

TASI 2003 lectures on AdS/CFT

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We give a short introduction to AdS/CFT and its plane wave limit.

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1. Introduction

In these lecture notes we provide a short introduction to the ideas related to the correspondence between gauge theories and gravity theories. For other reviews of the subject, including a more complete list of references, see [1,2,3,4,5].

We start by discussing the simplifications that occur in the large N limit of field theories. We discuss first the large N limit of vector theories, then the large N limit of theories where the fundamental fields are $N \times N$ matrices and we show that these theories are expected to be described in terms of strings [6]. If we start with a four dimensional gauge theory, we might naively expect to find a strings moving in four dimensions. But strings are not consistent in four flat dimensions. As we try to proceed, we are forced to introduce at least one more dimension [7]. If the gauge theory is conformal, then the original flat dimensions, together with this new extra dimension are constrained by the symmetries to form an Anti-de-Sitter spacetime. We will describe some basic properties of Anti-de-Sitter spacetimes. Then we present the simplest example of the relationship between a four dimensional field theory and a gravity theory. Namely, we discuss the relationship between Yang Mills theory with four supersymmetries to type IIB superstring theory on $AdS_5 \times S^5$ [8]. We later give the general prescription linking computations of correlation functions in the gauge theory to the computations of amplitudes in the gravity theory [9][10]. This is a general prescription that should hold for any field theory that has a gravity dual.

Finally we discuss a particular limit of the relationship between $\mathcal{N} = 4$ Yang Mills and $AdS_5 \times S^5$ where we consider particles with large angular momentum on the sphere [11]. In this limit the relevant region of the geometry looks like a plane wave where we can quantize strings exactly. Through a simple gauge theory computation one can reproduce the string spectrum.

2. Large N

There are theories that contain a large number of fields related by a symmetry such as $SO(N)$ or $U(N)$. These theories simplify when N is taken to infinity. For a more extended discussion of this subject see [12].

2.1. Large N for vector theories

Consider a theory with N fields η^i , where $i = 1, \dots, N$ with $O(N)$ symmetry. For example,

$$S = \frac{1}{2g_0^2} \int d^2\sigma (\partial\vec{n})^2, \quad \vec{n}^2 = 1 \quad (2.1)$$

First note that the effective coupling of the theory is $g_0^2 N$. The theory simplifies in the limit where we keep $g_0^2 N$ fixed and we take $N \rightarrow \infty$. In this limit only a subset of Feynman diagrams survives. A very convenient way to proceed is to introduce a Lagrange multiplier, λ , that enforces the constraint in (2.1) and integrate out the fields \vec{n} .

$$\begin{aligned} S &= \frac{1}{2g_0^2} \int d^2\sigma [(\partial\vec{n})^2 + \lambda(\vec{n}^2 - 1)] \\ S &= \frac{N}{2} \left[\log \det(-\partial^2 + \lambda) - \frac{1}{g_0^2 N} \int \lambda d^2\sigma \right] \end{aligned} \quad (2.2)$$

We get a classical theory for λ in this large N limit, so we set $\frac{\partial S}{\partial \lambda} = 0$. We get

$$\begin{aligned} 1 &= N g_0^2 \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + \lambda^2} = \frac{N g_0^2}{4\pi} \log \Lambda^2 / \lambda \\ \lambda &= \Lambda^2 e^{-\frac{4\pi}{N g_0^2}} = \mu^2 e^{-\frac{4\pi}{N g^2}} \end{aligned} \quad (2.3)$$

where g_0 is the bare coupling, by absorbing the Λ dependence in g_0 we define the renormalized coupling g . Notice that the cutoff dependence of g_0 is that of an asymptotically free theory. By looking again at (2.2) we find that the expectation value for λ in (2.3) gives a mass to the \vec{n} fields. Moreover, the model has an unbroken $O(N)$ symmetry. The fact the $O(N)$ symmetry is restored is consistent with the fact that in two dimensions we cannot break continuous symmetries. Note that the $g^2 N$ dependence of λ in (2.3) implies that the mass for \vec{n} is non-perturbative in $g^2 N$. Notice that, even though the dependence of the mass in $g^2 N$ looks non-perturbative, we have obtained this result by summing Feynman diagrams, in particular we obtained it through a one loop diagram contribution to the effective action and then balancing this term against a tree level term. Large N was crucial to ensure that no further diagrams contribute.

This theory is similar to QCD_4 since it is asymptotically free and has a mass gap. It has a large N expansion and the large N expansion contains the fact that the theory has a mass gap. This mass gap is non perturbative in $g^2 N$.

2.2. Matrix theories

Consider theories where the basic field is a hermitian matrix M . This arises, for example, in a $U(N)$ gauge theory, or a $U(N)$ gauge theory with matter fields in the adjoint representation. The Lagrangian has a schematic form

$$L = \frac{1}{g^2} \text{Tr}[(\partial M)^2 + M^2 + M^3 + \dots] = \frac{1}{g^2} \text{Tr}[(\partial M)^2 + V(M)] \quad (2.4)$$

The action is $U(N)$ invariant $M \rightarrow U M U^\dagger$. It is convenient to introduce a double line notation to keep track of the matrix indices

$$M_j^i \quad \begin{array}{c} \leftarrow \leftarrow i \\ \rightarrow \rightarrow j \end{array}$$

Fig. 1: Propagator

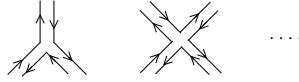


Fig. 2: Vertices

Each propagator, fig. 1, contributes a factor g^2 in the Feynman diagrams. Each vertex, fig. 2, contributes a factor of $1/g^2$. Finally, each closed line will contain a sum over the gauge index and will contribute a factor of N , see Fig. 3.

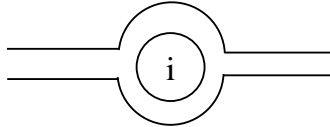


Fig. 3: Closed line contributes a factor of N .

Each diagram contributes with

$$(g^2)^{\#\text{Propagators} - \#\text{vertices}} N^{\#\text{Closed lines}} \quad (2.5)$$

We can draw these diagrams on a two dimensional surface and think of it as a geometric figure. We see that (2.5) becomes

$$N^{\#\text{Faces} - \#\text{Edges} + \#\text{vertices}} (g^2 N)^{\text{Power}} = N^{2-2h} (g^2 N)^{\text{Power}} \quad (2.6)$$

where h is the genus of the two dimensional surface. Namely, a sphere has genus $h = 0$, a torus has genus $h = 1$, etc.

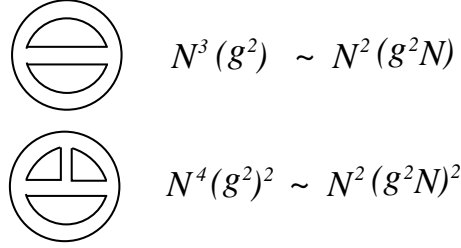


Fig. 4: Planar diagrams

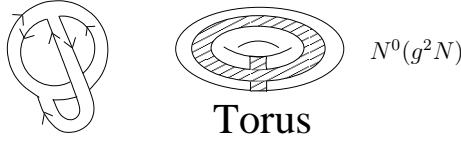


Fig. 5: A non-planar diagram

A few examples of diagrams that can be drawn on a plane or a sphere are shown in fig. 4, and example of a diagram that cannot be drawn on a sphere but can be drawn on a torus is shown in fig. 5. The sum of all planar diagrams gives

$$N^2[c_0 + c_1(g^2N) + c_2(g^2N)^2 + \dots] = N^2 f(g^2N) \quad (2.7)$$

where the c_i are numerical coefficients depending on the detailed evaluation of each Feynman graph. This detailed evaluation contains the momentum integrals. The full partition function has the form

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(g^2N) \quad (2.8)$$

The 't Hooft limit is

$$N \rightarrow \infty, \quad \lambda \equiv g^2N = \text{fixed} \quad (2.9)$$

λ is the 't Hooft coupling. In this limit only the planar diagrams contribute. As λ gets large a large number of diagrams contribute and they become dense on the sphere, so we might think that they describe a discretized worldsheet of some string theory. This worldsheet theory is defined to be whatever results from summing the planar diagrams. This argument is valid for any matrix theory. The argument does not give us a practical way of finding the worldsheet theory. In bosonic Yang Mills theory g^2 runs. In fact, the beta function has a smooth large N limit

$$\dot{\lambda} = \beta(\lambda) + o(1/N^2) \quad (2.10)$$

So we have $\lambda(E)$. The string description will be appropriate where $\lambda(E)$ becomes large. If we add matter in the fundamental, then we get diagrams with boundaries. These give rise to open strings which are mesons containing a quark and anti-quark.

Some features of QCD with $N = 3$ are similar to those of $N = \infty$, like the fact that mesons contain a quark and anti-quark, and that they have small interactions. Strings are also suggested by the existence of Regge trajectories. Namely that particles with highest spin for a given mass obey $\alpha' m^2 = J + \text{const}$. Confinement is also closely associated to a string that forms between the quark and anti-quark. Though we will see later that the string description does not necessarily imply confinement.

2.3. Large N correlators

Consider operators of the form

$$\mathcal{O} = N \text{tr}[f(M)] \quad (2.11)$$

diagrammatically represented in Fig. 6.

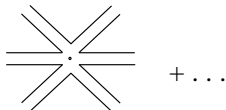


Fig. 6: Operator insertion

If we add them to the action, they have the same scaling as an extra interaction vertex. In the large N limit their correlation functions factorize

$$\langle \text{tr}[f_1(M)] \text{tr}[f_2(M)] \rangle = \langle \text{tr}[f_1(M)] \rangle \langle \text{tr}[f_2(M)] \rangle + o(1/N^2) \quad (2.12)$$

Notice that this implies that the leading contribution is a disconnected diagram. All connected correlation functions of operators normalized as in (2.11) go like N^2 . This means that

$$\langle \mathcal{O}\mathcal{O} \rangle_c \sim N^2, \quad \langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle_c \sim N^2, \quad \frac{\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle_c}{\langle \mathcal{O}\mathcal{O} \rangle_c^{3/2}} \sim \frac{1}{N} \quad (2.13)$$

where the subscript indicates the connected part. In the string description the insertion of these operators corresponds to the insertion of a vertex operator on the string worldsheet.

An interesting operator in gauge theories is the Wilson loop operator

$$W(\mathcal{C}) = N \text{Tr}[P e^{\oint_{\mathcal{C}} A}] \quad (2.14)$$

For a contour of large area the expectation value of this operator should go like $e^{-T(\text{Area})}$ for a confining theory. T is then the string tension.

3. Guessing the string theory

Rather than summing all Feynman diagrams one would like to guess what the final string theory description is. Naively, for $d = 4$ Yang Mills we expect to get a bosonic string theory that lives in four dimensions. We know this is not correct. The bosonic string is not consistent quantum mechanically in $d = 4$. It is consistent in $d = 26$ flat dimensions, but this is not the theory we are interested in.

The reason for this inconsistency is that the classical Polyakov action

$$S \sim \int \sqrt{g} g^{ab} \partial_a X \partial_b X \quad (3.1)$$

has a Weyl symmetry $g_{ab} \rightarrow \Omega g_{ab}$ which is not a symmetry quantum mechanically. In the quantum theory, under a change metric of the form $g_{ab} = e^\phi \hat{g}_{ab}$ the partition function

$$e^{-S_{eff}(g)} = \int DX D(bc) e^{-S[X,g] - S[b,c,g]} \quad (3.2)$$

changes as

$$S_{eff}(g) - S_{eff}(\hat{g}) = \frac{(26 - D)}{48\pi} \int \frac{1}{2} (\hat{\nabla} \phi)^2 + \hat{R}^{(2)} \phi + \mu^2 e^\phi \quad (3.3)$$

This action for ϕ is called ‘‘Liouville’’ action. Even though the initial classical action for the conformal factor in the metric was zero, a non-trivial action was generated quantum mechanically. Integrating over ϕ is like adding a new dimension.

For $D \leq 1$ this is the right answer. We start with a matrix integral or a matrix quantum mechanics and we get a string in one or two dimensions. Actually, it is necessary to do a particular scaling limit in the matrix quantum mechanics which involves $N \rightarrow \infty$ and a tuning of the potential that is analogous to taking the 't Hooft coupling to a region where there is a large number of Feynman diagrams that contribute, see [13].

For $D = 4$ it is not known how to quantize the Liouville action. Nevertheless the lesson we extract is that we need to include at least one extra dimension. So we introduce an extra dimension and look for the most general string theory. If we are interested in four dimensional gauge theories we look for strings in five dimensions. We need to specify the space where string moves. It should have 4d Poincare symmetry, so the metric has the form

$$ds^2 = w(z)^2 (dx_{1+3}^2 + dz^2) \quad (3.4)$$

we have used the reparametrization symmetry to set the coefficient of dz^2 equal to that of dx^2 .

Now suppose that we were dealing with a scale invariant field theory. $\mathcal{N} = 4$ Yang Mills is an example. Then

$$x \rightarrow \lambda x \tag{3.5}$$

should be a symmetry. But in string theory we have a scale, set by the string tension. So the only way that a string (with the usual Nambu action¹) could be symmetric under (3.5) is that this scaling is an isometry of (3.4). This means that $z \rightarrow \lambda z$ and that $w = R/z$. So we are dealing with 5 dimensional Anti-de-Sitter space

$$ds^2 = R^2 \frac{dx^2 + dz^2}{z^2} \tag{3.6}$$

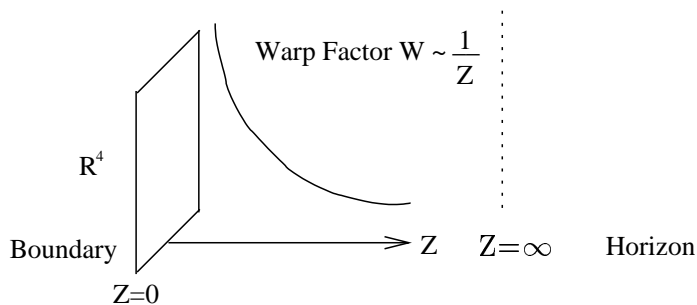


Fig. 7: A sketch of Anti-de-Sitter space. We emphasize the behavior of the warp factor.

This is a spacetime with constant negative curvature and it is the most symmetric spacetime with negative curvature. The most symmetric spacetime with positive curvature is de-Sitter. In Euclidean space the most symmetric positive curvature space is a sphere and the most symmetric negative curvature one is hyperbolic space. These are the Euclidean continuation of de-Sitter and Anti-de-Sitter respectively.

3.1. Conformal symmetry

A local field theory that is scale invariant is usually also conformal invariant. The change in the action due to a change in the metric is

$$\delta S = \int T^{\mu\nu} \delta g_{\mu\nu} \tag{3.7}$$

¹ It is possible to write a string action that is conformal invariant in four dimensions [14] but it is not know how to quantize it.

Under a coordinate transformation $x^\mu \rightarrow x^\mu + \zeta^\mu(x)$ the action changes by (3.7) with

$$\delta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu \quad (3.8)$$

If ζ^μ generates an isometry then the metric is left invariant, so that we have $\delta g_{\mu\nu} = 0$ in (3.8). The scale transformation (3.5) gives $\delta g_{\mu\nu} = 2\delta\lambda g_{\mu\nu}$. The action would be invariant if $T^\mu_\mu = 0$. In this case the action is also invariant under coordinate transformations such that $\delta g_{\mu\nu} = h(x)g_{\mu\nu}$ in (3.8), and h is any function. Coordinate transformations of this type are called conformal transformations. In $d=4$ they form the group $SO(2,4)$. This group is obtained by adding to the Poincare group the scale transformation and the inversion $\vec{x} \rightarrow -\vec{x}/x^2$. We see that the inversion maps the origin to infinity. It turns out that the conformal group acts more nicely if we compactify the space and we consider $S^3 \times R$ in the Lorentzian case or S^4 in the Euclidean case.

Note also that if the trace of the stress tensor is zero, then the theory is also Weyl invariant, it is invariant under a rescaling of the metric $g \rightarrow \Omega^2 g$. In the quantum case this symmetry will have a calculable anomaly and one can find the change in physical quantities under such a rescaling.

3.2. Isometries of AdS

In order to see clearly the AdS isometries we write *AdS* as a hypersurface in $R^{2,4}$

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = -R^2 \quad (3.9)$$

Note that even though the ambient space has 2 time directions the surface contains only one time direction, the other is orthogonal to the surface. You should not be confused by these two times, AdS is an ordinary Lorentzian space with one time!. We recover (3.6) after writing $X_{-1} + X_4 = R/z$, $X_\mu = Rx_\mu/z$ for $\mu = 0, \dots, 3$. By choosing an appropriate parameterization of (3.9) we can also write the metric

$$ds^2 = R^2[-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2] \quad (3.10)$$

These are called ‘‘global’’ coordinates. They cover the whole AdS space. In contrast the ‘‘Poincare’’ coordinates in (3.6) only cover a portion. Note that translations in τ correspond to rotations of X_{-1} and X_0 in (3.9). So in the construction based on (3.9) we would get closed timelike curves. Fortunately we can go to the covering space and consider (3.10)

with τ non-compact. When we talk about AdS we are always going to think about this covering space.

In the metric (3.10) we can take out a factor of $\cosh^2 \rho$, and define a new coordinate $dx = d\rho/\cosh \rho$. We see that the range of x is finite. This allows us to understand the Penrose diagram of AdS . It is a solid cylinder whose boundary is $S^3 \times R$ where R corresponds to the time direction. The field theory will be defined on this boundary. On this boundary the isometries of AdS act like the conformal group acts in four dimensions. The proper distance to the boundary along a surface of constant time is infinite.

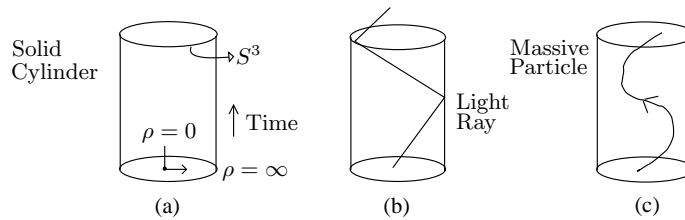


Fig. 8: (a) Penrose diagram of AdS. (b) Trajectory of a light ray. (c) Trajectory of massive geodesics.

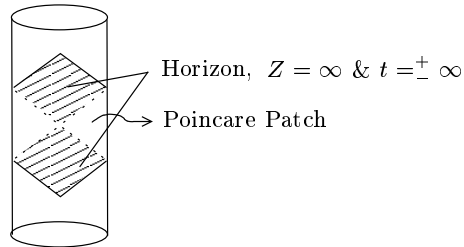


Fig. 9: The coordinates of (3.6) cover only the region of global AdS contained between the two shaded hyperplanes. These hyperplanes correspond to the horizons at $z = \infty$ and $t = \pm\infty$.

Finally note that Weyl transformations in the 4d theory correspond to picking different functions as conformal factors when we compute the Penrose diagram so that the boundary will have different metrics which differ by an overall function of the coordinates on the boundary.

3.3. Mapping of states and operators

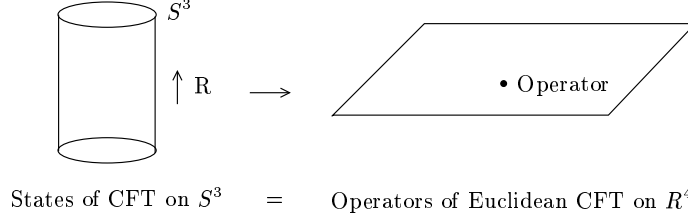


Fig. 10: We can map states of the field theory on $S^3 \times R$ to operators on R^4 .

In a CFT we have a correspondence between operators on R^4 and states on the cylinder $S^3 \times R$. This can be seen as follows. We start with a state on the cylinder, we go to Euclidean time and then notice that the cylinder and the plane differ only by a Weyl transformation so that the two theories are related. The vacuum on the cylinder corresponds to the identity on the plane. The energy of the state in the cylinder corresponds to the conformal dimension of the operator on the plane, $E_{cyl} = \Delta$.

3.4. $\mathcal{N} = 4$ $U(N)$ Yang Mills and strings on $AdS_5 \times S^5$

Consider a theory with four supersymmetries in 4 dimensions, namely sixteen real supercharges. This theory has a unique field content, there is a unique supermultiplet. Our only freedom is the choice of gauge group and coupling constants. The field content is as follows. One vector field or gauge boson A_μ , six scalars ϕ^I $I = 1, \dots, 6$ and four fermions $\chi_{\alpha i}, \chi_{\dot{\alpha} \bar{i}}$, where α and $\dot{\alpha}$ are four dimensional chiral and anti-chiral spinor indices respectively and $i = 1, 2, 3, 4$ is an index in the $\mathbf{4}$ of $SU(4) = SO(6)$ and \bar{i} in the $\bar{\mathbf{4}}$. (The $\mathbf{4}$ is the spinor of $SO(6)$). The theory has a global $SO(6)$ symmetry. This symmetry does not commute with the supercharges, since different components of the multiplet have different $SO(6)$ quantum numbers. In fact, the supercharges are in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$. A symmetry that does not commute with the supercharges is called an ‘‘R’’ symmetry. Note that $SU(4)$ is a chiral symmetry.

The Lagrangian is schematically of the form

$$L = \frac{1}{g^2} Tr \left[F^2 + (D\phi)^2 + \bar{\chi} \not{D} \chi + \sum_{IJ} [\phi^I \phi^J]^2 + \bar{\chi} \Gamma^I \phi^I \chi \right] + \theta Tr[F \wedge F] \quad (3.11)$$

It contains two parameters, the coupling constant and a theta angle. The theory is scale invariant quantum mechanically. Namely the beta function is zero to all orders. So it is also conformal invariant. The extra conformal symmetries commuted with the 16 ordinary

supersymmetries give 16 new supersymmetries. In any conformal theory we have this doubling of supersymmetries.

The theory has an S-duality under which

$$\tau_{YM} = \frac{\theta}{2\pi} + i \frac{2\pi}{g_{YM}^2} \quad (3.12)$$

transforms into $-1/\tau$. This combines with shifts in the θ angle into $SL(2, Z)$ acting on τ as it usually acts on the upper half plane. The 't Hooft coupling is $\lambda = g_{YM}^2 N$.

3.5. IIB strings on $AdS_5 \times S^5$.

Suppose that the radius is large. We will later find under which conditions this is true. Then we are looking for a solution of the type IIB supergravity equations of motion. These equations follow from the action²

$$S \sim \int \sqrt{g} R + F_5^2 \quad (3.13)$$

plus the self duality constraint for the fiveform field strength, $F_5 = *F_5$, which has to be imposed by hand. Due to the existence of $D3$ branes the flux of F_5 is quantized.

$$\int_{S^5} F_5 = N \quad (3.14)$$

Choosing a fiveform fieldstrength proportional to the volume form on S^5 plus the volume form on AdS_5 we find that $AdS_5 \times S^5$ is a solution. The radius R of the sphere and the radius of AdS are

$$R = (4\pi g_s N)^{1/4} l_s \sim N^{1/4} l_{pl} \quad (3.15)$$

where g_s is the string coupling and $2\pi l_s^2$ is the inverse of the string tension. It is clear from (3.13) that the radius in Planck units should have this form since the F_5^2 term in the action scales like N^2 , while the first term in (3.13) scales like R^8 . The equations of motion will balance these two terms giving (3.15).

It is also amusing to understand the energetics that gives rise to the negative cosmological constant in AdS_5 . For this purpose consider a compactification of the ten dimensional theory on S^5 to five dimensions with a fiveform fieldstrength on an S^5 of radius r . Then the five dimensional action is schematically

$$S = \int r^5 \sqrt{g_5} R^{(5)} - \sqrt{g_5} N^2 r^{-5} + \sqrt{g_5} r^3 = \int \sqrt{g_E} R_E - V(r) \quad (3.16)$$

with $V(r) = r^{-25/3} (N^2 r^{-5} - r^3)$

² We have suppressed the dependence of the action on the fields that are not important for our purposes.

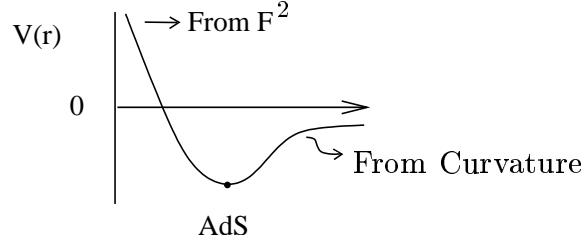


Fig. 11: Effective potential after compactifying ten dimensional supergravity on a five sphere with a flux of the fiveform field strength.

The second term comes from the flux and the third term come from the curvature of S^5 . Note that the sign in Einstein's action is such that an internal space with positive curvature gives rise to a negative contribution to the energy in the non-compact dimensions. The potential should be computed in 5d Einstein frame. This potential goes to zero as $r \rightarrow \infty$, which is a general feature of KK compactification. One can also see that there is a minimum that balances the two pieces with an r as in (3.15).

3.6. $\mathcal{N} = 4$ YM is the same as IIB on $AdS_5 \times S^5$

Now we want to relate the two theories we have just talked about. The general reason that they could be related is that in the 't Hooft limit we expect strings. This string will move in a space that has more than four dimensions. The field theory has 32 supersymmetries which is the same as the number of supersymmetries of type IIB string theory on this background. In fact, the two supergroups are the same. So it is reasonable that the two theories could be related.

There is an argument that relates these two theories which relies on looking at the near horizon geometry of D3 branes. The field theory on N D3 branes is $\mathcal{N} = 4$ U(N) Yang Mills at low energies. The near horizon geometry of $D3$ branes is $AdS_5 \times S^5$. Since excitations that live near the horizon have very small energies from the point of view of the outside observer we conclude that at low energies only these excitations will survive. So in the low energy limit we have two alternative descriptions which should be equivalent [8].

The coupling constants of YM are related to the string coupling and vev of the RR scalar

$$\frac{i}{g_{YM}^2} + \frac{\theta}{2\pi} = \tau_{YM} = \tau = \frac{i}{g_s} + \chi \quad (3.17)$$

This notation emphasizes that both theories have an $SL(2,Z)$ duality symmetry.

The relation between the two theories is a “duality”. There is a parameter, $g_{YM}^2 N$, such that in the region where it is very small one description (the Yang Mills one) is weakly coupled and the other (gravity) is strongly coupled, while the opposite is true when this parameter is large. Let us expand on this point. The gravity description is a good approximation to string theory if the radius of the space is much larger than l_s , since l_s is the intrinsic size of the graviton. We see from (3.15) (3.17) that this happens when $g_{YM}^2 N \gg 1$. It is good that the two weakly coupled descriptions are non-overlapping. Otherwise we would have blatant contradictions since the two theories have rather different properties in their respectively weakly coupled regimes. This fact also makes the conjecture hard to disprove, or hard to prove. In this supersymmetric case there are some quantities that are independent of the coupling which can be computed on both sides. Checking that these quantities agree we have checks of the duality. For a more detailed discussion of these checks see [4].

Finally note that α' is not a parameter in string theory so all physical quantities depend only on the size of AdS in string units. A useful way to think about it is to choose units where the radius $R = 1$, then $\alpha' = \frac{1}{\sqrt{g_{YM}^2 N}}$. All gravity computations depend only N , but not on $g_{YM}^2 N$. The reason is that if we write the action in Einstein frame then we have an overall factor of $l_{pl}^{-8} \sim N^2$. Then the α' expansion is an expansion in terms of $1/\sqrt{g_{YM}^2 N}$.

4. Establishing the dictionary

4.1. Correlation functions

We now consider correlation functions. We focus on the Euclidean case. The Euclidean CFT is dual to Euclidean AdS which is the same as hyperbolic space. We can write coordinates as in (3.6) where now dx^2 denotes the metric on R^4 .

We can evaluate the gravity partition function as a function of the boundary values of the fields. Since AdS has a boundary we need to specify the boundary conditions for the fields. The value of the partition function depends on these boundary conditions. If the gravity theory is weakly coupled we can approximate this by the value of the classical action

$$\mathcal{Z}_{bulk} [\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})] \sim e^{-N^2 S_{class}[\phi] + o(\alpha')} \times (\text{Quantum Corrections}) \quad (4.1)$$

We emphasized the fact that the classical gravity computation will always depend on N through this overall factor in the action and will be independent of $g^2 N$. Stringy corrections will correct the gravity action into the classical string action, whose form we do not know, but we know that it will have an expansion in powers of $\alpha' = \frac{1}{\sqrt{g^2 N}}$. Some terms in this expansion are known. For example, there is a well studied R^4 correction to the ten dimensional action.

For each field in the 5 dimensional bulk, we have a corresponding operator in the dual field theory. In general, figuring out which operator corresponds to which field is hard. But for some special operators it is easy due to their symmetries. For example, the graviton is associated to the stress tensor operator in the boundary theory. Similarly the dilaton is related to the Lagrangian of the theory, since we saw that the coupling is related to the dilaton and a change in coupling adds an operator proportional to the Lagrangian.

The *AdS/CFT* dictionary says that the quantity appearing in (4.1) is equal to the generating function of the correlation functions of the corresponding operators.

$$\mathcal{Z}_{bulk} [\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})] = \langle e^{\int d^4 x \phi_0(\vec{x}) \mathcal{O}(\vec{x})} \rangle_{\text{Field Theory}} \quad (4.2)$$

Note that $\phi_0(\vec{x})$ is an arbitrary function specifying the boundary values of the bulk field ϕ . Taking derivatives with respect to ϕ_0 and setting it to zero we obtain the correlation functions of the operator.

So the final conclusion is that changes in the boundary conditions of *AdS* correspond to changes in the Lagrangian of the field theory. Infinitesimal changes in the boundary condition correspond to the insertion of an operator.

Now let us discuss this more explicitly. We consider the metric (3.6). Consider a scalar field of mass m . Its action is

$$S = N^2 \int \frac{dx^4 dz}{z^5} [z^2 (\partial\phi)^2 + m^2 R^2 \phi^2] \quad (4.3)$$

The action contains higher order terms in ϕ . For the moment we focus on the quadratic terms.

In order to evaluate the classical action on a classical solution we need to solve the classical equations

$$z^3 \partial_z \left(\frac{1}{z^3} \partial_z \phi \right) - p^2 z^2 \phi - m^2 R^2 \phi = 0 \quad (4.4)$$

We have used translation symmetry to go to Fourier space for the R^4 part. This equation can be solved exactly in terms of Bessel functions. For the moment let us just understand

the behavior of the solutions near the boundary of AdS , at $z \sim 0$. We then look for solutions of (4.4) in terms of powers of the form $\phi \sim z^\alpha$, where α obeys the equation

$$\begin{aligned}\alpha(\alpha - 4) - m^2 R^2 &= 0 \\ \alpha_\pm &= 2 \pm \sqrt{4 + m^2 R^2}\end{aligned}\tag{4.5}$$

The solution with α_- dominates near $z \rightarrow 0$. The solution with α_+ always decays when $z \rightarrow 0$. We will impose the boundary condition on the dominating solution. More precisely we choose a boundary condition of the form

$$\phi(x, z)|_{z=\epsilon} = \epsilon^{\alpha_-} \phi_0^r(x)\tag{4.6}$$

In general we need to impose boundary conditions at $z = \epsilon$ and then take $\epsilon \rightarrow 0$ at the end of the computation. We see that if we keep $\phi_0^r(x)$ fixed as we take $\epsilon \rightarrow 0$ the solution in the bulk, at some fixed z will have a finite limit. So we call this $\phi_0^r(x)$ the “renormalized” boundary condition. This is related to the fact that in the field theory we need to renormalize the operator.

Under a rescaling of coordinates in the field theory, which in AdS is the isometry $x \rightarrow \lambda x$, $z \rightarrow \lambda z$, the original field ϕ does not get scaled, but due to the ϵ factor in (4.6) we see that ϕ_0^r has dimension α_- . Since we interpret the resulting gravity formulas through (4.2), where on the right hand side we have $\phi_0^r(x)$, we conclude that the dimension of the corresponding operator is

$$\begin{aligned}\Delta &= 4 - \alpha_- = \alpha_+ \\ \Delta &= 2 + \sqrt{4 + (mR)^2}\end{aligned}\tag{4.7}$$

Note that for the dilaton, which has $m = 0$, we get the correct dimension, $\Delta = 4$. Since the theory is exactly conformal for all values of g_{YM} the operator that changes infinitesimally the Yang Mills coupling should have dimension four. Similarly, the fact that the graviton is massless is related to the fact that the stress tensor has dimension four. This last fact does not depend on supersymmetry and is very general, valid for any local conformal QFT.

Note that in order to make the operator to field correspondence it is necessary to Kaluza-Klein reduce all fields to five dimensions. So when we talked about the dilaton, we were referring to the $l = 0$ mode on the five-sphere which gives rise to a massless field on AdS . A mode with angular momentum l on the five-sphere has mass given by $m^2 = l(l + 4)/R^2$ which leads to

$$\Delta = 4 + l.\tag{4.8}$$

The corresponding operators are roughly of the form [9,10]

$$Tr[F^2 \phi^{(I_1} \dots \phi^{I_l)}] \tag{4.9}$$

where the indices I_i are taken in a symmetric traceless combination which corresponds to the representations of the spherical harmonics. In (4.9) one should order the fields appropriately and it is also necessary to introduce the fermionic fields [15]. The dimension of (4.9) can be computed easily at weak coupling, we just sum the dimensions of the individual fields to get $\Delta = 4 + l$. We see that we get the same as the strong coupling result (4.8). The reason is that the operators (4.9) are in protected multiplets. A protected multiplet is a multiplet of supersymmetry that is smaller than the generic multiplet. Such multiplets have dimensions which are given in terms of their SO(6) quantum numbers. Therefore, such operators cannot have coupling dependent anomalous dimensions.

It turns out that in the case of $\mathcal{N} = 4$ YM all the operators that are in protected multiplets correspond to all the KK modes of the gravity fields. This provides a nice match since we have the same number of protected states on both sides. This matching goes beyond the matching of symmetries of the two theories. In fact, we could have obtained extra protected operators in the YM theory, for example. This would have killed the conjecture since we do not have any other light states in supergravity. As an example, note that if we changed the gauge group to SO(N) instead of U(N) then we have we do not have all the operators (4.9), we only have the ones with even l . In fact, the SO(N) theory corresponds to an orientifold of $AdS_5 \times S^5$ that maps antipodal points on S^5 [16]³.

Note that not all operators are protected. For example, the operator $tr[\phi^I \phi^I]$ has dimension two at weak coupling but there is no corresponding operator with dimension two at large coupling. What is its dimension at strong coupling?. As we discussed, all supergravity modes have dimensions that remain fixed as $g^2 N \rightarrow \infty$. But a string theory also contains massive string states, with masses $m \sim 1/l_s$, which according to (4.7) correspond to operators of dimension $\delta \sim R/l_s \sim (g_{YM}^2 N)^{1/4}$. So the dimension of this operator should at least be of this order of magnitude at strong 't Hooft coupling. In fact, the Yang Mills theory contains many operators with higher spin, like $tr[\phi^I \partial_{(\mu_i} \dots \partial_{\mu_s)} \phi^i]$. These operators have dimension $2 + s$ at weak coupling but they should also get dimensions of order $(g_{YM}^2 N)^{1/4}$ at strong coupling since the gravity theory only contains fields of spin

³ There are four distinct orientifolds, one gives SO(2N), one gives SO(2N+1) and the other two give two versions of the Sp(N) theory [16].

less than two. In any theory that has a weakly coupled gravity dual, with a radius of curvature much bigger than the string scale, operators with higher spin should have large dimensions.

The problem of solving the Laplace equation with fixed boundary condition is rather familiar from electrostatics. It is useful to introduce the bulk to boundary propagator, which is the solution of the problem where we put a boundary condition that is a delta function at a point on the boundary. In this context this bulk to boundary propagator is [10]

$$G_{\Delta}(z, \vec{x}, \vec{x}') = \frac{z^{\Delta}}{[(\vec{x} - \vec{x}')^2 + z^2]^{\Delta}} \quad (4.10)$$

We can use this to compute connected correlation functions. For example if we wanted to compute the connected three point function then we would have to include possible cubic terms in the action for the scalar field

$$S = N^2 \int (\nabla\phi)^2 + m^2 R^2 \phi^2 + \phi^3 + \dots \quad (4.11)$$

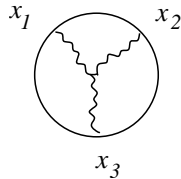


Fig. 12: Diagram contributing to a three point function. The vertex is a ϕ^3 interaction in the bulk theory. The lines going to the boundary are bulk to boundary propagators.

We need to evaluate the diagram in fig. 12. This leads to the following expression for the three point function

$$\int \frac{d^4 x dz}{z^5} G_{\Delta_1}(z, x, x_1) G_{\Delta_2}(z, x, x_2) G_{\Delta_3}(z, x, x_3) \quad (4.12)$$

with G as in (4.10). This integral gives, of course, the x dependence for a three point function that we expect on the basis of conformal invariance. Remember that in a conformal invariant theory the two and three point function are given by

$$\langle \mathcal{O}\mathcal{O} \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \quad (4.13)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{C_{123}}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3} |x_3 - x_2|^{\Delta_3 + \Delta_2 - \Delta_1} |x_1 - x_3|^{\Delta_1 + \Delta_3 - \Delta_2}} \quad (4.14)$$

Indeed the integral (4.12) gives (4.14), with $\Delta_i = \Delta$ [17].

Normally we can normalize the two point functions to one. There are some special two point functions for which the normalization is unambiguously defined. An example is the two point function of the stress tensor. This two point function is proportional to N^2 with a coefficient that agrees in gravity and field theory. In fact this coefficient does not depend on the coupling due to a supersymmetry argument [18].

4.2. Various remarks

Note that when we solve the equation (4.4) we need to impose two boundary conditions since it is a second order equation. One is the one we discussed so far. The second is that it should vanish as $z \rightarrow \infty$. This corresponds to the statement that there is nothing special happening at infinity. In Euclidean AdS the point $z = \infty$ is actually a point on the boundary. In general, when we solve the equation we need to impose the condition that the solution is not singular in the interior, this gives us the second boundary condition.

Note also that the renormalization in (4.6) is independent of p (or x). This is related to the fact that the dual theory is an ordinary local quantum field theory where the renormalization of an operator does not depend on the momentum. In other theories such as in linear dilaton backgrounds, or in non-commutative field theories, this is no longer true [19].

The fact that the theory on the boundary is local implies that the theory in the bulk should contain gravity. By the word ‘‘local’’ we mean that the theory contains a stress tensor. This in turn means that the theory can be defined on any manifold.

It is possible to have fields with negative mass squared in AdS_5 as long as the mass obeys

$$m^2 R^2 \geq -4 \quad (4.15)$$

These tachyons do not lead to instabilities and actually appear in $AdS_5 \times S^5$. They do not lead to instabilities because normalizable wavefunctions in global AdS, have positive energy. This is due to the fact that the wavefunction has to decrease as $\rho \rightarrow \infty$, which implies that it should have some kinetic energy which overwhelms the negative potential energy. These fields correspond to relevant operators. In fact, one can see that α_- is positive. This means that the perturbation they induce, which goes as $\phi \sim z^{\alpha_-}$, decreases

as $z \rightarrow 0$, which is the UV of the field theory⁴. Note also that a field of zero mass corresponds to a marginal deformation and in this case $\alpha_- = 0$ so that the perturbation produced by the operator is independent of z . Finally, if $m > 0$, the perturbation increases as $z \rightarrow 0$. This corresponds to an irrelevant operator.

The conformal group has a very special representation at $\Delta = 1$. Unitarity implies that the corresponding operator corresponds to a free field in R^4 . These operators arise in the $U(1)$ factor of the field theory. This is the $U(1)$ in $U(N)$, the operator is $Tr[\phi^I]$. These representations of the conformal group are called singletons. They are special because they do not correspond to an ordinary field propagating in the bulk. They should have no local degrees of freedom in the bulk, only on the boundary. Another example is the operator $Tr[F_{\mu\nu}]$. It corresponds in the bulk to the $l = 0$ modes of the RR and NS $B_{\mu\nu}$ fields. These fields have a long distance action governed by

$$N \int B^{NS} \wedge dB^{RR} \quad (4.16)$$

This is purely topological and gives rise to a field on the boundary if we put local boundary conditions [21] [22].

4.3. Physics of the warp factor

Let us try to understand the role of the radial coordinate z in the bulk theory. It is very important that the metric contains a redshift factor, or warp⁵ factor that multiplies the metric

$$ds^2 = w(z)^2(dx^2 + dz^2), \quad w(z) = \frac{1}{z} \quad (4.17)$$

The distances and times in the field theory are measured with the coordinates x . On the other hand proper times and proper distances in the bulk are related to the field theory coordinates by the warp factor $w \sim 1/z$. So a given object in the bulk, such as a

⁴ For some of these fields one sometimes effectively imposes the boundary conditions on the solution that decreases faster so that their dimension is equal to α_- instead of (4.7)[20]. This change in the boundary condition is not needed for $\mathcal{N} = 4$ YM but it is needed for some other theories.

⁵ A redshift factor multiplies dt and a warp factor is basically a redshift factor which multiplies several spacetime coordinates so that we have Poincare invariance.

massive string state, corresponds to field theory configurations of different size and energy depending on the value of the z coordinate of the bulk object. We have

$$E_{FT} = w(z)E_{proper} \quad (4.18)$$

$$(\text{size})_{FT} = \frac{1}{w(z)}(\text{proper size}) \quad (4.19)$$

where the $_{FT}$ subindex indicates a quantity in the field theory.

So we see that as we go to $z \rightarrow 0$ we have very small sizes in the field theory and very high energies. This is the UV of the field theory. Notice that going to small z corresponds to going to large distances from a point in the interior. In fact UV divergences in the field theory are related to IR divergences in the gravity theory [23]. The fact that short distances in the field theory correspond to long distances in the gravity description is called the “IR/UV correspondence”.

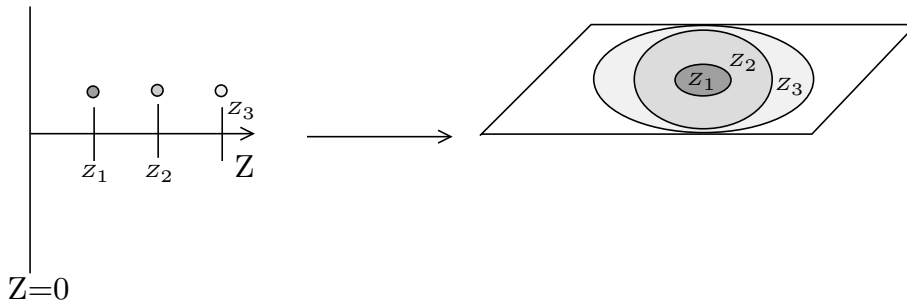


Fig. 13: The same bulk object at different z positions corresponds to an object in the CFT with different sizes.

The physics of the warp factor is also responsible for pointlike, or partonic, behavior of scattering amplitudes [24].

5. Thermal aspects

Consider black holes in AdS . The five sphere will not play an important role so we will not write it explicitly. In Poincare coordinates the simplest black hole is a black brane which is translation invariant along the three spatial directions of the boundary. The metric has the form

$$ds^2 = R^2 \frac{1}{z^2} \left[-\left(1 - \frac{z^4}{z_0^4}\right) dt^2 + d\vec{x}^2 + \frac{dz^2}{\left(1 - \frac{z^4}{z_0^4}\right)} \right] \quad (5.1)$$

The temperature is easily determined by going to Euclidean time and choosing a periodicity for Euclidean time to that there is no singularity at $z = z_0$. We see that this gives $\beta = \pi z_0$.

We can compute the entropy using the Bekenstein-Hawking formula $S = \frac{(\text{Area})}{4G_N}$ and we get the entropy per unit volume [25]

$$\frac{S}{V} = \frac{\pi^2}{2} N^2 T^3 \quad (5.2)$$

The weakly coupled field theory has entropy per unit volume

$$\frac{S}{V} = \frac{4}{3} \frac{\pi^2}{2} N^2 T^3 \quad (5.3)$$

Note that the temperature dependence is determined by conformal invariance. Note also that these black branes have positive specific heat.

The N dependence (for fixed $g^2 N$) is as we expect from large N counting. In principle one might have expected extra $g^2 N$ dependence at strong coupling, but we see that at very strong coupling the answer (5.2) becomes independent of $g^2 N$. Remember that we said that all gravity computations are independent of $g^2 N$. We see that the numerical coefficient is different. This is not surprising since this is not a protected computation and the answer could depend on the coupling $g^2 N$. In fact the leading $g^2 N$ correction to (5.3) has been computed [26]. The first α' correction to the gravity result (5.2) has also been computed, it comes from an R^4 term and is proportional to $\alpha'^3 \sim (g^2 N)^{-3/2}$ [27]. Both corrections go in the right direction but nobody has computed yet the whole interpolating curve. In other words, we expect a behavior of the form

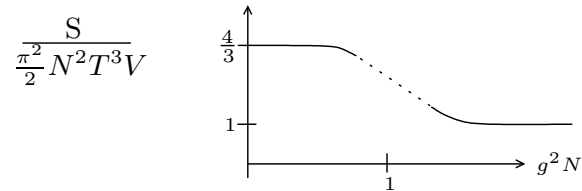


Fig. 14: Form of the Yang Mills free energy as a function of the 't Hooft coupling in the large N limit. The dotted line is a naive interpolation between the weakly coupled and the strongly coupled results.

If we consider AdS in global coordinates then we can consider solutions that correspond to localized black holes sitting at the center. If the Schwarzschild radius of these black holes is very large, larger than the radius of AdS , then they behave as the black branes we discussed above, but with the R^3 directions replaced by S^3 . On the other hand if their radius is much smaller than the radius of AdS they behave more as ordinary Schwarzschild black holes in flat space. They have negative specific heat and are unstable. It would be nice to find their precise description in the gauge theory.

5.1. Wilson Loops

A very interesting operator in gauge theories is

$$W(\mathcal{C}) = \text{Tr}[P e^{\oint_{\mathcal{C}} A}] \quad (5.4)$$

where \mathcal{C} is a closed curve in R^4 . The trace can be taken in any representation, but we will take it in the fundamental representation. This operator represents the Yang Mills contribution to the propagation of a heavy quark in the fundamental representation.

The general large N counting arguments that we reviewed above tell us that a quark in the fundamental will have a string ending on it. So we expect to have a string worldsheet with a boundary along the contour \mathcal{C} . As usual this worldsheet lives in 5 dimensions and ends on the boundary along \mathcal{C} .

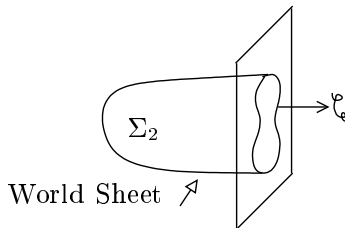


Fig. 15: String worldsheet ending on the contour \mathcal{C} corresponding to the trajectory of a heavy external quark.

In $AdS_5 \times S^5$ we also have to specify at what point of S^5 the string is sitting when it approaches the boundary. In fact the operator that corresponds to a string at a specific point on S^5 has the form [28] [29]

$$W = \text{tr}[P e^{\oint A_\mu dx^\mu + |dx| \phi^I(x) \theta^I}] \quad (5.5)$$

where ϕ^I are the scalar fields of $\mathcal{N} = 4$ YM and θ is a unit vector, $\theta^2 = 1$, which specifies a point on S^5 .

For large $g^2 N$ the leading contribution to the expectation value of the Wilson operator is of the form

$$\langle W \rangle \sim e^{-T(\text{Area})} \sim e^{-\sqrt{g^2 N} (\text{Area})_{R=1}} \quad (5.6)$$

Note that we have a factor of $1/\alpha'$ in the exponent. This has explicit $\sqrt{g^2 N}$ dependence because it involves a string, it goes beyond the supergravity fields.

The area in (5.6) is the proper area in the five (or ten) dimensional space. It is infinite since we have already seen that the proper distance to the boundary is infinite, as is the area. We need to regularize the expression and compute the area up to $z = \epsilon$. Then we find that the area goes as

$$\text{Area} = \frac{(\text{length})}{\epsilon} + A_r + o(\epsilon) \quad (5.7)$$

where A_r is finite and can be called the renormalized area. The divergent term is proportional to the length of the contour \mathcal{C} . This just renormalizes the mass of the external quark.

The simplest example is a circular contour where we get that A_r is a negative constant. The result is independent of the size of the circle due to conformal invariance. In fact, for a circular contour there is a trick that enables us to do the exact computation [30] [31]. This trick is based on the observation that the straight line would give zero since it corresponds to a BPS state. But the circle is related to the straight line by a conformal transformation. The mapping of the point at infinity is subtle so all the contribution comes from an anomaly in the transformation [31]. From this exact answer one can see that in the limit of large $g^2 N$ we get the right answer.

Another simple example is the computation of a quark anti-quark potential which can be obtained by considering a rectangular contour. This gives

$$V = -c_1 \frac{\sqrt{g_{YM}^2 N}}{L} \quad (5.8)$$

where c_1 is a numerical constant. The dependence on L follows from conformal invariance. The weak coupling result is

$$V = -c_2 \frac{g_{YM}^2 N}{L} \quad (5.9)$$

where c_2 is a numerical constant. Even though the Yang Mills theory has a string description the theory is not confining. (We will later see examples of confining theories). The reason this happens is because the string moves in five dimensions. The Wilson loop obeys the area law, but in 5-dimensions!

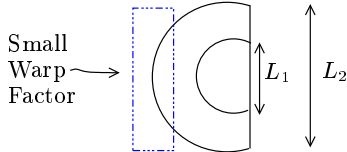


Fig. 16: String configurations relevant for computing the quark-anti-quark potential. As we separate the quarks the string moves into a region with smaller warp factor, which makes the renormalized area smaller.

The potential decreases when we increase L because the string moves into the region of large z where the warp factor is smaller, so that its proper renormalized area is smaller, see fig. 16.

6. Confining theories

Here we will consider the simplest example of a confining theory [32]. We start with 3+1 dimensional $\mathcal{N} = 4$ Yang Mills and compactify it on a circle of radius r_y down to 2+1 dimensions. So the theory is on the space $R^{2+1} \times S^1$. If we choose antiperiodic boundary conditions on the circle we will break supersymmetry. From the 2+1 dimensional point of view the fermions will be massive. Through quantum corrections, which are large if $g^2 N$ is large, the bosons will also get a mass. So the only massless fields will be the gauge bosons. So we have a pure Yang Mills theory in 2+1 dimensions at low energies. This theory is confining.

The supergravity description of this theory can be obtained in a simple way from our previous solution (5.1). This Euclidean black hole corresponds to the Euclidean field theory with one direction compact with an antiperiodic boundary condition for the fermions in this compact direction. We can then go back to Lorentzian signature by taking time to be one of the non-compact coordinates. The resulting metric is a double wick rotation of (5.1)

$$ds^2 = R^2 \frac{1}{z^2} \left[-dt^2 + dx_1^2 + dx_2^2 + \left(1 - \frac{z^4}{z_0^4}\right) dy^2 + \frac{dz^2}{\left(1 - \frac{z^4}{z_0^4}\right)} \right] \quad (6.1)$$

where we have suppressed the five sphere which will not play a role in our discussion. Now $y = y + 2\pi r_y$ and $z_0 = 2r_y$ is determined by demanding that (6.1) is nonsingular. Note that the topology of the boundary of (6.1) is $S^1 \times R^{2+1}$ and the topology of the full space

is $D^2 \times R^{2+1}$. So the circle is contractible in the full space. The radial direction z together with the circle y have the topology of a disk.

The crucial property of (6.1) is that the space terminates in the large z direction and the warp factor is bounded below, $w(z) \geq w(z_0) \sim 1/z_0$. Notice that the metric (6.1) does not have a horizon. A particle moving in the metric (6.1) feels a gravitational force towards $z = z_0$. So the lowest energy states live at $z = z_0$, the region of space where the warp factor is smallest. In fact, once we go to the quantum theory we expect that even massless particles get a non-zero mass due to the fact that they are moving in this gravitational potential well which forces the wavefunction to vary in the z direction since a normalizable wavefunction should go to zero at the boundary, $z \rightarrow 0$. In fact, all excitations on this geometry have positive mass from the $2 + 1$ dimensional point of view.

In order to find the $1 + 2$ dimensional particle spectrum we start from the five dimensional fields, let us say a scalar field of mass m . We then solve the classical equation

$$|g^{00}|\omega^2\phi + \frac{1}{\sqrt{g}}\partial_z(\sqrt{g}g^{zz}\partial_z\phi) - m^2R^2\phi = 0 \quad (6.2)$$

For simplicity we assumed that ϕ is independent of y . This equation should be solved with the boundary condition that it vanishes as $z \rightarrow 0$ and that it is regular at $z = z_0$, which means $\partial_z\phi|_{z_0} = 0$. So we really have an eigenvalue equation, i.e. the equation with these boundary conditions can have a non-zero solution only for special values of ω^2 . These special values are the masses of the particles in $2 + 1$ dimensions. By multiplying by ϕ and a suitable power of z and integrating (6.2) it is possible to see that the ω^2 eigenvalues are strictly positive. This can also be shown for the tachyons that we have in *AdS*.

So we see that the theory has a mass gap. i.e. all excitations have positive mass. This is a property we expect in confining theories. Note that these particles are glueballs from the point of view of the boundary theory. These have masses of the order of the mass gap, i.e. of order

$$M \sim 1/z_0 \sim 1/r_y \quad (6.3)$$

The fact that the warp factor is bounded below also implies that the Wilson loop will now lead to an area law in the boundary theory. What happens is that the string will sit at $z = z_0$ and it will have a finite tension there. I cannot move to a region where the warp factor is smaller.

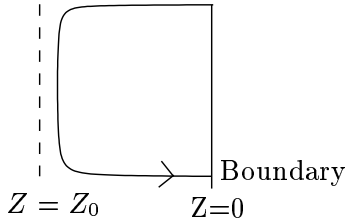


Fig. 17: String configuration relevant for the computation of the Wilson loop in a confining theory. Since the warp factor is bounded below we have a finite string tension from the boundary theory point of view.

The string tension from the point of view of the boundary theory is of order

$$T = \frac{1}{\alpha'} w(z_0)^2 \sim \frac{\sqrt{g^2 N}}{r_y^2} \quad (6.4)$$

6.1. Confinement-deconfinement transition

Consider now this 2 + 1 theory at finite temperature. In the Euclidean description we compactify time with period β . Going back to the 4d theory we now see that the theory is on the space $R^2 \times S^1_\beta \times S^1_{r_y}$, so that we have two circles. Then there are two Euclidean gravity solutions. We can take the above solution (6.1) and compactify Euclidean time. Or we could take the same type of solution but exchanging the two circles and adjusting z_0 appropriately. This gives two different ways of matching a five dimensional geometry to our four dimensional boundary. In the first solution the y circle is contractible but the Euclidean time circle is not. The opposite is true in the other solution. The first solution corresponds to just considering the original 5d space (6.1) at finite temperature, so we will excite thermally some of the particles we had discussed above. This is the solution that has lowest free energy at low temperatures. On the other hand the solution where the Euclidean time direction is contractible corresponds to having a black brane of the type we discussed above. This solution has lower free energy at high temperatures. There is a critical temperature at which they have the same free energy. At this temperature we have a first order phase transition. In this simple case the symmetries of the problem imply that the critical temperature is at $\beta_c = 2\pi r_y$. This is the confinement-deconfinement phase transition. It is of first order because the entropy changes dramatically. In the low temperature phase we have an entropy independent of N , we only excite color neutral glueballs whose spectrum is N independent. The high temperature phase has an entropy given by the Hawking Bekenstein formula which is proportional to N^2 . We interpret this as saying that now the gluons can move independently.

Note that in order to get this piece of physics correctly it was important to sum over all geometries which are asymptotic to a given boundary. This is a general principle in the duality. The choice of theory, i.e. the choice of Lagrangian, only fixes the boundary conditions of the gravity solution. Then we have to sum over all geometries with these boundary conditions. The leading contribution comes, of course, from classical solutions. In some situations there are several classical solutions. We should sum over all of them. The one with the lowest action will contribute the most. We have phase transitions when one dominates over the other. These are generically large N phase transitions, which can happen even in finite volume. Whether they are bona-fide phase transitions at finite N has to be thought about more carefully.

Is this really a solution for the pure bosonic Yang Mills theory in 2+1 dimensions?. Not really. The reason is the following. Bosonic Yang Mills theory in 2+1 has a dimensionfull coupling $g_3^2 N$. This theory is expected to have a confinement scale of order $\Lambda \sim g_3^2 N$. On the other hand if this theory arises as the low energy limit of some other theory, then this description in terms of 2 + 1 Yang Mills is quite reasonable if the scale of the new physics, let us call it Λ_{UV} is

$$\Lambda_{UV} \gg \Lambda = g_3^2 N \tag{6.5}$$

This means that the theory is weakly coupled for energies in between Λ and Λ_{UV} . In our case the three dimensional coupling that we obtain from dimensionally reducing the 4d theory is

$$g_3^2 N = \frac{g^2 N}{r_y} \tag{6.6}$$

while the scale at which we have new physics is $1/r_y \sim \Lambda_{UV}$. We see that we can never obey (6.5) if we want to trust supergravity, which requires $g^2 N \gg 1$.

So the theory that supergravity is describing is a confining theory but it is not pure 2 + 1 Yang Mills.

There are many examples of confining theories in four dimensions. Some are supersymmetric $\mathcal{N} = 1$ theories that are confining [33]. See M. Strassler's lectures.

Theories that have large radius supergravity duals are special. For example glueballs of spin greater than two will be much more massive than the lightest glueballs. Similarly the string tension will be much larger than the mass scale set by the lightest glueball, as we can see from (6.3)(6.4). It is believed that in large N bosonic Yang Mills in four dimensions these two scales are of the same order. This would imply that the theory does not have a large radius gravity dual. We expect a dual description in terms of strings moving on a space whose curvature is of the order of the string scale.

6.2. Remarks about more general field theories

Many theories have gravity duals and they do not need to be conformal. For local quantum field theories the corresponding gravity solution has the feature that the warp factor becomes large towards the boundary. If the theory is free in the UV, then the geometry becomes singular as the boundary is approached, the radius of curvature in string units becomes very small.

If a CFT is deformed by adding some relevant operators to the Lagrangian then there can be interesting effects in the IR. A simple possibility is that the theory flows in the IR to another CFT. In some cases people have found interpolating solutions that start as one AdS space near the boundary and end as another AdS space in the IR region [34]. One can prove that the supergravity equations imply that the radius of curvature in 5d Planck units of the AdS space decreases as you flow to the IR. This is a supergravity c theorem [34][35].

If the theory is confining, then one typically finds that the space ends as you go into the interior and that the warp factor is bounded below. How precisely it ends depends on the theory under consideration. It can end in a purely geometric way, as we saw above, or there can be some branes [36].

6.3. D-branes in the bulk

We have a $U(N)$ theory on the boundary. But the $U(1)$ is a free factor. The physics in the interior of the bulk is really described by the $SU(N)$ piece. In $SU(N)$ N quarks can combine into a neutral object. In the bulk we can have N fundamental strings that end on a D5 brane that is wrapping the S^5 [16].

Normally we cannot have a fundamental string ending on a brane with compact volume since the endpoints of fundamental strings act as electric charges for the $U(1)$ gauge field living on the D-brane worldvolume. Of course we could have a string ending and one “departing”, the orientation is important.

However, there is an interesting coupling on the D5 worldvolume of the form

$$S_{brane} \sim \int d^6x A \wedge F_5 \sim N \int dt A_0 \tag{6.7}$$

where A is the worldvolume gauge field. This is saying that the 5-form field strength induces N units of background electric charge on the 5-brane. This can be cancelled by N strings that end on it.

In other theories, which have dynamical quarks, we can have baryons as states in the theory.

Note that if we add flavors to the field theory we will get D-branes that are extended along all 5 dimensions of AdS . The open strings living on them are the mesons. The gauge fields living on them are associated to flavor symmetries.

Note that gauge symmetries in the bulk correspond to global symmetries in the boundary theory. In a gravitational theory only gauged symmetries have associated conserved charges.

7. The plane wave limit of AdS/CFT

Other reviews of this subject can be found in [5].

7.1. Plane waves

Plane waves are spacetimes of the form

$$ds^2 = -2dx^+ dx^- - A_{ij}(x^+) y^i y^j (dx^+)^2 + d\vec{y}^2 \quad (7.1)$$

The index $i = 1, \dots, D - 2$. These spacetimes have many isometries. One is obvious, ∂_- . To describe the other ones let us assume for the moment that there is only one coordinate y and call $A_{11} = \mu^2$. Then the other isometries have the form

$$a \equiv \zeta(x^+) \partial_y + \dot{\zeta}(x^+) y \partial_- \quad (7.2)$$

where $\zeta(x^+)$ is a complex solution of the equation

$$\ddot{\zeta} + \mu^2(x^+) \zeta = 0 \quad (7.3)$$

In the particular case that $\mu > 0$ is constant a simple solution is $\zeta = e^{-i\mu x^+}$. Since ζ is complex there are two isometries in (7.2) the other is the complex conjugate $a^\dagger = a^*$. We can normalize the solution of (7.3) so that the commutator of these two isometries is

$$[a, a^\dagger] = i\partial_- = -p_- \quad (7.4)$$

If we have n y coordinates, then we have n a, a^\dagger pairs.

The fact that the space has many isometries enables us to find the solution of the geodesic equation and also enables us to separate the Klein Gordon equation.

Let us see this more explicitly. The action for a particle is

$$\begin{aligned}
S &= \frac{1}{2} \int d\tau [e^{-1} \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} - m^2 e] \\
S &= \frac{1}{2} \int dx^+ \left\{ (-p_-)(\dot{y}^2 - \mu^2 y^2) - \frac{m^2}{(-p_-)} \right\}
\end{aligned} \tag{7.5}$$

where in the last line we have chosen lightcone gauge $x^+ = \tau$, found the conserved quantity $p_- = -e^{-1}$ and plugged it back into the action. Note that $p_- \leq 0$. The Hamiltonian corresponding to the action (7.5) is $H = i\partial_+ = -p_+$. We see that we get a harmonic oscillator, possibly with time dependent frequency. If μ is constant we get an ordinary harmonic oscillator. Note that μ^2 can be negative in some plane waves. In fact, if the metric (7.1) is a solution of the vacuum Einstein equations then the trace of A_{ij} is zero, so that there are both positive and negative eigenvalues. This correspond to the fact that tidal forces are focusing in one direction and defocusing in others.

The Klein Gordon equation for a field of mass m can also be solved by taking solutions with fixed p_- and then writing the equation for constant μ as

$$i\partial_+ \phi = \mu(a^\dagger a + 1/2)\phi + \frac{m^2}{2(-p_-)}\phi \tag{7.6}$$

For simplicity we consider only one y coordinate. If μ is x^+ independent then the spectrum is

$$-p_+ = \mu\left(n + \frac{1}{2}\right) + \frac{m^2}{2(-p_-)} \tag{7.7}$$

The ground state energy is $1/2$ (per dimension) for a scalar field, but it has other values of higher spin fields.

People who work in the lightcone gauge sometimes prefer to use p^+ and p^- . In our case since we are in a curved space it is more convenient to stick with p_\pm and live with the inconvenience that they are typically negative, p_- is always negative.

It turns out that string theory is also solvable on plane waves. We will discuss only one example.

7.2. Type IIB supergravity plane wave

We will be interested in the following plane wave solution of IIB supergravity [37]

$$ds^2 = -2dx^+ dx^- - y^2(dx^+)^2 + dy^i dy^i \tag{7.8}$$

with a constant field strength

$$F = dx^+(dy_1dy_2dy_3dy_4 + dy_5dy_6dy_7dy_8) \quad (7.9)$$

String propagation on this background can be solved exactly. by choosing light cone gauge in the Green-Schwarz action [38,39]. The lightcone action becomes

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^{\pi\alpha'|p_-|} d\sigma \left[\frac{1}{2}\dot{y}^2 - \frac{1}{2}y'^2 - \frac{1}{2}\mu^2 y^2 + i\bar{S}(\not{\partial} + \mu I)S \right] \quad (7.10)$$

where $I = \Gamma^{1234}$ and S is a Majorana spinor on the worldsheet and a positive chirality SO(8) spinor under rotations in the eight transverse directions. We quantize this action by expanding all fields in Fourier modes on the circle labeled by σ . For each Fourier mode we get a harmonic oscillator (bosonic or fermionic depending on the field). Then the light cone Hamiltonian is

$$-p_+ = H_{lc} = \sum_{n=-\infty}^{+\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha'|p_-|/2)^2}} \quad (7.11)$$

Here n is the label of the Fourier mode, $n > 0$ label left movers and $n < 0$ right movers. N_n denotes the total occupation number of that mode, including bosons and fermions. Note that the ground state energy of bosonic oscillators is canceled by that of the fermionic oscillators. The constraint on the momentum in the sigma direction reads

$$P = \sum_{n=-\infty}^{\infty} nN_n = 0 \quad (7.12)$$

In the limit that μ is very small, $\mu\alpha'|p_-| \ll 1$, we recover the flat space spectrum. It is also interesting to consider the opposite limit, where

$$\mu\alpha'p^+ \gg 1 \quad (7.13)$$

This limit corresponds to strong tidal forces on the strings, i.e. to strong curvatures. In this limit all the low lying string oscillator modes have almost the same energy. This limit corresponds to a highly curved background with RR fields. In fact we will later see that the appearance of a large number of light modes is expected from the Yang-Mills theory. In this limit different pieces of the string move independently.

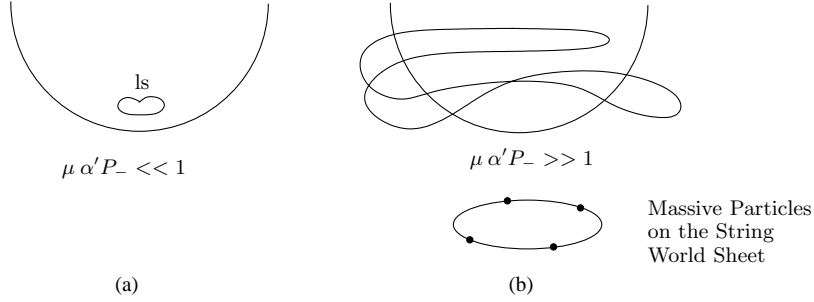


Fig. 18: Strings moving in a plane wave. (a) We see the weak field limit. (b) correspond to the strong field limit, where excitations along the string behave as very massive particles.

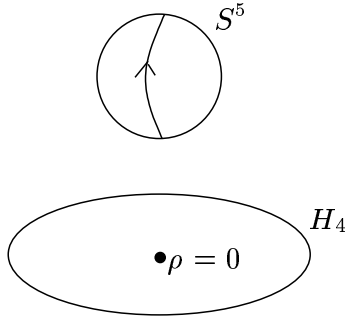


Fig. 19: In the plane wave limit we focus on the lightlike trajectory that goes around a great circle of S^5 and sits at the origin of the AdS spatial coordinates.

7.3. Type IIB plane wave from $AdS_5 \times S^5$

In this subsection we obtain the maximally supersymmetric plane wave of type IIB string theory as a limit of $AdS_5 \times S^5$. This is a so called “Penrose” limit. It consists on focusing on the spacetime region near a lightlike geodesic.

The idea is to consider the trajectory of a particle that is moving very fast along the S^5 and to focus on the geometry that this particle sees. See fig. 19.

We start with the $AdS_5 \times S^5$ metric written as

$$ds^2 = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega'_3{}^2 \right] \quad (7.14)$$

We want to consider a particle moving along the ψ direction and sitting at $\rho = 0$ and $\theta = 0$. We will focus on the geometry near this trajectory. We can do this systematically by introducing coordinates $\tilde{x}^- = t - \psi$, $x^+ = t$ and then performing the rescaling

$$x^- = R^2 \tilde{x}^-, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}, \quad R \rightarrow \infty \quad (7.15)$$

and x^+ is not rescaled. x^- , r , y are kept fixed as $R \rightarrow \infty$. In this limit the metric (7.14) becomes [40]

$$ds^2 = -2dx^+dx^- - (\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2 \quad (7.16)$$

where \vec{y} and \vec{r} parametrize points on R^4 . We can also see that only the components of F with a plus index survive the limit. The mass parameter μ can be introduced by rescaling $x^- \rightarrow x^-/\mu$ and $x^+ \rightarrow \mu x^+$ in eqn. (7.15). These solutions were studied in [37].

It will be convenient for us to understand how the energy and angular momentum along ψ scale in the limit (7.15). The energy in global coordinates in AdS is given by $E = i\partial_t$ and the angular momentum by $J = -i\partial_\psi$. This angular momentum generator can be thought of as the generator that rotates the 56 plane of R^6 .

7.4. The “plane wave” limit in gauge theory variables

In terms of the dual CFT these are the energy and R-charge of a state of the field theory on $S^3 \times R$ where the S^3 has unit radius. Alternatively, we can say that $E = \Delta$ is the conformal dimension of an operator on R^4 . We find that

$$\begin{aligned} -p_+ &= i\partial_{x^+} = i(\partial_t + \partial_\psi) = \Delta - J \\ -p_- &= -\frac{\tilde{p}_-}{R^2} = \frac{1}{R^2}i\partial_{x^-} = \frac{1}{R^2}i(-\partial_\psi) = \frac{J}{R^2} \end{aligned} \quad (7.17)$$

Configurations with fixed non zero p_- in the limit (7.15) correspond to states in AdS with large momentum along the S^5 , or large R charge in the field theory, $J \sim R^2 \sim (gN)^{1/2}$. It is useful also to rewrite (7.11) in terms of the Yang Mills parameters. Then we find that the contribution of each oscillator to $\Delta - J$ is

$$(\Delta - J)_n = w_n = \sqrt{1 + \frac{4\pi g_s N n^2}{J^2}} \quad (7.18)$$

Notice that $g_s N/J^2$ remains fixed in the $g_s N \rightarrow \infty$ limit that we are taking.

When we perform the rescalings (7.15) we can perform the limit in two ways. If we want to get the plane wave with finite string coupling then we take the $N \rightarrow \infty$ limit keeping the string coupling g_s fixed and we focus on operators with $J \sim N^{1/2}$ and $\Delta - J$ fixed.

On the other hand we could first take the 't Hooft limit $g \rightarrow 0$, $gN = \text{fixed}$, and then after taking this limit, we take the limit of large 't Hooft coupling keeping J/\sqrt{gN} fixed and $\Delta - J$ fixed. Taking the limit in this fashion gives us a plane wave background with

zero string coupling. Since we will be interested in these notes in the free string spectrum of the theory it will be more convenient for us to take this second limit. But to consider string interactions we need to consider the first.

From this point of view it is clear that the full supersymmetry algebra of the metric (7.14) is a contraction of that of $AdS_5 \times S^5$ [37]. This algebra implies that $p_{\pm} \leq 0$.

7.5. Strings from $\mathcal{N} = 4$ Super Yang Mills

After taking the 't Hooft limit, we are interested in the limit of large 't Hooft coupling $gN \rightarrow \infty$. We want to consider states which carry parametrically large R charge $J \sim \sqrt{gN}$.⁶ This R charge generator, J , is the $SO(2)$ generator rotating two of the six scalar fields. We want to find the spectrum of states with $\Delta - J$ finite in this limit. We are interested in single trace states of the Yang Mills theory on $S^3 \times R$, or equivalently, the spectrum of dimensions of single trace operators of the Euclidean theory on R^4 . We will often go back and forth between the states and the corresponding operators.

Let us first start by understanding the operator with lowest value of $\Delta - J = 0$. There is a unique single trace operator with $\Delta - J = 0$, namely $Tr[Z^J]$, where $Z \equiv \phi^5 + i\phi^6$ and the trace is over the N color indices. We are taking J to be the $SO(2)$ generator rotating the plane 56. At weak coupling the dimension of this operator is J since each Z field has dimension one. This operator is a chiral primary and hence its dimension is protected by supersymmetry. It is associated to the vacuum state in light cone gauge, which is the unique state with zero light cone Hamiltonian. In other words we have the correspondence

$$\frac{1}{\sqrt{JN^{J/2}}} Tr[Z^J] \iff |0, p_+\rangle_{l.c.} \quad (7.19)$$

We have normalized the operator as follows. When we compute $\langle Tr[\bar{Z}^J](x) Tr[Z^J](0) \rangle$ we have J possibilities for the contraction of the first \bar{Z} but then planarity implies that we contract the second \bar{Z} with a Z that is next to the first one we contracted and so on. Each of these contraction gives a factor of N . Normalizing this two point function to one we get the normalization factor in (7.19).⁷

⁶ Since we first took the 't Hooft limit then giant gravitons are not important.

⁷ In general in the free theory any contraction of a single trace operator with its complex conjugate one will give us a factor of N^n , where n is the number of fields appearing in the operator.

Now we can consider other operators that we can build in the free theory. We can add other fields, or we can add derivatives of fields like $\partial_{(i_1} \cdots \partial_{i_n)} \phi^r$, where we only take the traceless combinations since the traces can be eliminated via the equations of motion. The order in which these operators are inserted in the trace is important. All operators are all “words” constructed by these fields up to the cyclic symmetry, these were discussed and counted in [41]. We will find it convenient to divide all fields, and derivatives of fields, that appear in the free theory according to their $\Delta - J$ eigenvalue. There is only one mode that has $\Delta - J = 0$, which is the mode used in (7.19). There are eight bosonic and eight fermionic modes with $\Delta - J = 1$. They arise as follows. First we have the four scalars in the directions not rotated by J , i.e. ϕ^i , $i = 1, 2, 3, 4$. Then we have derivatives of the field Z , $D_i Z = \partial_i Z + [A_i, Z]$, where $i = 1, 2, 3, 4$ are four directions in R^4 . Finally there are eight fermionic operators $\chi_{J=\frac{1}{2}}^a$ which are the eight components with $J = \frac{1}{2}$ of the sixteen component gaugino χ (the other eight components have $J = -\frac{1}{2}$). These eight components transform in the positive chirality spinor representation of $SO(4) \times SO(4)$. We will focus first on operators built out of these fields and then we will discuss what happens when we include other fields, with $\Delta - J > 1$, such as \bar{Z} .

The state (7.19) describes a particular mode of ten dimensional supergravity in a particular wavefunction [10]. Let us now discuss how to generate all other massless supergravity modes. On the string theory side we construct all these states by applying the zero momentum oscillators a_0^i , $i = 1, \dots, 8$ and S_0^b , $b = 1, \dots, 8$ on the light cone vacuum $|0, p_+\rangle_{l.c.}$. Since the modes on the string are massive all these zero momentum oscillators are harmonic oscillators, they all have the same light cone energy. So the total light cone energy is equal to the total number of oscillators that are acting on the light cone ground state. We know that in $AdS_5 \times S^5$ all gravity modes are in the same supermultiplet as the state of the form (7.19)[42]. The same is clearly true in the limit that we are considering. More precisely, the action of all supersymmetries and bosonic symmetries of the plane wave background (which are intimately related to the $AdS_5 \times S^5$ symmetries) generate all other ten dimensional massless modes with given p_- . For example, by acting by some of the rotations of S^5 that do not commute with the $SO(2)$ symmetry that we singled out we create states of the form

$$\frac{1}{\sqrt{J}} \sum_l \frac{1}{\sqrt{J} N^{J/2+1/2}} Tr[Z^l \phi^r Z^{J-l}] = \frac{1}{N^{J/2+1/2}} Tr[\phi^r Z^J] \quad (7.20)$$

where ϕ^r , $r = 1, 2, 3, 4$ is one of the scalars neutral under J . In (7.20) we used the cyclicity of the trace. Note that we have normalized the states appropriately in the planar limit. We can act any number of times by these generators and we get operators roughly of the form $\sum Tr[\dots Z\phi^r Z \dots Z\phi^k]$. where the sum is over all the possible orderings of the ϕ s. We can repeat this discussion with the other $\Delta - J = 1$ fields. Each time we insert a new operator we sum over all possible locations where we can insert it. Here we are neglecting possible extra terms that we need when two $\Delta - J = 1$ fields are at the same position, these are subleading in a $1/J$ expansion and can be neglected in the large J limit that we are considering. We are also ignoring the fact that J typically decreases when we act with these operators. In other words, when we act with the symmetries that do not leave Z invariant we will change one of the Z s in (7.19) to a field with $\Delta - J = 1$, when we act again with one of the symmetries we can change one of the Z s that was left unchanged in the first step or we can act on the field that was already changed in the first step. This second possibility is of lower order in a $1/J$ expansion and we neglect it. We will always work in a “dilute gas” approximation where most of the fields in the operator are Z s and there are a few other fields sprinkled in the operator.

For example, a state with two excitations will be of the form

$$\sim \frac{1}{N^{J/2+1}} \frac{1}{\sqrt{J}} \sum_{l=0}^J Tr[\phi^r Z^l \psi_{J=\frac{1}{2}}^b Z^{J-l}] \quad (7.21)$$

where we used the cyclicity of the trace to put the ϕ^r operator at the beginning of the expression. We associate (7.21) to the string state $a_0^{\dagger k} S_0^{\dagger b} |0, p_+\rangle_{l.c.}$. Note that for planar diagrams it is very important to keep track of the position of the operators. For example, two operators of the form $Tr[\phi^1 Z^l \phi^2 Z^{J-l}]$ with different values of l are orthogonal to each other in the planar limit (in the free theory).

The conclusion is that there is a precise correspondence between the supergravity modes and the operators. This is of course well known [9,10,43]. Indeed, we see from (7.11) that their $\Delta - J = -p_+$ is indeed what we compute at weak coupling, as we expect from the BPS argument.

In order to understand non-supergravity modes in the bulk it is clear that what we need to understand the Yang Mills description of the states obtained by the action of the string oscillators which have $n \neq 0$. Let us consider first one of the string oscillators which creates a bosonic mode along one of the four directions that came from the S^5 , let's say $a_n^{\dagger 8}$. We already understood that the action of $a_0^{\dagger 8}$ corresponds to insertions of an

operator ϕ^4 on all possible positions along the “string of Z ’s”. By a “string of Z s” we just mean a sequence of Z fields one next to the other such as we have in (7.19). We propose that $a_n^{\dagger 8}$ corresponds to the insertion of the same field ϕ^4 but now with a position dependent phase

$$\frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{\sqrt{J} N^{J/2+1/2}} \text{Tr}[Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \quad (7.22)$$

In fact the state (7.22) vanishes by cyclicity of the trace. This corresponds to the fact that we have the constraint that the total momentum along the string should vanish (7.12), so that we cannot insert only one $a_n^{\dagger i}$ oscillator. So we should insert more than one oscillator so that the total momentum is zero. For example we can consider the string state obtained by acting with the $a_n^{\dagger 8}$ and $a_{-n}^{\dagger 7}$, which has zero total momentum along the string. We propose that this state should be identified with

$$a_n^{\dagger 8} a_{-n}^{\dagger 7} |0, p_+\rangle_{l.c.} \iff \frac{1}{\sqrt{J}} \sum_{l=1}^J \frac{1}{N^{J/2+1}} \text{Tr}[\phi^3 Z^l \phi^4 Z^{J-l}] e^{\frac{2\pi i n l}{J}} \quad (7.23)$$

where we used the cyclicity of the trace to simplify the expression. The general rule is pretty clear, for each oscillator mode along the string we associate one of the $\Delta - J = 1$ fields of the Yang-Mills theory and we sum over the insertions of this field at all possible positions with a phase proportional to the momentum. States whose total momentum is not zero along the string lead to operators that are automatically zero by cyclicity of the trace. In this way we enforce the $L_0 - \bar{L}_0 = 0$ constraint (7.12) on the string spectrum.

In summary, each string oscillator corresponds to the insertion of a $\Delta - J = 1$ field, summing over all positions with an n dependent phase, according to the rule

$$\begin{aligned} a^{\dagger i} &\longrightarrow D_i Z \quad \text{for } i = 1, \dots, 4 \\ a^{\dagger j} &\longrightarrow \phi^{j-4} \quad \text{for } j = 5, \dots, 8 \\ S^a &\longrightarrow \chi_{J=\frac{1}{2}}^a \end{aligned} \quad (7.24)$$

In order to show that this identification makes sense we want to compute the conformal dimension, or more precisely $\Delta - J$, of these operators at large 't Hooft coupling and show that it matches (7.11). First note that if we set $\frac{gN}{J^2} \sim 0$ in (7.18) we find that all modes, independently of n have the same energy, namely one. This is what we find at weak 't Hooft coupling where all operators of the form (7.23) have the same energy, independently

of n . Expanding the string theory result (7.18) we find that the first correction is of the form

$$(\Delta - J)_n = w_n = 1 + \frac{2\pi g N n^2}{J^2} + \dots \quad (7.25)$$

This looks like a first order correction in the 't Hooft coupling and we can wonder if we can reproduce it by a simple perturbative computation.

In order to compute the corrections it is useful to view the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 1$ theory. As an $\mathcal{N} = 1$ theory we have a Yang Mills theory plus three chiral multiplets in the adjoint representation. We denote these multiplets as W^i , where $i = 1, 2, 3$. We will often set $Z = W^3$ and $W = W^1$. The theory also has a superpotential

$$\mathcal{W} \sim g_{YM} Tr(W^i W^j W^k) \epsilon_{ijk} \quad (7.26)$$

The potential for the Yang Mills theory is the sum of two terms, $V = V_F + V_D$, one coming from F terms and the other from D-terms. The one coming from F terms arises from the superpotential and has the form

$$V_F \sim \sum_{ij} Tr([W^i, W^j][\bar{W}^i, \bar{W}^j]) \quad (7.27)$$

On the other hand the one coming from D terms has the form

$$V_D \sim \sum_{ij} Tr([W^i, \bar{W}^i][W^j, \bar{W}^j]) \quad (7.28)$$

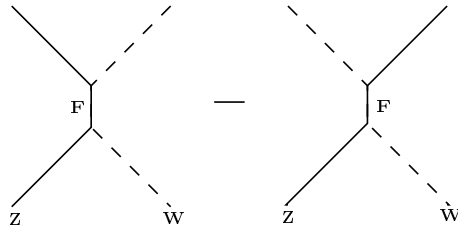


Fig. 20: Diagrams that come from F terms. The two diagrams have a relative minus sign. The F terms propagator is a delta function so that we could replace the three point vertex by a four point vertex coming from (7.27). If there are no phases in the operator these contributions vanish.

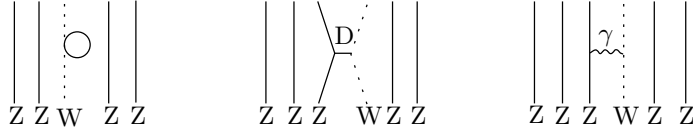


Fig. 21: Diagrams that do not lead to phase dependence and cancel out.

We will concentrate in computing the contribution to the conformal dimension of an operator which contains a W insertion along the string of Z s. There are various types of diagrams. There are diagrams that come from D terms, as well as from photons or self energy corrections. There are also diagrams that come from F terms. The diagrams that come from F terms can exchange the W with the Z . The F term contributions cancel in the case that there are no phases, see fig. 20. This means that all other diagrams should also cancel, since in the case that there are no phases we have a BPS object which receives no corrections. All other one loop diagrams that do not come from F terms do not exchange the position of W , this means that they vanish also in the case that there are phases since they will be insensitive to the presence of phases. In the presence of phases the only diagrams that will not cancel are then the diagrams that come from the F terms. These are the only diagrams that give a momentum, n , dependent contribution.

In the free theory, once a W operator is inserted at one position along the string it will stay there, states with W 's at different positions are orthogonal to each other in the planar limit (up to the cyclicity of the trace). We can think of the string of Z s in (7.19) as defining a lattice, when we insert an operator W at different positions along the string of Z s we are exciting an oscillator b_l^\dagger at the site l on the lattice, $l = 1, \dots, J$. The interaction term (7.27) can take an excitation from one site in the lattice to the neighboring site. So we see that the effects of (7.27) will be sensitive to the momentum n . In fact, one can precisely reproduce (7.25) from (7.27) including the precise numerical coefficient. Below we give some more details on the computation.

We will write the square of the Yang-Mills coupling in terms of what in AdS is the string coupling that transforms as $g \rightarrow 1/g$ under S-duality. The trace is just the usual trace of an $N \times N$ matrix.

We define $Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6)$ and similarly for W . Then the propagator is normalized as

$$\langle Z_i^j(x) \bar{Z}_k^l(0) \rangle = \delta_i^l \delta_k^j \frac{2\pi g}{4\pi^2} \frac{1}{|x|^2} \quad (7.29)$$

In (7.27) there is an interaction term of the form $\frac{1}{\pi g} \int d^4x \text{Tr}([Z, W][\bar{Z}, \bar{W}])$, where W is one of the (complex) transverse scalars, let's say $W = W^1$. The contribution from the F terms shown in (7.27) give

$$\langle O(x)O^*(0) \rangle = \frac{\mathcal{N}}{|x|^{2\Delta}} \left[1 + N(4\pi g)(-2 + 2 \cos \frac{2\pi n}{J})I(x) \right] \quad (7.30)$$

where \mathcal{N} is a normalization factor and $I(x)$ is the integral

$$I(x) = \frac{|x|^4}{(4\pi^2)^2} \int d^4y \frac{1}{y^4(x-y)^4} \sim \frac{1}{4\pi^2} \log|x|\Lambda + \text{finite} \quad (7.31)$$

We extracted the log divergent piece of the integral since it is the one that reflects the change in the conformal dimension of the operator.

In conclusion we find that for large J and N the first correction to the correlator is

$$\langle O(x)O^*(0) \rangle = \frac{\mathcal{N}}{|x|^{2\Delta}} \left[1 - \frac{4\pi g N n^2}{J^2} \log(|x|\Lambda) \right] \quad (7.32)$$

which implies that the contribution of the operator W inserted in the “string of Z s” with momentum n gives a contribution to the anomalous dimension

$$(\Delta - J)_n = 1 + \frac{2\pi g N n^2}{J^2} \quad (7.33)$$

which agrees precisely with the first order term computed from (7.25).

There are similar computations we could do for insertions of $D_i Z$, \bar{W} or the fermions $\chi_{J=1/2}^a$. In the case of the fermions the important interaction term will be a Yukawa coupling of the form $\bar{\chi}\Gamma_z[Z\chi] + \bar{\chi}\Gamma_{\bar{z}}[\bar{Z}, \chi]$. We have not done these computations explicitly since the 16 supersymmetries preserved by the state (7.19) relate them to the computation we did above for the insertion of a W operator.

The full square root in (7.18) was derived in a paper by Santambrogio and Zanon [44], see also [45].

In summary, the “string of Z s” becomes the physical string and each Z carries one unit of J which is one unit of $-p_-$. Locality along the worldsheet of the string comes from the fact that planar diagrams allow only contractions of neighboring operators. So the Yang Mills theory gives a string bit model (see [46]) where each bit is a Z operator. Each bit carries one unit of J which is one unit of $-p_-$.

The reader might, correctly, be thinking that all this seems too good to be true. In fact, we have neglected many other diagrams and many other operators which, at weak 't

Hooft coupling also have small $\Delta - J$. In particular, we considered operators which arise by inserting the fields with $\Delta - J = 1$ but we did not consider the possibility of inserting fields corresponding to $\Delta - J = 2, 3, \dots$, such as \bar{Z} , $\partial_k \phi^r$, $\partial_{(l} \partial_{k)} Z$, etc.. The diagrams of the type we considered above would give rise to other 1+1 dimensional fields for each of these modes. These are present at weak 't Hooft coupling but they should not be present at strong coupling, since we do not see them in the string spectrum. We believe that what happens is that these fields get a large mass in the $N \rightarrow \infty$ limit. In other words, the operators get a large conformal dimension. One can compute the first correction to the energy (the conformal weight) of the of the state that results from inserting \bar{Z} with some “momentum” n . In contrast to our previous computation for $\Delta - J = 1$ fields we find that besides an effective kinetic term as in (7.25) there is an n independent contribution that goes as gN with no extra powers of $1/J^2$ [11]. This is an indication that these excitations become very massive in the large gN limit. In addition, we can compute the decay amplitude of \bar{Z} into a pair of ϕ insertions. This is also very large, of order gN .

Though we have not done a similar computation for other fields with $\Delta - J > 1$, we believe that the same will be true for the other fields. In general we expect to find many terms in the effective Lagrangian with coefficients that are of order gN with no inverse powers of J to suppress them. In other words, the Lagrangian of Yang-Mills on S^3 acting on a state which contains a large number of Z s gives a Lagrangian on a discretized spatial circle with an infinite number of KK modes. The coefficients of this effective Lagrangian are factors of gN , so all fields will generically get very large masses.

The only fields that will not get a large mass are those whose mass is protected for some reason. The fields with $\Delta - J = 1$ correspond to Goldstone bosons and fermions of the symmetries broken by the state (7.19). Note that despite the fact that they morally are Goldstone bosons and fermions, their mass is non-zero, due to the fact that the symmetries that are broken do not commute with p_+ , the light cone Hamiltonian. The point is that their masses are determined, and hence protected, by the (super)symmetry algebra.

Having described how the single string Hilbert space arises it is natural to ask whether we can incorporate properly the string interactions. Clearly string interactions come when we include non-planar diagrams. There has been a lot of recent work relating the string interactions to the leading non-planar contributions in Yang Mills [47].

Finally we should note that there is another interesting limit where we consider operators with large spin [48]. In this case one finds that for large spin the operators have

dimensions $\Delta - S \sim (const)\sqrt{g^2 N} \log S$. At weak coupling one has a similar relation but in front of the logarithm we have a factor of $g^2 N$.

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