 1$; $R_1 = 1$; $R_2 = 1$; $R_3 = 1$. Euler characteristic $\chi = 0$.

Homology classes	Description of typical p-cycle in this homology class
$\{c^0\}$	A point; all points are homologous
$\{c^1\}$	Closed line of force that goes once through the wormhole[b]
$\{c^2\}$	Closed surface about the mouth M_1[c]
$\{c^3\}$	The 3-space itself.

[a] A closed curve that can be shrunk to zero can be constructed as the combination of two such lines drawn in opposite senses about the torus.

[b] A closed curve that can be shrunk to zero is the linear combination with opposite senses—that is, the difference—of two such lines. A closed curve that goes around the mouth, M_1, of the wormhole can be moved away from this location and shrunk to zero. Three dimensions gives freedom to conduct this operation. It is impossible on the lower dimensional space of Fig. 3.

[c] A closed surface that can be shrunk to zero is the linear combination of two such surfaces with opposite sign.

ing distinct homology classes, we therefore always omit the class of cycles which are homologous zero. Normally there exist infinitely many homology classes. None is more fundamental than another. The situation is reminiscent of the displacements of a crystal lattice—say a simple cubic lattice—which carry it from one position to an equivalent position. One can have a displacement a in the x-direction; or a displacement a in the y-direction; or a knight's move of a in the x-direction simultaneously with $2a$ in the y-direction; etc. The general allowed move can be represented as a linear combination of three basic or linearly independent moves. However, there is no unique choice of these basic moves. Similarly, by number of distinct homology classes we mean the number of linearly independent homology classes in the following sense: A set of homology classes

$\{c_i^p\}$ is *linearly independent* if and only if every relation of the form

$$\sum_i s^i\{c_i^p\} = \left\{\sum_i s^i c_i^p\right\} = \{0\},$$

with real coefficients s^i, implies that all the s^i are zero; that is, if and only if no linear combination of the cycles c_i^p is a boundary. When the homology classes $\{c_i^p\}$ are linearly independent we shall say that the cycles c_i^p representing these homology classes are *independent cycles.*

The basic relationships between chains, cycles and boundaries which we have been discussing may be summarized in a purely algebraic form. Since we can add chains together, the set of chains in a manifold $\mathfrak{M}$ forms a group. We may also multiply a p-chain by a real number. Consequently the p-chains form an infinite dimensional vector space, $C_p(\mathfrak{M}, R)$, called the *group of real p-chains*. The chains c^p satisfying $\partial c^p = 0$ form a subspace $Z_p(\mathfrak{M}, R)$ called the group of p-cycles. Every chain c^p which is a boundary, $c^p = \partial c^{p+1}$, is a cycle. Consequently the set of all bounding p-cycles $B_p(\mathfrak{M}, R)$ is a vector space which is a subspace of $Z_p(\mathfrak{M}, R)$. The quotient space

$$H_p(\mathfrak{M}, R) = Z_p(\mathfrak{M}, R)/B_p(\mathfrak{M}, R) \tag{134}$$

is called the p-dimensional *homology group* of $\mathfrak{M}$ with real coefficients. The Betti number R_p is the dimension of the vector space $H_p(\mathfrak{M}, R)$ and is finite for any compact manifold.

It may happen that in a closed (compact) n-manifold, every n-chain has a boundary so that $R_n = 0$. In this case the manifold is called nonorientable; in the opposite case $R_n = 1$ and the manifold is orientable. This distinction points up a difference between the idea of boundary expressed by ∂ and the idea of a boundary as consisting of points on the "edge" of a set. In a manifold every point has a neighborhood homeomorphic to Euclidean space, so no point can be considered as being on an "edge" or "boundary" of the manifold. When, however, we break the manifold up into cubes,

$$\mathfrak{M} = c^n = \sum_{i=1}^{N} C_i^n, \tag{135}$$

we may not be able to choose the orientation of the cubes in such a way that the faces of all the cubes cancel out in pairs when we calculate ∂c: This is the situation in a nonorientable manifold. An example is given in Fig. 5 where we compute the Betti numbers of the projective plane P^2.

Stokes Theorem.

We are now in a position to state Stokes theorem concerning the integral of a p-form $\mathbf{a}$ over the p-dimensional boundary ∂c^{p+1} of a $(p + 1)$-chain c^{p+1}:

$$\int_{c^{p+1}} \mathbf{da} = \int_{\partial c^{p+1}} \mathbf{a}. \tag{136}$$

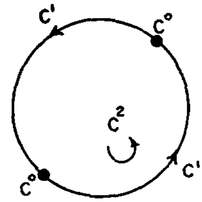

FIG. 5. The projective plane P^2 as an example of a nonorientable manifold. The points of the projective plane are pairs of antipodal points on the 2-sphere S^2. In the figure we show the upper hemisphere c^2 of S^2, and its equator. Two points on the equator of S^2, such as those marked c^0 in the figure, correspond to exactly *one* point c^0 in P^2. Similarly a single line c^1 in P^2 appears twice as shown in the figure on the equator of S^2. Regarding c^0, c^1, c^2, as chains, i.e., as being broken up into a large finite number of small "cubes", we see that $\partial c^2 = 2c^1$ when c^2 and c^1 are oriented as indicated by the arrows in the figure. Since c^2 has a boundary $2c^1$ in the ∂ sense of boundary, the Betti number $R_2(P^2)$ is zero. The curve c^1 has no boundary: $\partial c^1 = c^0 - c^0 = 0$. However c^1 is the boundary of the 2-chain $\frac{1}{2}c^2$, so every 1-cycle bounds and $R_1(P^2) = 0$. The point c^0 has no boundary of course, and it is not the boundary of any 1-chain, so there is one class of 0-cycles, $\{c^0\}$, different from zero, and $R_0(P^2) = 1$.

The integral of a form over a chain is defined in terms of the integral in Eq. (118) over the standard unit cube. Therefore the proof of Stokes theorem reduces to integration by parts on this cube, I^{p+1}. The simplest case of Stokes theorem is that of a 0-form or function, f, whose gradient is integrated over a curve c^1 with end points x_1 and x_0:

$$\int_{c^1} \mathrm{d}f = \int_{\partial c^1 = x_1 - x_0} f \equiv f(x_1) - f(x_0). \tag{137}$$

Here we make the convention that the integral of a function over a 0-cube, or point, is the value of the function at that point. For a 1-form, such as the magnetic field $\mathbf{h} = h_i \mathrm{d}x^i$ in 3-space, Stokes theorem reads

$$\int_{c^2} \mathbf{dh} \equiv \frac{1}{2} \int_{c^2} \left(\frac{\partial h_j}{\partial x^i} - \frac{\partial h_i}{\partial x^j} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j = \int_{\partial c^2} h_i \, \mathrm{d}x^i. \tag{138}$$

This is the formula which usually carries Stokes' name. Gauss' theorem is also included in formula (136) as we may see by considering a 2-form

$$*\mathbf{e} = \frac{1}{2!} *e_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j. \tag{139}$$

We have in this case, for a volume c^3 in 3-space,

$$\begin{aligned} \int_{c^3} \mathrm{d}{*\mathbf{e}} &\equiv \frac{1}{3!} \int_{c^3} \left(\frac{\partial *e_{jk}}{\partial x^i} + \frac{\partial *e_{ki}}{\partial x^i} + \frac{\partial *e_{ij}}{\partial x^k} \right) \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &= \int_{c^3} \left(\frac{\partial *e_{23}}{\partial x^1} + \frac{\partial *e_{31}}{\partial x^2} + \frac{\partial *e_{12}}{\partial x^3} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ &= \frac{1}{2} \int_{\partial c^3} *e_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j \equiv \int_{\partial c^3} *\mathbf{e}. \end{aligned} \tag{140}$$

This equation takes the usual form of Gauss' theorem in flat space when we write e_1 for $*e_{23}$, etc.

Applications of Stokes Theorem: Electric Charge, Magnetic Charge, and the Vector Potential.

Two particular cases of Stokes theorem are of principal interest:
(1) If c^{p+1} is a *closed chain*, i.e., a cycle, and $\mathbf{a} = \mathbf{a}^p$ is a p-form, then Eq. (136) reads:

$$\int_{c^{p+1}} \mathbf{da}^p = 0 \qquad \text{if} \qquad \partial c^{p+1} = 0. \tag{141}$$

(2) If $\mathbf{da}^p = 0$ we call $\mathbf{a}^p$ a *closed form*. In this case Stokes theorem gives:

$$\int_{\partial c^{p+1}} \mathbf{a}^p = 0 \qquad \text{if} \qquad \mathbf{da}^p = 0. \tag{142}$$

This second case may be put in the form of a *conservation law*: Let c_a and c_b be homologous p-cycles, so $c_b - c_a = \partial c^{p+1}$. Then Eq. (142) becomes

$$\int_{\partial c^{p+1}} \mathbf{a}^p = \int_{c_b} \mathbf{a}^p - \int_{c_a} \mathbf{a}^p = 0. \tag{143}$$

In other words, *the integral of a closed form over a closed surface depends only upon the homology class of the surface*. The value of this integral is called the *period* of $\mathbf{a}$ on $\{c_a\}$, the common homology class of c_a and c_b. According to Maxwell's equations, (104) and (108), both $\mathbf{f}$ and $*\mathbf{f}$ are closed forms. To the periods of these tensors we give names[15], to be justified in the next paragraph:

$$\int_{c^2} \mathbf{f} = 4\pi p^* = 4\pi \text{ (magnetic charge of } \{c^2\}) \tag{144a}$$

$$\int_{c^2} *\mathbf{f} = 4\pi q^* = 4\pi \text{ (electric charge of } \{c^2\}) \tag{144b}$$

When the electromagnetic field is derived from a vector potential $\mathbf{a} = a_\mu \mathbf{d}x^\mu$, then there is zero net flux through every surface c^2 that is closed ($\partial c^2 = 0$):

$$\int_{c^2} \mathbf{f} = \int_{c^2} \mathbf{da} = \int_{\partial c^2} \mathbf{a} = 0. \tag{145}$$

[15] The superscript * indicates that we are measuring both kinds of charge in the purely geometrical units of length. As we pass from the geometrical measure of field strength, $\mathbf{f}$ (cm^{-1}) to the conventional measure of field, $\mathbf{F}$ (gauss or g$^{1/2}$/cm$^{1/2}$ sec) by multiplication with the factor $c^2/G^{1/2}$(g$^{1/2}$ cm$^{1/2}$/sec), so the same factor translates from the geometrical measure of charge, q^* (cm), to the conventional measure of charge in electrostatic units (g$^{1/2}$ cm$^{3/2}$/sec):

$$q(\text{esu}) = (c^2/G^{1/2})q^* \text{ (cm)}.$$

In other words, *the existence of a vector potential implies that there is no magnetic charge.*

In reasonable examples the homology class of the surface c^2 includes (a) similar appearing surfaces of larger or smaller size at the same time, that is, in the same space like surface and (b) surfaces like these at earlier and later times. Both **f** and ***f** are closed. Then Eq. (143) says that the flux through all of these surfaces is the same—independent of the size of the surface (Gauss' law) and independent of time (law of conservation of charge). Therefore charge as defined in Eqs. (144) has the properties that one is accustomed to demand. The four dimensional character of the analysis has combined into a single law two laws normally regarded as separate. In brief, we have shown from Maxwell's equations that charge—regarded as lines of force trapped in the topology—stays constant with time regardless of changes in the details of the electromagnetic field, and regardless of changes in space. No mention of the metric even entered the proof of the conservation theorem.

To see in a little more detail the reasonableness of the identification (144), consider flat space with the usual time and space coordinates. Then the forms **f** and ***f** are related to electric and magnetic fields in the following way:

$$\mathbf{f} = -\mathbf{d}t \wedge (e_i\, dx^i) + (*h)_{ij}\mathbf{d}x^i \wedge \mathbf{d}x^j, \tag{146a}$$

$$*\mathbf{f} = \mathbf{d}t \wedge (h_i\, dx^i) + (*e)_{ij}\mathbf{d}x^i \wedge \mathbf{d}x^j. \tag{146b}$$

Here e_i and h_i are the components of the usual electric and magnetic fields, and $(*h)_{xy} = h_z$, $(*e)_{yz} = e_x$, etc. When c^2 is a closed surface, say a sphere, in the $T=$const. hyperplane, and when **f** is the electromagnetic field produced by a particle inside this sphere, then the magnetic pole strength p^* and the electric charge q^* of this particle are given by

$$4\pi p^* = \int_{c^2} (*\mathbf{h})_{ij}\, \mathbf{d}x^i \wedge \mathbf{d}x^j = \int_{c^2} \mathbf{f}, \tag{147a}$$

$$4\pi q^* = \int_{c^2} (*\mathbf{e})_{ij}\, \mathbf{d}x^i \wedge \mathbf{d}x^j = \int_{c^2} *\mathbf{f}, \tag{147b}$$

The first equalities in these equations are familiar expressions of Gauss' law in flat space; the second equalities hold because the dT terms in Eqs. (146) contribute nothing to an integral over a surface such as c^2 on which T is constant. In other words, the vector $\mathbf{d}T$ = grad T is orthogonal to any surface on which T is constant, and therefore the projection of $\mathbf{d}T$ on that surface is zero. In this way we see that Eqs. (144) are direct generalizations of familiar equations to a curved space-time.

Examples of Spaces Permitting Charge.

To fill the gap between the very general language of the conservation laws in the last section and the simple intuitive picture of charge in Fig. 3 as lines of force trapped in the topology, it is appropriate to present here two examples of spaces permitting charge; that is, spaces endowed with one or more wormholes: $R_2 \geqslant 1$. These examples are the manifolds $R \times W_k$ and $R \times T^3$ which were defined earlier in part B of this section (Examples of Manifolds, Product of Manifolds). To define the differentiable structure of W_k (see Differentiable Manifolds) we introduced on W_1 a function $f(x)$; in W_k we use k such functions $f_i(x)$, $i = 1, 2, \cdots, k$. Let $c_i^2(r)$ be the sphere in W_k defined by $f_i(x) = r$. In the two dimensional analog of W_k shown in Fig. 4, the sphere is doubly pierced ($k = 2$). The typical surface $c_1(r)$ is a circle that runs around the mouth of one of the tubes that connects opposite faces of the sphere. The typical surface $c_2(r)$ runs around the mouth of the other tube. In the product manifold $R \times W_k$, of a real line, $-\infty < T < \infty$, with W_k, there is on each hypersurface $T =$ const. a copy of $c_i^2(r)$ which we will call $t_i^2(r, T)$. Moreover, the pth Betti number $R_p(R \times W_k)$ of the product manifold is the same as the pth Betti number $R_p(W_k)$ of W_k itself. Now consider on $R \times W_k$ a 2-form $\mathbf{f}$ or $*\mathbf{f}$. Then the flux

$$4\pi p_i^*(r, T) = \int_{c_i^2(r,T)} \mathbf{f} \tag{148a}$$

or

$$4\pi q_i^*(r, T) = \int_{c_i^2(r,T)} *\mathbf{f} \tag{148b}$$

apparently depends on r and T. To see that $p_i^*(r, T)$ or $q_i^*(r, T)$ in fact depends only on i we need (a) to require that $\mathbf{df} = 0$ or $\mathbf{d*f} = 0$ and (b) to see that all the surfaces $c_i^2(r, T)$ for fixed i belong to the same homology class. The surface $c_i^2(0, 0)$ is a 2-sphere. As we let r increase this sphere sweeps out a 3-surface $c_i^3(R)$ $0 \leqslant r \leqslant R$ which is topologically identical to the volume $1 \leqslant x^2 + y^2 + z^2 \leqslant 2$ in Euclidean 3-space. From this mental image of the 3-space $c_i^3(R)$ in the familiar Euclidean 3-space R^3, we see the boundary of $c_i^3(R)$:

$$\partial c_i^3(R) = c_i^2(R, 0) - c_i^2(0, 0). \tag{149}$$

We conclude that $c_i^2(r, 0)$ is homologous to $c_i^2(0, 0)$. No metric notions have entered yet. Therefore we can let $c_i^2(0, T)$ sweep out a 3-surface $c_i^3(T)$ just as we did above for r, and find

$$\partial c_i^3(T) = c_i^2(0, T) - c_i^2(0, 0) \tag{150}$$

so that $c_i^2(0, T)$ and $c_i^2(0, 0)$ are homologous. A similar construction shows that $c_i^2(r, T)$ is homologous to $c_i^2(r, 0)$. More generally, we conclude that all the surfaces $c_i^2(r, T)$ for the same i are homologous. Moreover, the equations $\mathbf{df} = 0$ and $\mathbf{d*f} = 0$ say that no flux gets lost between homologous surfaces. Therefore the charges $p_i^*(r, T)$ and $q_i^*(r, T)$ are independent of r and T.

The preceding discussion made no use of any metric on $R \times W_k$, and is therefore valid no matter what metric we introduce. For example, at one value of T the space-like hypersurface T = const may have the appearance of the doubly-pierced sphere of Fig. 4(c). At a later T the two ends of one of the wormholes may move together preparatory to annihilation as in Fig. 3. But as long as the topology does not actually change, the lines of force remain trapped and the flux $4\pi q_i^*$ through the closed surface $c_i^2(r, T)$ is unaltered by disturbances of the metric.

In this analysis of the wormhole space-time $R \times W_k$, we *assumed* the existence of a closed form $*\mathbf{f}$ or $\mathbf{f}$ for which the charges and pole strengths in Eqs. (148) do not all vanish. In the following section on de Rham's theorem we shall see that many such forms do exist. In a simpler example, $R \times T^3$, we can display them explicitly.

On the torus, T^3, functions $x^i(i = 1, 2, 3)$ cannot be defined continuously everywhere except to within an additive multiple of 2π. However, the *gradients* $\mathbf{d}x^i$ are differentiable vector fields. In the simpler case of a 2-torus one of these gradients may be a field of unit vectors (red!) that circle about the minor circumference of the torus. The other field of unit vectors (green!) may be taken to be parallel to the major circumference. Similarly vector fields are definable in the space $R \times T^3$. From them one can construct a form,

$$*\mathbf{f} = \mathbf{d}x^2 \wedge \mathbf{d}x^3,$$

which is obviously closed ($\mathbf{d}*f = 0$). It represents a uniform electric field in the x^1 direction [see Eq. (146b)]. Integrating this 2-form over the surface $c_1^2(x^1, T)$ defined by T = const, x^1 = const, we get the value

$$4\pi q_1^* = \int_{c_1^2(x^1,T)} \mathbf{d}x^2 \wedge \mathbf{d}x^3 = (2\pi)^2. \tag{151}$$

Thus we verify directly that in this case q_1^* is independent of x^1 and T. We might hesitate to call q_1^* a charge since we cannot easily imagine any metric that makes all the lines of force appear to originate in a *small* region of space. However q_1^* clearly measures the total flux of the electric field around the 3-torus through the closed surface c_1^2. Similarly, there are two other independent directions on the 3-torus, in each of which the flux can be given what value one pleases. There this space is endowed with three arbitrary constants of charge.

De Rham's Theorem Proves the Existence of 2-Forms Endowed with Charge.

To have given an example of a charge-like solution is far from having proven that such solutions are always possible in a four space endowed with wormholes; that is, in a space with Betti number $R_2 \geqslant 1$. The existence of such solutions is established by the *first theorem of de Rham* (*36*): If c_i^p are R_p independent p-cycles of a manifold $\mathfrak{M}$, and Q_i any R_p real numbers, there exists a p-form $\mathbf{a}^p$ which is closed ($\mathbf{da}^p = 0$) and differentiable throughout $\mathfrak{M}$ and which satisfies

$$\int_{c_i^p} \mathbf{a}^p = Q_i \qquad (i = 1, 2, \cdots, R_p). \tag{152}$$

To translate into physical terms, we note that $\mathbf{a} = \mathbf{f}^2$ or $*\mathbf{f}^2$ is the form that represents the electromagnetic field or its dual. The Betti number R_2 is the number of wormholes, and $Q_i = 4\pi p_i^*$ or $4\pi q_i^*$ is the magnetic or electric flux through the ith wormhole. More precisely, Q_i is the flux through surfaces of the ith independent homology class, or in mathematical terms, the so-called *period* on this class. We have to recall again the arbitrariness in the choice of the R_2 independent or basic homology classes of order 2, linear combinations of which give all homology classes of order 2. There are situations where it is reasonably clear that two wormholes are well separated from each other, so that it might be particularly appropriate for Q_1 to designate the flux through one, and Q_2 the flux through the other. However, it is also possible to give instances where one wormhole is imbedded inside the other and where $Q'_1 = Q_1 + Q_2$ and $Q'_2 = Q_1 - Q_2$, or some other combinations, are just as natural as Q_1 and Q_2 as measures of charge. Thus de Rham's theorem says that there exists a solution $\mathbf{f}$ of the Maxwell equation, $\mathbf{df} = 0$ *or* a solution, $*\mathbf{f}$, of the Maxwell equations, $\mathbf{d}*\mathbf{f} = 0$, which has arbitrarily assignable real magnetic or electric charges.

Now demand that the magnetic charge of every wormhole vanish; in other words, demand that the period of $\mathbf{f}$ on every homology class $\{c^2\}$ shall be zero:

$$\int \mathbf{f} = 0 \quad \text{for all } c^2. \tag{153}$$

Then the *second theorem of de Rham* guarantees the existence of a vector potential, $\mathbf{a}$:

When $\mathbf{f}^p$ is a p-form on $\mathfrak{M}$, which is closed ($\mathbf{df}^p = 0$) and all of whose periods are zero, then $\mathbf{f}^p$ is derived from a differentiable $(p - 1)$-form:

$$\mathbf{f}^p = \mathbf{da}^{p-1}. \tag{154}$$

In a case such as this, where there exists a potential, so that the one form, $\mathbf{f}$, can be written as the curl of another form, $\mathbf{a}$, then the form $\mathbf{f}$ is said to be *exact.* To be exact is a more demanding requirement on a form than merely to be closed (Table VII).

Concepts Based on the Metric: Duality and Divergence: Completion of Maxwell's Equations.

We have just derived Gauss' theorem and the law of conservation of charge from Maxwell's eight equations, $\mathbf{df} = 0$ and $\mathbf{d}*\mathbf{f} = 0$, without making any appeal whatsoever to the notions of metric or length. The burden of the proof lay on

TABLE VII

SIMILARITIES BETWEEN THE CONCEPTS OF CYCLES OR CLOSED CHAINS (SUCH AS CLOSED SURFACES) AND CLOSED FORMS[a]

	Surfaces
c	General p-chain; a surface for $p = 2$.
$\partial c = 0$	Means that p-chain is closed; c is then called a p-cycle; signifies for $p = 2$ a closed surface.
$c^p = \partial c^{p+1}$	Means that c^p is the boundary of c^{p+1}; it then follows that $\partial c^p = 0$, in other words, $\partial^2 = 0$. However, from $\partial c^p = 0$ it does not necessarily follow that there exists any c^{p+1} of which c^p is the boundary.
Stokes' theorem:	Given *one* chain c^p that is a boundary: $c^p = \partial c^{p+1}$ (whence $\partial c^p = 0$ and c^p = a cycle); then on this cycle the period $\int_{c^p} \mathbf{f}$ vanishes for *every* p-form $\mathbf{f}$ that is closed ($\mathbf{df} = 0$).
de Rham's theorem:	Given *one* chain c^p which is closed (c^p = a cycle; $\partial c^p = 0$); and given that the period $\int_{c^p} \mathbf{f}$ vanishes for *every* p-form $\mathbf{f}$ that is closed; then c^p is itself a boundary: $c^p = \partial c^{p+1}$.
	Forms
$\mathbf{f}$	General p-form; a vector for $p = 1$; alternating tensor for $p = 2$.
$\mathbf{df} = 0$	Then $\mathbf{f}$ is called a closed form. The Maxwell field tensor is described by a closed form.
$\mathbf{f} = \mathbf{da}$	Then $\mathbf{f}$ is said to be an *exact* form, or $\mathbf{f}$ is said to *bound* $\mathbf{a}$ (the four potential, for example). It then follows from the relation $\mathbf{d}^2 = 0$ that $\mathbf{f}$ is automatically closed. However, from the closure of a form $\mathbf{f}$ it does not necessarily follow that there exists any $(p - 1)$ form $\mathbf{a}$ of which $\mathbf{f}$ is the curl or boundary.
Stokes' theorem:	Given *one* p-form $\mathbf{f}$ that is exact: $\mathbf{f} = \mathbf{da}^{p-1}$ (whence $\mathbf{df} = 0$); then the period $\int_{c^p} \mathbf{f}$ vanishes on *every* p-cycle ($\partial c^p = 0$).
de Rham's theorem:	Given one p-form $\mathbf{f}$ that is closed ($\mathbf{df} = 0$) and given that the period $\int_{c^p} \mathbf{f}$ vanishes on every p-cycle ($\partial c^p = 0$), then $\mathbf{f}$ is exact: $\mathbf{f} = \mathbf{da}^{p-1}$.

[a] The concept of chain as defined earlier and used here means a linear combination of a finite number of cubes—a form of definition important for open or non-compact manifolds.

topology. We avoided any appeal to Riemannian geometry by treating the Maxwell field

$$\mathbf{f} = \tfrac{1}{2} f_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \tag{155a}$$

and its dual

$$*\mathbf{f} = \tfrac{1}{2} {*f_{\mu\nu}} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \tag{155b}$$

as two completely independent objects. We extract extra content from Maxwell's equations, and impose additional restrictions on the field **f**, when we note the one-to-one relation between one field and the other that is established by the metric. We shall therefore now define duality and divergence, and formulate the remaining content of Maxwell's equations.

The connection (11) between the electromagnetic tensor **f** and its dual ***f** is generalizable to a connection between any p-form

$$\mathbf{a} = (p!)^{-1} a_{\alpha_1 \cdots \alpha_p} \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_p} \tag{156}$$

and its dual, the $(n - p)$-form

$$*\mathbf{a} = \sum_{\substack{\alpha_1 < \cdots < \alpha_p \\ \beta_1 < \cdots < \beta_{n-p}}} (|g|)^{1/2} a^{\alpha_1 \cdots \alpha_p} [\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_{n-p}] \, \mathrm{d}x^{\beta_1} \wedge \cdots \wedge \mathrm{d}x^{\beta_{n-p}} \tag{157}$$

Here n is the dimensionality of the manifold and p and $(n - p)$ the orders of the two forms, g is the determinant of the metric tensor $g_{\alpha\beta}$, and the quantities with upper indices are the contravariant components of the original tensor:

$$a^{\alpha_1 \cdots \alpha_p} = g^{\alpha_1 \mu_1} \cdots g^{\alpha_p \mu_p} a_{\mu_1 \cdots \mu_p}. \tag{158}$$

Applied to a special 0-form, the constant function 1, the duality gives the *volume element*

$$*1 = |g|^{1/2} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n. \tag{159}$$

The following general formulas are most readily proven by applying the definition of the duality operation in an orthonormal reference system:

$$* *\mathbf{a}^p = (-1)^{np+p+s} \mathbf{a}^p \tag{160}$$

(double duality)

$$\mathbf{a}^p \wedge *\mathbf{b}^p = \mathbf{b} \wedge *\mathbf{a} = (p!)^{-1} a^{\alpha_1 \cdots \alpha_p} b_{\alpha_1 \cdots \alpha_p} *1 \tag{161}$$

(relation between exterior product and scalar product)

$$(*\mathbf{a}) \wedge *(*\mathbf{b}) = (-1)^s \mathbf{a} \wedge *\mathbf{b} \tag{162}$$

(scalar product of dual forms expressed in terms of scalar product of forms themselves). In these equations s stands for the signature of the metric: $s = 0$ on space-like hypersurfaces, where the metric is of the form $+ + +$; $s = 1$ in four space with the metric $- + + +$; and more generally, s is the maximum number of orthogonal vectors with negative norm.

The p-form and its dual are distinguished in the following way, that the integral

$$\int_{c_p} \mathbf{a} \tag{163}$$

may be said to be the integral of the tangential component of **a** over a surface or p-chain c^p, whereas the quantity

$$\int_{c^{n-p}} *\mathbf{a} \tag{164}$$

represents the integral of the normal component of **a** over an $(n - p)$-chain c^{n-p}.

Take the electric field **e**, a 1-form in 3-space; form its dual, a 2-form; take its exterior derivative and secure a 3-form; and finally take its dual and obtain a 0-form or scalar. This scalar represents the divergence of the electric field, as seen most quickly by writing out the indicated sequence of operations in flat space:

$$\begin{aligned} \mathbf{e} &= e_k \mathbf{d}x^k; \\ *\mathbf{e} &= e_1 \mathbf{d}x^2 \wedge \mathbf{d}x^3 + \text{two similar terms;} \\ \mathbf{d}*\mathbf{e} &= (\partial e_1/\partial x^1)\mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 + \text{two similar terms} \\ &= (\text{div } \mathbf{e})\mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3; \\ *\mathbf{d}*\mathbf{e} &= \text{div } \mathbf{e}. \end{aligned} \tag{165}$$

Generalizing to a curved space and a p-form of any order, the theory of forms defines a general *divergence* or *co-differential* operation $\boldsymbol{\delta}$ which takes a p-form **a** into a $(p - 1)$ form,

$$\mathbf{b}^{p-1} = \boldsymbol{\delta}\mathbf{a}^p \equiv (-1)^{np+n+s+1} *\mathbf{d}*\mathbf{a}^p, \tag{166}$$

where the $\pm$ sign is introduced to simplify the statement of Green's theorem. For a $(p - 1)$ form **a** and a p-form **b** this theorem connects the n-fold integral of two scalars over all or part of the manifold $\mathfrak{M}$ with the $(n - 1)$ fold integral of a normal vector, $\mathbf{a}*\mathbf{b} \equiv \mathbf{a} \wedge *\mathbf{b}$, over the boundary of this region:

$$\int_{\partial \mathfrak{M}} \mathbf{a}*\mathbf{b} = \int_{\mathfrak{M}} \mathbf{d}(\mathbf{a}*\mathbf{b}) = \int_{\mathfrak{M}} \mathbf{b}*\mathbf{d}\mathbf{a} - \int_{\mathfrak{M}} \boldsymbol{\delta}\mathbf{b}*\mathbf{a}. \tag{167}$$

The negative d'Alembertian operator

$$\Delta = \boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta} (= -\text{"}\nabla^2\text{"} \text{ or } -\text{"}\Box\text{"} \text{ for a scalar}) \tag{168}$$

carries a p-form into a p-form. In particular it carries the scalar, φ, into the scalar

$$\Delta\varphi = \boldsymbol{\delta}\mathbf{d}\varphi = -\text{div grad } \varphi = -\varphi^{;k}{}_{;k}\,. \tag{169}$$

The properties of the differential operators **d**, $\boldsymbol{\delta}$, and Δ are summarized in Table VIII.

TABLE VIII

PROPERTIES OF THE DUALITY OPERATION*, THE EXTERIOR DERIVATIVE **d**, AND THE DIVERGENCE OPERATION **δ** IN SPACE TIME AND ON A SPACE-LIKE HYPERSURFACE. THE NOTATION **u*v** STANDS FOR THE EXTERIOR PRODUCT $\mathbf{u} \wedge * \mathbf{v} = \mathbf{v} \wedge * \mathbf{u}$

Property	4-manifold	3-manifold
Signature of metric	$- + + +$	$+ + +$
Dual of dual of p-form	$** = (-)^{p+1}$	$** = 1$
Divergence of p-form	$\boldsymbol{\delta} = *\mathbf{d}*$	$\boldsymbol{\delta} = *\mathbf{d}* (-)^p$
Second order differential operator	"$-\Box$" $= \Delta \equiv \boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta}$	"$-\nabla^2$" $= \Delta \equiv \boldsymbol{\delta}\mathbf{d} + \mathbf{d}\boldsymbol{\delta}$

General formulas, n-manifold $\mathfrak{M}^n$, any metric

$$*\mathbf{d} = \boldsymbol{\delta} * (-1)^{p+1}; \quad *\boldsymbol{\delta} = \mathbf{d} * (-1)^p$$

$$*\Delta = \Delta *; \quad \mathbf{d}\Delta = \Delta\mathbf{d}; \quad \boldsymbol{\delta}\Delta = \Delta\boldsymbol{\delta}$$

Stokes' theorem

$$\int_{\partial c} \mathbf{a} = \int_c \mathbf{da}$$

Integral formulas for case $\partial\mathfrak{M}^n = 0$

$\boldsymbol{\delta}$ is adjoint of $\mathbf{d}$	$\int_{\mathfrak{M}} \mathbf{b}*\mathbf{da} = \int_{\mathfrak{M}} \boldsymbol{\delta}\mathbf{b}*\mathbf{a}$	$(\mathbf{a} = \mathbf{a}^{p-1}; \quad \mathbf{b} = \mathbf{b}^p)$
Δ is self adjoint	$\int_{\mathfrak{M}} \mathbf{b}*\Delta\mathbf{a} = \int_{\mathfrak{M}} \Delta\mathbf{b}*\mathbf{a}$	$(\mathbf{a} = \mathbf{a}^p; \quad \mathbf{b} = \mathbf{b}^p)$
For $n = 4m - 1$, $*\mathbf{d}$ is self adjoint on $(2m - 1)$ forms	$\int_{\mathfrak{M}^{4m-1}} \mathbf{b}*(*\mathbf{da}) = \int_{\mathfrak{M}^{4m-1}} (*\mathbf{db})*\mathbf{a}$	$(\mathbf{a} = \mathbf{a}^{2m-1}; \quad \mathbf{b} = \mathbf{b}^{2m-1})$

In terms of the divergence operation $\boldsymbol{\delta}$, the entire content of Maxwell's equations can now be summarized in the form

$$\mathbf{df} = 0 = \boldsymbol{\delta}\mathbf{f}. \tag{170}$$

D. CHARACTERIZATION OF SOLUTIONS OF MAXWELL'S EQUATIONS BY INITIAL VALUE DATA: CHARGE AND WAVE NUMBER

Alternative Formulations of the Initial Value Problem: Summary of Analysis. On a space like surface endowed with a metric it is well known to be enough to specify arbitrary continuous electric and magnetic fields—satisfying div $\mathbf{e} = 0$, div $\mathbf{h} = 0$—in order to be able to continue the fields to the future and the past by way of Maxwell's equations[16]. This type of initial value data is the natural analogue of the x and $\dot{x}$, or x and p, needed to predict the future of a classical particle. A different choice of data is made in the Lagrange formulation. Applied to a particle, it demands that x be known at the initial and final times, with no

[16] This is a familiar application of the standard Cauchy-Kowaleski theory (*37*).

information needed about either $\dot{x}$ or p. Applied to the Maxwell field, the Lagrange formulation asks for example for the magnetic field—or the electric field, but not both—on the two space like hypersurfaces $T = T_1$ and $T = T_2$. This "two surface" type of boundary condition is appropriate when one is interested in the classical action of the field integrated over the space-time interval between the two surfaces, or when one wants to know Feynman's quantum propagator: the probability amplitude to pass from configuration C_1 on σ_1 to configuration C_2 on σ_2. When the two cotimes have a finite separation one knows how to analyze this classical boundary value problem in flat space and in some types of curved space (see below) but not in the general type of curved space. However, when the two surfaces in question are so close together that the boundary value data can be considered to define the magnetic field and its first cotime derivative, then the two surface problem reduces to the one surface initial value problem. To it we shall henceforth limit our attention in either form A or form B:

A: Give $\mathbf{h}$ and $\partial\mathbf{h}/\partial T$ on σ; and give the R_2 electric charges or wormhole fluxes q_i^* once and for all. (171A)

B: Give $\mathbf{h}$ and $\mathbf{e}$ on σ. (171B)

We shall first analyze the relation between a form in 4-space and its projection or trace on a 3-space; then the connection between the metric on the hypersurface and the metric of 4-space; and then Maxwell's equations expressed in terms of forms on the hypersurface. We show that specifications (171A) and (171B) are equally appropriate to determine uniquely the future evolution of the system. Finally we treat briefly the specification of the initial value data, not directly in a space like form as in (171A, B), but by way of a natural generalization of Fourier analysis to curved space.

The Concept of Projection or Trace.

A hypersurface σ in the manifold $\mathfrak{M}$ can be defined by an equation of the form $\varphi(x) = 0$, where φ is a function with $\mathbf{d}\varphi \neq 0$ at σ. We may, if we wish, choose φ as one of our coordinates and call it T or x^0. We now consider an operation which enables us to obtain from any covariant tensor on the 4-manifold $\mathfrak{M}$, a corresponding tensor on the 3-manifold σ. In the special case where $\varphi(x)$ is any scalar function on $\mathfrak{M}$, its *projection* or *trace* on σ is the function $\varphi^\sigma(x)$ defined for x on σ by the equation:

$$\varphi^\sigma(x) = \varphi(x),\ x \text{ on } \sigma. \qquad (172)$$

Representing φ in terms of coordinates as $\varphi(T, x^1, x^2, x^3)$, where σ is given by

$$x^0(x) = T(x) = T_0\,,\ x \text{ on } \sigma, \qquad (173)$$

we can rewrite Eq. (172) in the particularly simple form

$$\varphi^\sigma(x^1, x^2, x^3) = \varphi(T_0\,, x^1, x^2, x^3). \qquad (174)$$

Similarly, for a p-form

$$\mathbf{a} = (p!)^{-1} a_{\mu_1 \cdots \mu_p}(x^0, x^1, x^2, x^3) \mathbf{d}x^{\mu_1} \wedge \cdots \wedge \mathbf{d}x^{\mu_p}, \tag{175}$$

we obtain the *trace* $\mathbf{a}^\sigma$ of $\mathbf{a}$ on σ by substituting in Eq. (175) the values $x^0 = T_0$ $\mathbf{d}x^0 = 0$. Likewise, the quadratic form

$$\mathbf{ds}^2 = g_{\mu\nu} \mathbf{d}x^\mu \mathbf{d}x^\nu, \; (\mu, \nu = 0, 1, 2, 3) \tag{176}$$

on σ reduces to

$$\mathbf{dl}^2 \equiv g_{ij}^\sigma \mathbf{d}x^i \mathbf{d}x^j = (\mathbf{ds}^2) = g_{ij} \mathbf{d}x^i \mathbf{d}x^j \qquad (i, j = 1, 2, 3), \tag{177}$$

which displays the components g_{ij}^σ of the metric we use on the 3-manifold σ.

The exterior derivative $\mathbf{d}^\sigma$ on σ involves no differentiations with respect to x^0, but the identity

$$\mathbf{d}^\sigma \mathbf{a}^\sigma = (\mathbf{da})^\sigma \tag{178}$$

assures us that it will cause no confusion to write $\mathbf{d}$ for $\mathbf{d}^\sigma$.

Now we may define, *relative to the space like hypersurface* σ, the dual *magnetic* field, $*\mathbf{h}$, and the dual *electric* field, $*\mathbf{e}$, by the equations

$$*\mathbf{h} = \mathbf{f}^\sigma, \tag{179a}$$

$$*\mathbf{e} = (*\mathbf{f})^\sigma. \tag{179b}$$

We have written $*\mathbf{h}$ and $*\mathbf{e}$ here so that $\mathbf{e}$ and $\mathbf{h}$ will be vectors or 1-forms of the familiar kind on σ. The duality to be used here is based on the metric g_{ij}^σ of σ, and so could be written $*_\sigma$. This allows us to define

$$\mathbf{h} = *_\sigma \mathbf{f}^\sigma \tag{180a}$$

$$\mathbf{e} = *_\sigma (*\mathbf{f})^\sigma, \tag{180b}$$

consistent with Eqs. (179) because of the identity $*_\sigma *_\sigma = 1$. We shall never apply $*_\sigma$ to a form on $\mathfrak{M}$, so the meaning will be clear from the context if we drop the subscription on $*_\sigma$ as in Eqs. (179). From Maxwell's equations on $\mathfrak{M}$, $\mathbf{df} = 0 = \mathbf{d}*\mathbf{f}$, and the identity (178) we obtain the trace of Maxwell's equations on σ, $\mathbf{df}^\sigma = 0 = \mathbf{d}(*\mathbf{f})^\sigma$. In terms of the definitions of (180) and of Table VIII, *the trace of Maxwell's equations on a space like surface* takes the form

$$\boldsymbol{\delta}_\sigma \mathbf{h} = 0 = \boldsymbol{\delta}_\sigma \mathbf{e}. \tag{181}$$

We will normally omit the subscript σ on $\boldsymbol{\delta}$ because it will be clear from the context that we are dealing with a space like surface. We use subscripts on $\boldsymbol{\delta}_\sigma$ and $*_\sigma$ because they are not simply the traces on σ of the corresponding operations on $\mathfrak{M}$; in other words, no identity holds for $\boldsymbol{\delta}$ or $*$ like (178) for $\mathbf{d}$; for instance, $*_\sigma \mathbf{f}^\sigma$ is a 1-form, whereas $(*\mathbf{f})^\sigma$ is a 2-form.

Maxwell's Equation in 3-Dimensional Form.

To reconstruct the electromagnetic field **f** from the 3-dimensional fields **e** and **h** it simplifies the analysis to make a more careful choice of a cotime coordinate T than the one that was demanded in the previous section. Let T be the cotime measured along a geodesic which starts out normal to a space like hypersurface σ_0. Then σ_0 is given by $T(x) = 0$ and each σ_T, defined by $T(x) = \text{const}$, is again a space like hypersurface. The quadratic form $\mathbf{ds}^2$ can be written

$$\mathbf{ds}^2 = -(\mathbf{d}T)^2 + \mathbf{dl}^2, \tag{182}$$

where $\mathbf{dl}^2$ is the positive definite quadratic form defined by the metric on σ_T. Our construction of geodesically parallel surfaces σ_T may carry us into 4-space for only a small interval of T (see ref. *38*), but in the 3-spaces σ_T we nevertheless have a well-defined global topology. We know from the definitions of **e** and **h** that at σ, **f** and ***f** must have the form

$$\mathbf{f} = *_\sigma\mathbf{h} - \mathbf{d}T \wedge \boldsymbol{\varepsilon} \tag{183a}$$

$$*\mathbf{f} = *_\sigma\mathbf{e} + \mathbf{d}T \wedge \boldsymbol{\kappa}, \tag{183b}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are 1-forms which remain to be determined. When we use T, and any coordinates $x^i (i = 1, 2, 3)$ on σ_0, as coordinates x^μ in 4-space ($\mu = 0, 1, 2, 3$) with $x^0 = T$, the metric components have the property $g_{00} = -1$, $g_{0i} = 0$. Then from Eqs. (157) and (183) we compute ***f** and $*(*\mathbf{f}) = -\mathbf{f}$. In this way we discover that $\boldsymbol{\varepsilon} = \mathbf{e}$ and $\boldsymbol{\kappa} = \mathbf{h}$, that is

$$\mathbf{f} = *_\sigma\mathbf{h} - \mathbf{d}T \wedge \mathbf{e} \tag{184a}$$

$$*\mathbf{f} = *_\sigma\mathbf{e} + \mathbf{d}T \wedge \mathbf{h} \tag{184b}$$

Since **d** and $\mathbf{d}^\sigma$ are related by the formula

$$\mathbf{d} = \mathbf{d}^\sigma + \mathbf{d}T \wedge \frac{\partial}{\partial T}, \tag{185}$$

we can use this expression in Maxwell's equations, $\mathbf{df} = 0 = \mathbf{d}*\mathbf{f}$, together with Eqs. (184) and (100) to find

$$\mathbf{d}^\sigma(*_\sigma\mathbf{h}) + \mathbf{d}T \wedge \left(\frac{\partial}{\partial T} *_\sigma\mathbf{h} + \mathbf{d}^\sigma\mathbf{e}\right) = 0, \tag{186a}$$

$$\mathbf{d}^\sigma(*_\sigma\mathbf{e}) + \mathbf{d}T \wedge \left(\frac{\partial}{\partial T} *_\sigma\mathbf{e} - \mathbf{d}^\sigma\mathbf{h}\right) = 0. \tag{186b}$$

Here $\partial/\partial T$ is the derivative along the geodesic normal to σ. The terms in $\mathbf{d}T$ in Eqs. (186) must vanish independently of the other terms, so we may rewrite Eqs. (186) as 3-dimensional equations (Table IX), treating T as a parameter

TABLE IX

THE TRACE OF MAXWELL'S EQUATIONS ON A SPACE LIKE SURFACE

	"Electric equations"		"Magnetic equations"	
	Intrinsic notation	3-vector notation on flat space	Intrinsic notation	3-vector notation on flat space
Equations in 4-space	$\boldsymbol{\delta}\mathbf{f} = 0$ or $\mathbf{d}{*}\mathbf{f} = 0$	$\operatorname{div}\mathbf{e} = 0$ and $\partial\mathbf{e}/\partial T = \operatorname{curl}\mathbf{h}$	$\mathbf{d}\mathbf{f} = 0$ (or $\boldsymbol{\delta}{*}\mathbf{f} = 0$)	$\operatorname{div}\mathbf{h} = 0$ and $\partial\mathbf{h}/\partial T = -\operatorname{curl}\mathbf{e}$
Trace on σ	$\mathbf{d}({*}\mathbf{f})^\sigma = 0$ Define $\mathbf{e} = {*}_\sigma({*}\mathbf{f})^\sigma$ Then $\mathbf{d}({*}_\sigma\mathbf{e}) = 0$ or $\boldsymbol{\delta}_\sigma\mathbf{e} = 0$	$\operatorname{div}\mathbf{e} = 0$	$\mathbf{d}(\mathbf{f}^\sigma) = 0$ Define $\mathbf{h} = {*}_\sigma(\mathbf{f}^\sigma)$ Then $\mathbf{d}({*}_\sigma\mathbf{h}) = 0$ or $\boldsymbol{\delta}_\sigma\mathbf{h} = 0$	$\operatorname{div}\mathbf{h} = 0$

and omitting the σ's:

$$\boldsymbol{\delta}\mathbf{h} = 0, \qquad \partial({*}\mathbf{h})/\partial T = -\mathbf{de}, \tag{187a}$$

$$\boldsymbol{\delta}\mathbf{e} = 0, \qquad \partial({*}\mathbf{e})/\partial T = \mathbf{dh}. \tag{187b}$$

Here we have written $\boldsymbol{\delta}\mathbf{h} = 0$ in place of the $\mathbf{d}{*}\mathbf{h} = 0$ of Eq. (186a) to agree with the customary three dimensional forms of Maxwell's equations.

Formulation B of the Initial Value Problem.

Equations (187) are now in the form to which the Cauchy-Kowalewski theorem applies. More explicitly, the equations

$$\partial({*}\mathbf{h})/\partial T = -\mathbf{d}{*}({*}\mathbf{e}), \tag{188a}$$

$$\partial({*}\mathbf{e})/\partial T = \mathbf{d}{*}({*}\mathbf{h}), \tag{188b}$$

are six equations, solved for the time derivatives, for six unknown functions, the components of ${*}\mathbf{e}$ and ${*}\mathbf{h}$. They determine ${*}\mathbf{e}$ and ${*}\mathbf{h}$ uniquely when the initial values of ${*}\mathbf{e}$ and ${*}\mathbf{h}$ on the hypersurface σ_0 are given. When in addition these initial values satisfy

$$\mathbf{d}{*}\mathbf{h} = 0 = \mathbf{d}{*}\mathbf{e} \tag{189}$$

on σ_0, then these same requirements of zero divergence will also be satisfied on σ_T as a consequence of Eqs. (188):

$$\partial(\mathbf{d}{*}\mathbf{h})/\partial T = 0 = \partial(\mathbf{d}{*}\mathbf{e})/\partial T. \tag{190}$$

The proof depends on the facts that $\mathbf{d} = \mathbf{d}^\sigma$ commutes with $\partial/\partial T$, and that $\mathbf{d}^2 = 0$. De Rham's theorem assures us that we can find initial values ${*}\mathbf{h}$ and ${*}\mathbf{e}$

on σ_0 which (1) satisfy the divergence requirements (189) and (2) have any desired values for the magnetic and electric charges

$$4\pi p_i^* = \int_{c_i^2} *\mathbf{h} \tag{191a}$$

and

$$4\pi p_i^* = \int_{c_i^2} *\mathbf{e} \tag{191b}$$

on R_2 independent wormholes in σ_0. Granted such initial value data, we are assured[17] of the existence for some finite time beyond σ_0 of a unique solution of the source-free Maxwell equations that displays constantly the specified charges.

Formulation A of the Initial Value Problem.

So much for the case where $\mathbf{e}$ and $\mathbf{h}$ are specified on σ; now for the alternative case where (1) we know the electric charges q_i^* once and for all, and where in addition (2) we know on σ the quantities $*\mathbf{h}$ and $\partial*\mathbf{h}/\partial T$, as the limit of information about the magnetic field on two infinitesimally close space like surfaces—information of the kind suited to the Lagrangian formulation of quantum mechanics.

The initial value data must satisfy the divergence conditions,

$$\mathbf{d}*\mathbf{h} = 0 \tag{192}$$

and

$$\mathbf{d}(\partial*\mathbf{h}/\partial T) = 0 \tag{193}$$

and the requirement that the magnetic fluxes—if any—through all wormholes shall stay constant in time:

$$\int_{c_i^2} (\partial*\mathbf{h}/\partial T) = 0. \tag{194}$$

This granted, we find the electric field $\mathbf{e}$ on the space like surface uniquely from the three pieces of information

$$\boldsymbol{\delta}\mathbf{e} = 0, \tag{195}$$

$$\mathbf{d}\mathbf{e} = -\partial*\mathbf{h}/\partial T, \tag{196}$$

$$\int_{c_i^2} \mathbf{e} = 4\pi q_i^*, \tag{197}$$

[17] The Cauchy-Kowalewski theorem requires analyticity, while the de Rham theorem supplies only differentiable initial values $\mathbf{e}$ and $\mathbf{h}$. To remedy this fault in the demonstration one may employ the existence theorem of Fourès-Bruhat (*39*).

TABLE X

CLASSIFICATION OF REAL NONSINGULAR VECTOR FIELDS IN A COMPACT 3-SPACE ENDOWED WITH POSITIVE DEFINITE METRIC; THE TRANSVERSE WAVES ARE SAID TO HAVE RIGHT- OR LEFT-HANDED POLARIZATION ACCORDING AS THE SCALAR WAVE NUMBER, k, IS POSITIVE OR NEGATIVE

Type of vector field		Longitudinal	Transverse	Coulomb (harmonic)
Symbol		$\mathbf{L}$	$\mathbf{X}$	$\mathbf{C}$
Definition		$\Delta\mathbf{L} = \kappa^2\mathbf{L}$ $d\mathbf{L} = 0$	$*d\mathbf{X} = k\mathbf{X}$ $\delta\mathbf{X} = 0$ $\lvert k \rvert > 0$	$d\mathbf{C} = 0 = \delta\mathbf{C}$
Association with eigenvalues or surfaces		$\mathbf{L}_1\ \mathbf{L}_2 \cdots$ $\kappa_1^2\ \kappa_1^2 \cdots$	$\mathbf{X}_1\ \mathbf{X}_2 \cdots$ $k_1\ k_2 \cdots$	$\mathbf{C}_a\ \mathbf{C}_b \cdots \mathbf{C}_{R_2}$ $S_a\ S_b \cdots S_{R_2}$
Consequence		$\Delta(\delta\mathbf{L}) = \kappa^2(\delta\mathbf{L})$	$\Delta\mathbf{X} = k^2\mathbf{X}$	$\Delta\mathbf{C} = 0$
Potential		scalar φ	2-form $\mathbf{Y}$	none
Potential from field		$\varphi = \kappa^{-1}\delta\mathbf{L}$	$\mathbf{Y} = *\mathbf{X} = k^{-1}d\mathbf{X}$	—
Field from potential		$\mathbf{L} = \kappa^{-1}d\varphi$	$\mathbf{X} = k^{-1}\delta\mathbf{Y} = *\mathbf{Y}$	—
Potential equations		$\Delta\varphi = k^2\varphi$	$*\delta\mathbf{Y} = k\mathbf{Y}$ $d\mathbf{Y} = 0$	—
Potential simplifies analysis?		Yes	No	—
Standardization		$\int \mathbf{L}*\mathbf{L} = 1$	$\int \mathbf{X}*\mathbf{X} = 1$	$\int_{S_a} *\mathbf{C}_b = 4\pi\delta_{ab}$
Scalar product with	$\mathbf{L}$	$\int \mathbf{L}_m*\mathbf{L}_n = \delta_{mn}$	$\int \mathbf{L}_m*\mathbf{X}_n = 0$	$\int \mathbf{L}_m*\mathbf{C}_b = 0$
	$\mathbf{X}$	$\int \mathbf{X}_m*\mathbf{L}_n = 0$	$\int \mathbf{X}_m*\mathbf{X}_n = \delta_{mn}$	$\int \mathbf{X}_m*\mathbf{C}_b = 0$
	$\mathbf{C}$	$\int \mathbf{C}_a*\mathbf{L}_n = 0$	$\int \mathbf{C}_a*\mathbf{X}_n = 0$	$\int \mathbf{C}_a*\mathbf{C}_a \doteq 8\pi/R_{a\,\min}$ $\int \mathbf{C}_a*\mathbf{C}_b \doteq 4\pi \cdot \sum_{(\pm)_a (\pm)_b} (\pm 1)_a\,(\pm 1)_b / r_{(a\pm)(b\pm)}$

—for example, by the methods of Fourier analysis described in the following section (Table X). Knowing both **e** and **h**, we are back at an initial value problem that has already presented itself, and that possesses a unique solution for some finite time beyond σ_0.

Generalization of Fourier Analysis to Curved Space.

When one is dealing with electrodynamics in curved space it is most natural to specify initial value data as a function of *position* on σ, as in (171A, B). An ordinary Fourier analysis is completely out of the question. However, one can give a generalization of Fourier analysis to curved and even multiply-connected space which is of use in considering special situations where the metric is constant, or nearly constant, with respect to a suitably chosen cotime coordinate, T. It permits the initial value data to be specified in terms of Fourier coefficients.

On the space-like hypersurface σ consider initial value data such as the electric field $\mathbf{e}$, the magnetic field $\mathbf{h}$, or more generally any vector field or 1-form, $\mathbf{v}$. According to the analysis of Hodge (*30*), we can decompose $\mathbf{v}$ uniquely into the sum of three parts, characterized as in Table X by the names *longitudinal*, *transverse*, and *harmonic*—or, as we may say, *Coulomb*. We will be limiting attention to 3-space. Therefore it will be permissible as in Table X to omit the superscripts σ that evidences the space-like character of the operations of curl, $\mathbf{d}^\sigma$, and divergence, $\boldsymbol{\delta}^\sigma$. For simplicity of analysis we shall assume that the space is closed.

A longitudinal field has zero curl, but a nonconserved flux because it has a nonzero divergence. Electric or magnetic fields of this type would correspond to distributions of "real" charge or magnetic poles. Therefore we exclude such fields from attention.

A transverse field has zero divergence and zero flux through any wormhole, but its curl is nonzero. Such a field can be decomposed into *proper modes* $\mathbf{X}_m$, each endowed with a characteristic *scalar* wave number $k_m(\neq 0)$ of its own:

$$*\mathbf{d}\mathbf{X}_m(=\text{curl}\,\mathbf{X}_m) = k\mathbf{X}_m\,. \tag{198}$$

Modes with positive k we describe as right-handedly polarized; conversely for modes with negative k.

A harmonic or Coulomb field has zero divergence and zero curl, but a nonvanishing flux through at least one wormhole. Such a vector field can be decomposed uniquely into a linear combination of *characteristic harmonic vector fields*, $\mathbf{C}_a(a = 1, 2, \cdots, R_2)$, such that $\mathbf{C}_a$ attributes a unit charge to the ath wormhole, and contributes no flux at all to any other wormholes—that is, to any other of R_2 basic linearly independent homology classes.

The general vector field in curved multiply-connected 3-space can be Fourier-analyzed in the form

$$\mathbf{v} = \sum_m \lambda_m \mathbf{L}_m + \sum_m \xi_m \mathbf{X}_m + \sum_{a=1}^{R_2} Q_a \mathbf{C}_a \tag{199}$$

The orthogonality relations and the normalization conventions summarized in Table X permit one to find all the coefficients in this expansion by relations of which the following is typical:

$$\xi_m = \int \mathbf{X}_m * \mathbf{v}. \tag{200}$$

Without attempting to derive here the general expansion (199), we can note some of the underlying principles. (1) The inner product of two vector fields,

$$\langle \mathbf{a}, \mathbf{b} \rangle = \int_\sigma \mathbf{a} * \mathbf{b}, \tag{201}$$

is positive definite, so that

$$\langle \mathbf{a}, \mathbf{a} \rangle > 0 \tag{202}$$

unless $\mathbf{a} = 0$. (2) It follows that the set of all vector fields or 1-forms of finite norm generate a Hilbert space. (3) The operators Δ and $*\mathbf{d}$ are self adjoint when applied to vectors in 3-space (Table IX) and therefore have eigenvalues and complete sets of eigenfunctions. (4) From the property

$$(\mathbf{v}, \Delta\mathbf{v}) = (\boldsymbol{\delta}\mathbf{v}, \boldsymbol{\delta}\mathbf{v}) + (\mathbf{dv}, \mathbf{dv}) \geqslant 0 \tag{203}$$

it follows that the eigenvalues of Δ are non-negative numbers, k^2. (5) Given any nonzero solution of the equation

$$\Delta\mathbf{v} = k^2\mathbf{v} \tag{204}$$

with $k > 0$, one can decompose it into two parts,

$$\begin{aligned} \mathbf{v} &= \mathbf{a} + \mathbf{b}, \\ \mathbf{a} &= k^{-2}\mathbf{d}\boldsymbol{\delta}\mathbf{v}, \\ \mathbf{b} &= k^{-2}\boldsymbol{\delta}\mathbf{d}\mathbf{v}, \end{aligned} \tag{205}$$

both of which satisfy the same space-like version (204) of the wave equation, and one of which is longitudinal; the other, transverse:

$$\mathbf{da} = 0, \tag{206}$$

$$\boldsymbol{\delta}\mathbf{b} = 0. \tag{207}$$

(6) For those eigenvalues Δ for which there exists a nonzero transverse field $\mathbf{b}$, there can be constructed out of this field a curl,

$$\mathbf{c} = k^{-1}*\mathbf{db}(= k^{-1} \operatorname{curl} \mathbf{b}), \tag{208}$$

such that

$$k^{-1}*\mathbf{dc}(= k^{-1} \operatorname{curl} \mathbf{c}) = k^{-2}\Delta\mathbf{b} = \mathbf{b}. \tag{209}$$

From $\mathbf{b}$ and $\mathbf{c}$ we can build right-handedly and left-handedly polarized fields

$$\begin{aligned} \mathbf{X}_+ &= \mathbf{b} + \mathbf{c}, \\ k^{-1}*\mathbf{dX}_+(= k^{-1} \operatorname{curl} \mathbf{X}_+) &= \mathbf{X}_+, \end{aligned} \tag{210}$$

and

$$\begin{aligned} \mathbf{X}_- &= \mathbf{b} - \mathbf{c} \\ -k^{-1}*\mathbf{dX}_-(&= -k^{-1}\,\mathrm{curl}\,\mathbf{X}_-) = \mathbf{X}_-, \end{aligned} \tag{211}$$

not both of which can vanish. As example, consider in flat space the transverse field or 1-form,

$$\mathbf{b} = \mathbf{d}x^2 \sin kx^1,$$

$$\mathbf{c} = k^{-1}\,\mathrm{curl}\,\mathbf{b} = k^{-1}*\mathbf{db} = \mathbf{d}x^3 \cos kx^1,$$

from which we build the circularly polarized fields

$$\mathbf{X}_\pm = \mathbf{d}x^2 \sin kx^1 \pm \mathbf{d}x^3 \cos kx^1. \tag{212}$$

Relation of Harmonic Field to Familiar Pattern of Electric Lines of Force. The harmonic or Coulomb fields are unique to multiply-connected space, with Betti number $R_2 \geqslant 1$. A typical one of these fields has a pattern of lines of force like that shown in Fig. 3. Consider the case where the space has an enormous radius of curvature, so that it is nearly flat, except in the immediate vicinity of the + and − mouths of typical wormholes, a, b, with exceedingly small radii R_a, R_b, $\cdots$. Let the length "underground" of the connection between matching wormhole mouths be very short, and also of the order R_a, R_b, $\cdots$. Let the fluxes $4\pi q_a^*$, $4\pi q_b^*$, $\cdots$ go through the respective wormholes. In this very special model the field almost everywhere looks like that of R_2 pairs of equal and opposite charges:

$$\begin{aligned} \mathbf{e} &= q_a^*\mathbf{C}_a + q_b^*\mathbf{C}_b + \cdots \\ &\sim q_a^*(\mathbf{r}_{a^+}/r_{a^+}^{\;3} - \mathbf{r}_{a^-}/r_{a^-}^{\;3}) + q_b^*(\cdots) + \cdots . \end{aligned} \tag{213}$$

The electrostatic energy of the fields,

$$\int \mathbf{e}*\mathbf{e}/8\pi, \tag{214}$$

consists of (1) self energy terms of the order

$$q_a^{*2}/R_a + q_b^{*2}/R_b + \cdots \tag{215}$$

and (2) interaction terms of the form

$$q_a^*Q_b^*(r_{a^+b^+}^{\;-1} - r_{a^+b^-}^{\;-1} - r_{a^-b^+}^{\;-1} + r_{a^-b^-}^{\;-1}) + \cdots . \tag{216}$$

From this circumstance one can immediately deduce the integrated value of the products of individual harmonic forms,

$$\int \mathbf{C}_a*\mathbf{C}_b, \tag{217}$$

as indicated in Table X. One sees that these harmonic forms give one all the machinery needed to describe *classical* charge in a completely consistent and divergence free manner.

Fourier Coefficients as Initial Value Data; Dynamic Equations for Their Rate of Change.

To specify the initial values of **e** and **h** on a space like surface σ it is enough, we conclude, to know the Fourier coefficients in the following expansions:

$$\begin{aligned} \mathbf{e} &= \sum_{m}^{\infty} e_m \mathbf{X}_m + \sum_{a=1}^{R_2} q_a{}^* \mathbf{C}_a , \\ \mathbf{h} &= \sum_{m}^{\infty} h_m \mathbf{X}_m . \end{aligned} \tag{218}$$

We have omitted magnetic pole terms from the second expression as without interest.

The evolution of the electromagnetic field with time proceeds most simply when the metric does not change with increase of the geodesically normal cotime coordinate T. Then the dynamical equations (188) reduce to an immediately integrable form,

$$\begin{aligned} dq_i{}^*/dT &= 0; \; q_i{}^* = \text{const} \\ de_m/dT &= k_m h_m ; \\ dh_m/dT &= -k_m e_m ; \end{aligned} \tag{219}$$

$$\begin{aligned} e_m &= (\tfrac{1}{2})(e_m + ih_m)_0 \exp(-ik_m T) + \text{c.c.}; \\ h_m &= (\tfrac{1}{2})(e_m + ih_m)_0 \exp(-ik_m T) + \text{c.c.} \end{aligned} \tag{220}$$

Here the generalizations of the familiar normal mode decomposition of the field are simple and obvious.

No such simplicity reigns when the metric changes with time in a complicated way. Then the normal modes themselves alter. Consequently the dynamical equations (219) acquire new terms,

$$\begin{aligned} de_m/dT &= k_m h_m + \sum K_{mn} e_n + \sum_a L_{ma} q_a{}^* , \\ dh_m/dT &= -k_m e_m + \sum K_{mn} h_n , \end{aligned} \tag{221}$$

which might loosely be considered to represent (1) the emission of energy into the transverse field, or absorption of energy from the transverse field, by the Coulomb fields due to the motion of the charges or other changes in the metric (2) the pumping or scattering of energy into one transverse mode out of another mode or out of the metric itself, as a result of changes in the metric. It is true that one can define the local density of electromagnetic field energy in an unambiguous way. However, the same is not true of the integrated field energy (*40*). There is no common set of space and time coordinates with respect to which one could even hope to refer the components of a total energy-momentum four vector. Moreover, the local conservation laws, integrated over a closed space such as we are considering here, produce nothing of interest, only the trivial identity $0 = 0$. Consequently it is necessary to use with caution the idea of energy transfer from one mode to another. This caution in no way depreciates either the dynamical equations (221) or what they have to say about the most interesting coupling between modes of the field and modifications of the metric.

IV. CHARGE AND MASS AS ASPECTS OF GEOMETRY

We have just finished examining the response *of* an electromagnetic field to curvature and multiple connectedness in the metric, without asking—or having to ask—about the influence of this field *on* the metric. However, an adequate analysis of classical charge and mass (in the sense of Table I) demands that both sides of the interaction be examined. For this reason (1) we consider a space like surface σ and ask what initial value data has to be given on this surface—and what conditions this data must satisfy—to specify uniquely the future evolution of *both* the electromagnetic field and the metric (2) we examine the properties of the Schwarzschild and Reissner-Nordstrom solutions as solutions of these equations that manifest mass, and mass together with charge (3) we note an exact solution of the initial value conditions that generalizes the Reissner-Nordstrom solution to a space endowed with many charges and masses, all momentarily at rest at the moment of observation and (4) we remark that such wormhole solutions, together with geons, furnish two techniques of building disturbances in empty space, out of which one can make rich combinations.

The Initial Value Conditions and the Existence of Solutions to the Maxwell-Einstein Equations

Consider a space-like surface σ. On it let the trace of the metric *tensor*, $\mathbf{dl}^2$, take on prescribed values as a function of position,

$$\mathbf{dl}^2 = g_{ik}{}^{\sigma}\mathbf{d}x^i\mathbf{d}x^k. \tag{222}$$

Also let $\mathbf{e}$ and $\mathbf{h}$ have prescribed values on σ. Then there will in general exist no acceptable solution either of the Maxwell equations or of the Einstein equations. The equations of Maxwell will have a well defined solution if and only if the initial value conditions (189) are satisfied:

$$\boldsymbol{\delta}_\sigma\mathbf{e} = 0 = \boldsymbol{\delta}_\sigma\mathbf{h}. \tag{223}$$

Similarly, the field equations of Einstein will have a well-defined solution if and only if certain initial value requirements are satisfied (Eqs. (224) and (225) below). *Granted* that these conditions as well as Eqs. (223) are fulfilled, then a solution of the combined Einstein-Maxwell equations with the prescribed initial values will exist for some finite time.

The initial value problem for general relativity has been studied extensively by Lichnerowicz (*41*), and an existence theorem which does not require analyticity has been given by Fourès (*42*). In general relativity, the four of Einstein's equations involving $R_\mu^{\ 0} - \frac{1}{2}\delta_\mu^{\ 0}R$ *provide the initial value requirements analogous to Eqs. (223) in electromagnetic theory*—requirements which connect the time rate of change of curvature on σ with the Poynting flux and Maxwell energy density:

$$(P_i^{\ j} - \delta_i^{\ j}P)_{;j} = -2(g^{(3)})^{1/2}[ijk]e^jh^k = -2[*(\mathbf{e} \wedge \mathbf{h})]_i \qquad (224)$$

$$P^2 - P^{ij}P_{ij} + R^{(3)} = 2(e_ie^i + h_ih^i) = 2*(\mathbf{e}*\mathbf{e} + \mathbf{h}*\mathbf{h}). \qquad (225)$$

Here we follow the notation of Mme. Fourès (*43*), who has demonstrated the existence of solutions of Eqs. (224) and (225) in the case where there is no electromagnetic field. We let the surface σ be given by $x^0 = T = 0$, and write $g_{00} = -V^2$, $g_{0i} = V\eta_i (i = 1, 2, 3)$. Then $g_{ij} = g_{ij}^{(3)}$ provides the metric $\mathbf{dl}^2$ on σ which is used to raise indices and define covariant derivatives in Eqs. (224) through (228). With $\eta^2 = \eta_i\eta^i$, we define

$$h^i = \tfrac{1}{2}(g^{(3)})^{1/2}[ijk]f_{jk}\,, \qquad (226a)$$

$$e_i = (1 + \eta^2)^{-1/2}(V^{-1}f_{i0} - f_{ij}\eta^j), \qquad (226b)$$

$$P_{ij} = \tfrac{1}{2}(1 + \eta^2)^{-1/2}[V^{-1}\partial g_{ij}/\partial x^0 - (\eta_i\partial/\partial x^j + \eta_j\partial/\partial x^i)V - (\eta_{i;j} + \eta_{j;i})], \qquad (227)$$

and

$$P = P_i^{\ i} \qquad (228)$$

Thus P_{ij} is essentially the time derivative of the metric, while $R^{(3)}$ is the scalar curvature of the metric $g_{ij}^{(3)}$ on σ. The formulas (226) translate Eqs. (180) into components relative to the coordinates chosen here.

The Schwarzschild and Reissner-Nordstrom Solutions

The Schwarzschild metric furnishes the most familiar example of a solution of the field equations and therefore a solution of the initial value requirements (224) and (225). In this example the electric and magnetic fields vanish,

$$\mathbf{e} = 0 = \mathbf{h}. \qquad (229)$$

The metric

$$ds^2 = -(1 - 2m^*/r)dT^2 + (1 - 2m^*/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (230)$$

satisfies the field equations $R_{\mu\nu} = 0$. In contrast, the Ricci curvature tensor for the metric on the space like surface T = const has the nonzero value

$$R_i^{(3)k} = \begin{pmatrix} -2m^*/r^3 & & \\ & m^*/r^3 & \\ & & m^*/r^3 \end{pmatrix} \tag{231}$$

but still a zero trace,

$$R^{(3)} = 0. \tag{232}$$

Also the quantities P_{ik} vanish in virtue of the constancy of the metric. Thus the initial value requirements (224) and (225) are fully satisfied by the Schwarzschild solution.

The Schwarzschild metric appears at first sight to have a singularity at the point $r = 2m^*$. However the Ricci curvature tensor (231) is perfectly well behaved at this point, as is also the full Riemann curvature tensor in the appropriate mixed covariant-contravariant representation. To bring part of this regularity into evidence, we make a coordinate transformation of a type suggested before by more than one writer (*44*)[18].

$$r = [1 + (m^*/2\rho)]^2\rho. \tag{233}$$

Then the metric takes the form:

$$ds^2 = -\left(\frac{1 - m^*/2\rho}{1 + m^*/2\rho}\right)^2 dT^2 + (1 + m^*/2\rho)^4(d\rho^2 + \rho^2\, d\Omega^2). \tag{234}$$

Here

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{235}$$

is the metric of a unit sphere. This result can be interpreted as follows: In making the transformation (233) we have redefined the differentiable structure of the

[18] The regularization in all of these cases removes only the singularity in the space part of the metric, not that in the time part, and therefore falls short in every instance of the regularization described in the text. Einstein and Rosen, supposing that they would not be able to eliminate the singularity in the space part of the metric by a coordinate transformation alone, also changed the sign of the constant of gravitation. Of course this step gives a negative value for the electromagnetic contribution to the mass, and for the energy of electromagnetic fields in general, in contradiction with experience. Einstein and Rosen pictured their regularizing transformation in terms of a space nearly flat except near particles, and a mirror space, close to and parallel to the first space, with a bridge across at the Schwarzschild singularity. On this picture the charge or charges in the primary space need not add up to zero. In contrast, the analysis in terms of harmonic forms (1) imposes no such mirror property on space (2) treats all points of space as on the same footing (3) brings in the idea of a real continuation of electric lines of force through a wormhole (4) normally contemplates a closed space, with no excess unbalanced lines of force that have to end at infinity.

manifold we are considering, so that *now ρ, rather than r, is to be called a differentiable function.* As a result, the metric dl^2 on the space-like surface $T = 0$ is now nonsingular at the Schwarzschild radius. The *space-time* metric ds^2 remains singular, however, since g_{TT} vanishes at the Schwarschild radius, i.e., $g^{TT} \to \infty$. We do not know how to eliminate this singularity explicitly for all time, but according to the initial value theory of Lichnerowicz and Fourès, we can remove it for some undetermined, but nonzero, length of time. This we see by noting that the initial value conditions (224) and (225) leave $g_{TT} = -V^2$ completely arbitrary on the initial surface. We take as initial conditions therefore, the following:

$$ds^2 = -V^2 dT^2 + dl^2, \tag{236}$$

$$dl^2 = (1 + m^*/2\rho)^4(d\rho^2 + \rho^2\, d\Omega^2), \tag{237}$$

$$\partial g_{\mu\nu}/\partial T = 0. \tag{238}$$

We may, for instance, choose $V = 1$. Since V need only be differentiable, not analytic, we may also choose $V = V_{\text{Schwarzchild}}$ as in (234) when ρ is a small distance ϵ away from the singular surface $\rho = m^*/2$, but fill in over the previously singular surface with differentiable, nonzero, values of V. With g_{TT} specified on the initial surface $T = 0$ in a manner such as this, the solution that results from integrating the Einstein equations is at least asymptotically stable: It will exist, and coincide (*45*) with the Schwarzschild solution for a cotime T which is arbitrarily large as $\rho \to \infty$. Furthermore, for any fixed $\rho_0 > m^*/2$ a solution of this type may be found which exists for all $\rho > \rho_0$ for at least any prescribed length of time, say 10^{100} years.

These same arguments apply equally well when the center of attraction is endowed with charge in addition to mass. We start with the Reissner-Nordstrom (*46*) solution for this problem. It differs from the Schwarzschild solution (230) only in the replacement of the factor $(1 - 2m^*/\text{r})$ everywhere by

$$(1 - 2m^*/r + q^{*2}/r^2).$$

The dual of the electric field in this case is

$$*e = q^* \sin\theta d\theta \wedge d\varphi. \tag{239}$$

Then we write

$$r = \rho[(1 + m^*/2\rho)^2 - q^{*2}/4\rho^2] \tag{240}$$

and find for the metric

$$ds^2 = -V_{RN}{}^2 dT^2 + dl^2, \tag{241}$$

where

$$V_{\text{RN}}{}^2 = \left(1 - \frac{m^{*2} - q^{*2}}{4\rho^2}\right)^2 \Big/ \left[\left(1 + \frac{m^{*2}}{2\rho}\right)^2 - \left(\frac{q^*}{2\rho}\right)^2\right]^2$$

and

$$dl^2 = [(1 + m^*/2\rho)^2 - (q^*/2\rho)^2]\,(d\rho^2 + \rho^2\, d\Omega^2). \tag{243}$$

The singularity of the *space-time* metric due to the vanishing of V_{RN} when $2\rho = (m^{*2} - q^{*2})^{1/2}$, may again be removed by modifying $g_{TT} = -V^2$ on an initial surface near this critical value of ρ. To make possible this regularization we must demand that the mass exceed the minimum limit that is associated by general relativity with the charge in question:

$$m^*(\text{cm}) > q^*(\text{cm})$$

or

$$m(g) > G^{-1/2}q = (3.88 \times 10^3 g/\text{esu})q(\text{esu}). \tag{244}$$

This condition is quite incompatible with the properties of the "dressed" particles studied by experiment. This circumstance stresses again that we are dealing here exclusively with a description of *classical* charge and mass that has nothing whatsoever directly to do with the particles found in nature.

Thus far we have shown how the singular surface in the Schwarzschild metric may be removed, but there apparently remains in Eqs. (237) and (243) a singularity as $\rho \to 0$. We shall now see that we cannot call this a singularity unless we are willing to say that the metric is also singular as $\rho \to +\infty$. The transformation

$$\frac{2\rho}{(m^{*2} - q^{*2})^{1/2}} \xrightarrow{J} \frac{(m^{*2} - q^{*2})^{1/2}}{2\rho} \tag{245}$$

maps the entire manifold, $\rho > 0$, onto itself in a differentiable, 1-1, *metric preserving* way. Therefore the vicinity of $\rho = 0$ is in every meaningful way equivalent to the vicinity of $\rho = \infty$. The initial surface $T = 0$ is a *complete* Riemannian manifold, that is, every geodesic can be continued to infinite length. One might nevertheless wish a stronger property to specify the global meaning of "nonsingular". One might insist that the manifold be either closed or that there be only one region (*either* $\rho \to \infty$ *or* $\rho \to 0$) where the space is asymptotically flat. An example of this type may be constructed from the charge-free Schwarzschild solution (237). We complete the definition of the mapping J of Eq. (246) by defining

$$T \xrightarrow{J} T, \tag{246}$$

$$\theta \xrightarrow{J} \pi - \theta, \tag{247}$$

$$\varphi \xrightarrow{J} \varphi + \pi. \tag{248}$$

The mapping J then has no fixed points, and we may identify the point x with the point Jx to obtain a new manifold with only one region, $\rho \rightarrow +\infty$, which is asymptotically flat. The surface $T = 0$ in this manifold is topologically equivalent to projective 3-space P^3 (see Sec. III B) with one point removed, "the point at ∞". This procedure cannot be applied to the Reissner-Nordstrom solution, for although the metric is invariant under J, the electromagnetic field changes sign and is therefore not consistent with the identifications. The time dependence of this nonsingular modification of the charge-free Schwarzschild solution has been investigated by Dubman (*47*) to first order in T^2. He finds that the area of the critical sphere begins to decrease.

Many Charges and Masses

The Reissner-Nordstrom initial conditions (239) and (241–243) can be generalized to the case of N wormhole mouths endowed with charge and mass in the following way: Set

$$ds^2 = -V^2 dT^2 + (\chi^2 - \phi^2)^2(dx^2 + dy^2 + dz^2), \tag{249}$$

$$*e = [ijk]\left(\phi \frac{\partial \chi}{\partial x^i} - \chi \frac{\partial \phi}{\partial x^i}\right) dx^j \wedge dx^k, \tag{250}$$

$$\partial g_{\mu\nu}/\partial T = 0 \tag{251}$$

at $T = 0$, let V be any nonsingular, positive definite function, and let each χ, ϕ, satisfy the flat space Laplace equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)f = 0. \tag{252}$$

These initial values make the initial surface a complete Riemannian manifold when we choose

$$\chi = 1 + \sum_a \alpha_a/r_a, \phi = \sum_a \beta_a/r_a, \tag{253}$$

with $r_a = |\mathbf{r} - \mathbf{r}_a|$ and $\alpha_a \geqslant |\beta_a|$, $a = 1, 2, \cdots N$. The points $\mathbf{r}_0 = (x_a, y_a, z_a)$ are of course excluded from the manifold, The restriction $\alpha_a \geqslant |\beta_a|$ arises from the fact that the metric would be singular if $(\chi^2 - \phi^2) = 0$ at any point. The completeness requirement also *excludes negative masses* as well as multipole terms in either χ or ϕ. The α's and β's are related to mass and charge respectively, but do not give the masses and charges directly[19]. In the special case where there is no charge, $\phi = 0$, one obtains a class of problems which has been studied by Lichnerowicz (*49*). As long as only instantaneously static solutions of the initial value conditions are desired, a similar $(\chi^2 - \phi^2)^2$ factor may be used to modify

[19] See Lindquist and Wheeler (*48*) for the relationship between m_a^* and the α_a, and for a study of the time dependence of some solutions of this type.

any three dimensional metric in place of the $dx^2 + dy^2 + dz^2$ of Eq. (249). The initial value conditions then reduce to the form

$$\bar{\Delta}\chi + \tfrac{1}{8}\bar{R}\chi = 0 \tag{254}$$

plus a similar equation for ϕ that replaces Eq. (252). Here $-\bar{\Delta}$ and $\bar{R}$ are the Laplacian and scalar curvature of the unmodified metric.

Relation to Geons

Electromagnetic geons (*50*) are objects built out of electromagnetic radiation and held together by mutual gravitational attraction. The metric is greatly altered in the region occupied by the geon but the topology of space is still isomorphic to the normal Euclidean topology. Shortwavelength radiation sent directly at the geon comes out again whether it experiences a small or a large alteration in direction in the encounter. In contrast, shortwavelength radiation directed at one mouth of a wormhole will come out the other mouth. Despite this difference in properties, the two objects curve space in the same way and are indistinguishable, mass for mass, as regards their $1/r^2$ gravitational attraction.

These two techniques for constructing mass-like solutions of the equations of geometrodynamics need not be employed separately. One can envisage an object endowed both with circulating radiation and with wormhole mouths, drawn together and moving about in a limited region of space as a consequence of mutual gravitational attraction. In addition, a sufficient number of wormhole mouths of the same sign moving on nearly identical orbits will simulate a current. They can create magnetic fields strong enough to have a substantial or even a dominating effect on the structure. It follows that the variety of objects that can be built out of curved empty space is exceedingly rich and very far from having been explored or even surveyed.

Equations of Motion

Masses and charges are not objects distinct from the fields in the theory we are presenting here—not even to the extent of being singularities in the fields. It is therefore obvious that the field equations determine the motions of the masses and charges. We must, however, insist that these motions correspond in the appropriate limit to the Newtonian and Lorentz force laws, for these laws express most basic and well tested properties of idealized classical point particles. In geometrodynamics, mass and charge are not idealized as properties of *point* particles, they are rather aspects of the geometrical structure of space. To discuss equations of motion it will be necessary to view space-time with less resolving power than heretofore, and collapse the entire structure of a mass or charge down to a point whose motion we may then compare with the laws of Newton and Lorentz. For instance, an entire region about one mouth of a wormhole, as in Fig. (3), may be *called* one point x_1, and a similar region about the other mouth may be called x_2. The tube connecting these two region we simply leave out altogether. A similar idealization may be made for a geon, and likewise for an object that is built out of radiation and wormholes. In this way the manifold of Fig. (3) is mapped in a *singular* way onto a topologically Euclidean space, and the metric which is carried over will have two singularities, at x_1 and x_2. Similarly for many masses and charges: by collapsing each to a point we ignore all details of inner structure and pass to a limit where it becomes appropriate to compare motions with those predicted by the force laws of Newton and Lorentz. The problem so defined is the problem so carefully studied by Einstein, Infeld, and Hoffman. Those and other investigators *derive* the mechanical equations of motion of singularities[20] from the field equations. Consequently we can conclude that the objects discussed here—charges and masses built out of curved empty space and nothing more—satisfy in the appropriate limit the equations of Newton and Lorentz.

No Units but Length in Purely Classical Physics

The purely geometrical character of classical physics shows itself in the circumstance that space curvatures are measured in cm^{-2}, electromagnetic fields in cm^{-1}, charges in cm, and masses in cm. There is no place for any units other than length. A parable may be pardoned, of a kingdom where distances to the north were sacred and measured in miles, while those to the east and west and up and down were measured in feet. A special education was needed to calculate diagonal distances from coordinate readings until the discovery was made that a single constant of nature sufficed for the theory of the calculation. Thereafter much attention was devoted to "explaining" how nature happened to be endowed with a natural "slope" of 5280 feet per mile. The parable perhaps makes one charitable towards similar attempts to "explain" why the speed of light should be 3×10^{10} cm/sec. That Boltzmann's constant k is only a conversion factor between two chance units of energy is of course a familiar idea. It is less familiar that grams and centimeters are two equivalent units for length; that the Schwarzschild radius of an object, $r_{\text{schw}} = Gm/c^2 \equiv m^*(\text{cm})$, is a purely geometrical way to characterize its inertia. We have seen that *classical* electromagnetism likewise requires nothing but units of length for its simplest expression. The field is only a manifestation of curved empty space. Classical physics in the sense of Table I reduces to pure geometry.

[20] Einstein, Infeld, and Hoffman (*51*) deal with uncharged singularities of finite mass in slow motion; Infeld and Schild (*52*) treat uncharged singularities of infinitesimal mass moving at arbitrary velocities; and D. M. Chase (*53*) analyzes charged singularities of infinitesimal mass moving at arbitrary velocities.

V. PROBLEMS AND PROSPECTS OF GEOMETRODYNAMICS

Certain gaps remain to be filled in the logical structure of geometrodynamics and almost everything remains to be done to exploit the richness of this subject.

Case of Null Fields

The algebraic relations (4), (5) of already unified field theory were developed quite generally, but the differential equations (7), (8) were derived only on the assumption that the Ricci curvature tensor is non-null. For this simplifying assumption to fail, for the tensors $R_{\mu\nu}$ (and $F_{\mu\nu}$) to be null, it is necessary that the two electromagnetic field invariants should simultaneously vanish:

$$\mathbf{e}\cdot\mathbf{h} = 0; \qquad \mathbf{h}^2 - \mathbf{e}^2 = 0. \tag{256}$$

At a given moment of time this condition will ordinarily be fulfilled only on certain isolated lines in space. As time advances these lines will trace out surfaces in space time,

$$x^\beta = x^\beta(\xi, \eta). \tag{257}$$

On such surfaces Eq. (7) gives no well defined value for the vector α_β, the gradient of the complexion of the electromagnetic field. Is α itself well defined? Can the equations of already unified theory be formulated in such a way as to hold right across such a surface? Does the presence of such surfaces impose any additional topological or periodicity requirements on the Ricci curvature tensor? Do any special problems arise when the invariants (256) vanish, not merely on surfaces in space time, but throughout regions of greater dimensionality? These questions all obviously hang together.

Why Only One Kind of Charge?

A second group of questions concerns charge. The interpretation of charge in terms of lines of force trapped in the topology allows all charges to be as well purely magnetic as purely electric. However, the difference between the two possibilities is well known to be only one of names. The duality transformation

$$\mathbf{f}' = *\mathbf{f}; \qquad *\mathbf{f}' = **\mathbf{f} = -\mathbf{f}$$

or

$$\mathbf{h}' = \mathbf{e}; \qquad \mathbf{e}' = -\mathbf{h} \tag{258}$$

renames magnetic charges so that they are all electric, in conformity with the usual convention. The charges associated with all wormholes—or homology classes in dimension two—can again be renamed as purely electric when in the original frame of reference each is a *mixture* of an electric charge e_i and a magnetic pole p_i, provided that the ratio of the two has for each wormhole the same value

$$e_i/p_i = \cos \beta/\sin \beta. \tag{259}$$

Then the duality rotation

$$\mathbf{f}' = e^{*\beta}\mathbf{f} \tag{260}$$

accomplishes the renaming in accordance with tradition. However, this renaming of all charges as electric is only possible when the *ratio* (259) of the two kinds of charges is identical for all the homology classes or wormholes in the original duality reference system. Can this ratio condition be restated in other terms?

When the regions over which space is appreciably curved are small compared to the distances between different wormhole mouths, then there is an approximate localizability of the typical charge. We are invited to look at it in a local Lorentz frame where it appears momentarily to be at rest. In this frame the field close to the charge is practically purely electric. Consequently there will be a surface, either around the mouth of the wormhole, or deep down in its throat, where the complexion, α, is either everywhere zero or equal to some integral multiple of 2π. Moreover, we expect the surface $\alpha = 2\pi n$ to remain tied to the neck of the wormhole in the same way as time advances; similarly for other wormholes, i. Insofar as the characteristic complexion, $\alpha_{\text{char}} = \alpha^{(i)}$, for each is well defined, the condition that all charges be electric therefore says

$$\alpha^{(i)} - \alpha^{(j)} = 2\pi \cdot \begin{pmatrix} \text{positive or negative} \\ \text{integer or zero} \end{pmatrix}. \tag{261}$$

This condition reminds one of the periodicity requirement,

$$\oint \alpha_\mu \, dx^\mu = 2\pi \cdot \text{integer} \tag{262}$$

(Eq. 77) imposed by already unified field theory upon the Ricci curvature tensor. Nevertheless, we have seen no way to derive (261) from (262). Therefore we are not clear whether the requirement that all charges be electric is a part of the theory as it exists, or whether it has to be added to the theory.

For all magnetic poles to be zero[21] it is enough, according to Eq. (145), that the field should be derivable from a 4-vector potential:

$$\mathbf{f} = \mathbf{da} \tag{263}$$

or

$$f_{\mu\nu} = \partial a_\nu/\partial x^\mu - \partial a_\mu/\partial x\nu. \tag{264}$$

[21] See Malkus (*54*) for the convincing experimental evidence against the existence of free magnetic poles.

This is an assumption that goes beyond Maxwell's equations, for in Sec. III-D we saw that there exist solutions of Maxwell's equations which display electric *and* magnetic charge side by side. There, however, electromagnetism was considered within the arena of a *prescribed* metric. Only in this framework of ideas is it clear that the exclusion of magnetic poles—or the existence of a 4-potential—is a demand independent of, and supplemental to, the Maxwell equations themselves.

When we turn to the full coupled Einstein-Maxwell equations, and note that the Ricci curvature $R_{\mu\nu}$ is completely determined by the field, then it is *not* clear that the nullity of magnetic poles, or the existence of a 4-potential, has to be added as a supplementary condition. Moreover, no example is known of a solution of the coupled Einstein-Maxwell equations in which the magnetic poles cannot all be eliminated by a duality rotation. It is obviously an important issue of principle to decide whether the existence of a 4-potential really has to be *added* to Eqs. (4), (5), (7), and (8) of already unified field theory. Can it be *derived* from those equations?

Another issue presents itself: Can Eqs. (4), (5), (7), and (8) of geometrodynamics, with or without a possible supplementary condition about a 4-potential, all be derived from a single variational principle? Some work has been done that is relevant to this issue (*55*) but the problem itself appears never even to have been formulated in the literature.

Now we turn from the question of the best formulation of the field equations to the problem how best to deal with the initial value requirements. For the case of pure electromagnetism one has long ago learned to satisfy automatically the requirement div $\mathbf{h} = -\delta \mathbf{h} = 0$ on the initial hypersurface by introducing as primary data, not $\mathbf{h}$ itself, but a 3-potential $\mathbf{a}$ that generates an acceptable $\mathbf{h}$. In the case of the coupled equations of geometrodynamics, we meet nonlinear initial value requirements (224) and (225) on the measures, P_{ij}, of the time derivative of the metric. Does there exist any kind of freely choosable superpotential, analogous to $\mathbf{a}$, which will generate a tensor P_{ij} which in turn will automatically satisfy (224) and (225)? If so, the properties of such a superpotential should reveal much about the truly independent variables of geometrodynamics. To clarify this point is essential for the understanding of already unified theory, for its most efficient application, and for illumination on what it means to quantize it.

Still another question of principle raises itself. In electromagnetism we can make a linear combination of two solutions to obtain a third. Geometrodynamics is of course nonlinear. Does there nevertheless exist a method to combine two solutions to obtain a third? To search for such a combinatorial scheme would seem to demand an investigation of the continuously infinite dimensional space of field histories. On raising this issue, we were kindly advised by our colleague, Professor V. Bargmann, that the methods of Lie (*56*) should suffice to obtain a definitive answer to our question.

To the questions of principle that we have raised, about null fields, about the existence of a 4-potential, about a possible superpotential, and about the combinatorial properties of the infinite dimensional space of field histories, there should be added many issues about the consequences of the classical theory, on which we shall only touch[22]. (1) How wide a variety of wormholes and multiple connectedness is topologically conceivable. (2) When the deterministic evolution of the metric with time leads at a certain moment to fission or coalescence of wormhole mouths or to any other change in topology, what new phenomena occur? (3) What can one do to construct closed mathematical expressions for the metrics of spaces that show as much as possible of the richness of geometrodynamics–gravitational waves, wormholes, shocks, trapped radiation, and combinations of all these features?

One's attention is inevitably drawn beyond these fascinating questions to the still deeper issue, what is the nature of quantum geometrodynamics[23] and what ideas have to be added to quantum geometrodynamics for the description of nature?

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[22] See Power and Wheeler (*57*) for a table of the analogies between geometrodynamics and hydrodynamics, and the problems that are suggested by this analogy.

[23] In this connection see papers by Everett, Misner, and Wheeler (*58*).

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