Derivation of the Schwarzschild metric from the weak field metric

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*The purpose of this paper is to proof that the Schwarzschild metric for a spherical symmetric static gravitational field can be derived starting from the metric of a weak field by multiplying the first space component of this metric with a certain function **h(r)**. By using general physical arguments it can be proven, that this function becomes identical to the one occurring in the well-known Schwarzschild metric without calculating all the components of the Ricci-tensor and equating them with zero.*

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1. The metric of the weak field

To establish the idea, we start from a weak and static gravitational field generated by a mass located in the origin of the coordinate system.
If a particle moves in such a force field, this particle will follow a geodetic path in the Riemann space. For a weak gravitational field, the metric of that space will not differ much from the Minkowski-metric and if the velocity of the particle is much smaller than the speed of light c (for $v/c << 1$), this metric can be written as

$$\overline{ds}^2 = c^2(1+\varepsilon_{00})\overline{dt}^2 - \overline{dx}^2 - \overline{dy}^2 - \overline{dz}^2$$

So that we have $g_{00} = (1+\varepsilon_{00})$ instead of $g_{00} = 1$ and we will show that it is the metric coefficient that produces the gravitational forces.
We introduce $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon_{\mu\nu}$ in which $\varepsilon_{\mu\nu}$ is a small time independent perturbation. The line element can then be written as

$$d\overline{s}^2 = \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + \varepsilon_{\mu\nu}dx^\mu dx^\nu$$

We can also write the last term as $\varepsilon_{\mu\nu}\dfrac{dx^\mu}{cdt}\dfrac{dx^\nu}{cdt}d\left(ct\right)^2 = \varepsilon_{\mu\nu}\dfrac{v^\mu}{c}\dfrac{v^\nu}{c}d\left(x^0\right)^2$

For μ en $\nu \neq 0$ we can neglect the terms $\varepsilon_{\mu\nu}\dfrac{v^\mu}{c}\dfrac{v^\nu}{c}$ because the $\varepsilon_{\mu\nu}$'s are small for $v^2 \ll c^2$. For $\mu = \nu = 0$ we get $\varepsilon_{00}\left(dx^0\right)^2$ because $v^0 = \dfrac{dx^0}{dt} = c$.
The term then reduces to the above. In spherical coordinates this becomes

$$\overline{ds}^2 = c^2\left(1+\varepsilon_{00}(r)\right)\overline{dt}^2 - \overline{dr}^2 - r^2\overline{d\theta}^2 - r^2\sin\theta\overline{d\varphi}^2$$

Because the gravitational field is static it will depend only on the space coordinates and not on time. Furthermore, the ε_{00}'s will be much smaller than 1.

We from this metric we derive de $g^{\mu\nu}$

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ en } g^{\alpha\beta} = \begin{pmatrix} \frac{1}{g_{00}} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

With the knowledge of the g's we can calculate the Christoffel symboles $\Gamma^{\alpha}_{\mu\nu}$. These are generally equal to zero because the g's are constant, except g_{00} and $\frac{\partial g_{00}}{\partial x^i} \neq 0$

So we just have to calculate Γ^i_{00}

$$\Gamma^i_{00} = \frac{1}{2}\sum_n g^{in}\left(\frac{\partial g_{0n}}{\partial x^0} + \frac{\partial g_{n0}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^n}\right) = -\frac{1}{2} g^{ii} \frac{\partial g_{00}}{\partial x^i}$$

Now $\frac{\partial g_{00}}{\partial x^i}$ is equal to 0 for $i = 0$ (g_{00} is independent of x^0) and $g^{ii} = -1$ for $i = 1,2,3$ so that

$$\Gamma^i_{00} = \frac{1}{2}\frac{\partial g_{00}}{\partial x^i} = \frac{1}{2}\frac{\partial \varepsilon_{00}}{\partial x^i}$$

The equation of the geodesic then becomes $\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2 = 0$ with $x^0 = ct$

The relationship between $d\tau$ and dt follows from the definition of the proper time

$$c^2 d\tau^2 = ds^2 = c^2dt^2\left(1 + \varepsilon_{00} - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\right) = c^2dt^2\left(1 + \varepsilon_{00} - \frac{v^2}{c^2}\right)$$

or $\frac{v^2}{c^2} \ll 1$ we can approximate this by $d\tau \approx \sqrt{(1+\varepsilon_{00})}\, dt$ and $\frac{dt}{d\tau} = (1+\varepsilon_{00})^{-½}$.

We change the derivatives to τ in the derivatives to t in the equation of the geodesic through $\frac{d^2x^i}{d\tau^2} = \frac{d}{dt}\left(\frac{dx^i}{dt}\frac{dt}{d\tau}\right)\frac{dt}{d\tau} = \frac{d^2x^i}{dt^2}\left(\frac{dt}{d\tau}\right)^2$

The geodesic equation now becomes $\frac{d^2x^i}{dt^2} + c^2\Gamma^i_{00} = 0$ so that $\frac{d^2x^i}{dt^2} = -\frac{c^2}{2}\frac{\partial \varepsilon_{00}}{\partial x^i}$

In vector notation this can be written as

$$\frac{d^2\vec{r}}{dt^2} = -\frac{c^2}{2}\vec{\nabla}\varepsilon_{00}$$

and this is simply Newton's equation of motion in a classical gravitational field derived from the scalar potential Φ if we identify scalar potential with

$$\Phi = \frac{c^2}{2}\varepsilon_{00}$$

Conversely, given a classical potential Φ, the particle's motion will follow the four-dimensional geodesic if the metric tensor's term g_{00} satisfies the relationship

$$g_{00} = 1 + \frac{2\Phi}{c^2}$$

Let us apply this to the gravitational potential on the surface of the Earth.

$$\Phi_a = \frac{GM_a}{r_a} = \frac{6{,}674 \cdot 10^{-11} \cdot 6 \cdot 10^{24}}{6{,}38 \cdot 10^6} = 6{,}276 \cdot 10^7 \text{ N.m.kg}^{-1} = \text{m}^2 s^{-2}$$

which gives

$$\varepsilon_{00} = \frac{2\Phi_a}{c^2} = \frac{12{,}55 \cdot 10^7}{9 \cdot 10^{16}} \approx 10^{-9}$$

With such a small value for ε_{00}, we must conclude that the space in the vicinity of the earth is almost flat, after all g_{00} hardly differs from 1.
The fact that such a space nevertheless gives rise to a noticeable acceleration on Earth is due to the factor $c^2 \approx 10^{17}$ by which $\left.\frac{1}{2}\frac{\partial \varepsilon_{00}}{\partial r}\right|_{r=r_a} = \frac{10^{-9}}{12{,}8 \cdot 10^6} \approx 10^{-16} \text{ m}^{-1}$ must be multiplied to obtain the gravitational acceleration $g \approx 10^{17}.10^{-16} = 10 \text{ ms}^{-2}$.

(Note that the foregoing does not use Einstein's field equations and the assumption that for the empty space the Ricci-tensor $R_{\mu\nu} = 0$)

2. Correction to the metric of the weak field

Indeed, the results of a Mathematica notebook ‘Curvature’ show that all components of the Ricci tensor for the weak field metric are different from zero, so that the metric is not a solution of Einstein's field equation. We will now figure out what change we need to make to the metric so that it becomes a solution to that field equation and we now write that metric as

$$\overline{ds}^2 = c^2\left(1+\tfrac{2\Phi}{c^2}\right)\overline{dt}^2 - \overline{dr}^2 - r^2\overline{d\theta}^2 - r^2\sin\theta\overline{d\varphi}^2$$
$$= c^2\left(1-\tfrac{2GM}{c^2 r}\right)\overline{dt}^2 - \overline{dr}^2 - r^2\overline{d\Omega}^2$$

in which we introduce the Schwarzschild-radius $r_s = \frac{2GM}{c^2}$.

$$\overline{ds}^2 = c^2\left(1-\frac{r_s}{r}\right)\overline{dt}^2 - \overline{dr}^2 - r^2\overline{d\Omega}^2$$

We now adjust this metric by multiplying the space component $\overline{dr}^2$ by a function $h(r)$ with the intention of figuring out which expression to assign to this function so that the metric is a solution of Einstein's field equation.

$$\overline{ds}^2 = c^2\left(1-\frac{r_s}{r}\right)\overline{dt}^2 - h(r)\overline{dr}^2 - r^2\overline{d\Omega}^2$$

One can of course try to guess for a possible expression for the unknown function $h(r)$ by imposing a number of conditions. We will distinguish two conditions that are physically acceptable;

C1 -A first property that we can certainly demand from this function is that it satisfies a metric, which is asymptotically flat. This means that for $r \to \infty$ the coefficient $h(r) \to 1$. However, there are a very large number of functions that meet this condition. We just mention a few of them: $\left(1-\frac{r_s}{r}\right)^{2n}$, $1+\frac{r_s}{r}+\left(\frac{r_s}{r}\right)^2$, $e^{\pm\frac{r_s}{r}}$, etc.

C2 - A further property that this function should have, is that it changes sign so that the metric's signature is preserved in the interior of the black hole when the metric time component transforms into a space component. There are then a smaller number of functions that meet both conditions : $\left(1-\frac{r_s}{r}\right)^{2n+1}$, $\left(1-\frac{r_s}{r}\right)^{-(2n+1)}$, $\tanh\frac{r-r_s}{r_s}$, ect.

Unfortunately, these conditions are not sufficient to unambiguously determine $h(r)$ and if we cannot impose any further physically based condition on this function, we will be forced to calculate the Ricci tensor $R_{\mu\nu}$ for the chosen metric.

3. The Schwarzschild metric derived from the field equation

With the 'Curvature' notebook we can now quickly determine the components of the Ricci-tensor for this metric.

R[1, 1]	$-\frac{Rs\,(Rs\,h[r]+r\,(r-Rs)\,h'[r])}{4\,r^3\,(r-Rs)\,h[r]^2}$
R[2, 2]	$\frac{(4\,r-3\,Rs)\,(Rs\,h[r]+r\,(r-Rs)\,h'[r])}{4\,r^2\,(r-Rs)^2\,h[r]}$
R[3, 3]	$1-\frac{1}{h[r]}+\frac{Rs}{-2\,r\,h[r]+2\,Rs\,h[r]}+\frac{r\,h'[r]}{2\,h[r]^2}$
R[4, 4]	$\frac{1}{2}\,Sin[\theta]^2\left(2+\frac{-2+\frac{Rs}{-r+Rs}}{h[r]}+\frac{r\,h'[r]}{h[r]^2}\right)$

from which we learn that R_{tt} and R_{rr} become zero for

$$r_s h(r) + r(r-r_s)h'(r) = 0$$

or

$$\frac{h'(r)}{h(r)}=\frac{r_s}{r(r_s-r)}\rightarrow \ln h(r)=\int\frac{r_s}{r(r_s-r)}dr$$

The substitution $u=1-\frac{r_s}{r}$ gives $\int\frac{du}{-u}=-\ln u$ leads to

$$\ln h(r)=-\ln\left(1-\tfrac{r_s}{r}\right)\rightarrow h(r)=\frac{1}{1-\frac{r_s}{r}}$$

With this expression for the function $h(r)$, also $R_{\theta\theta}$ and $R_{\varphi\varphi}$ become equal to zero and the metric becomes identical to the static Schwarzschild metric, which we have now determined by a single integration.

4. **Determination of the as yet unknown function** $h(r)$

We consider the free fall of a small particle (e.g. an apple) at a certain height above the surface of the earth starting from a standstill. The orbit becomes a geodesic in the space-time for a weak gravitational field. We make a correction to this space-time by multiplying the first space component by an as yet unknown function $h(r)$. The orbit then becomes a geodesic in the gravitational field of a black hole with a Schwarzschild radius r_s equal to that of the Earth (approximately equal to 1 cm)
We can now write the metric

$$\overline{ds}^2=(1-\tfrac{r_s}{r})\overline{ct}^2-h(r)\overline{dr}^2-r^2\overline{d\theta}^2-r^2\sin^2(\theta)\overline{d\varphi}^2$$

In the equatorial plane this becomes ($d\theta=0$ en $\theta=\pi/2$)

$$\overline{ds}^2=(1-\tfrac{r_s}{r})\overline{ct}^2-h(r)\overline{dr}^2-r^2\overline{d\varphi}^2$$

We now apply the Euler-Lagrange formalism to calculate a time-like geodesic using this metric.

We start with the Lagrange-function $\mathcal{L}=-\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ met $\dot{x}=\frac{dx}{d\tau}$

The equation of the geodesic then becomes

$$\frac{d}{d\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}-\frac{\partial\mathcal{L}}{\partial x^\mu}=0$$

For $x=ct$ this gives $0=\frac{\partial}{\partial\tau}\left(-\left(1-\frac{r_s}{r}\right)c\dot{t}-0\right)\rightarrow\frac{E}{c}=c\left(1-\frac{r_s}{r}\right)\dot{t}\rightarrow\dot{t}=\frac{E}{c^2\left(1-\frac{r_s}{r}\right)}$

For the constant we choose E / c so that the dimensions are correct. E/c is the energy per unit mass and it is also the time component of the four-momentum.

For x=φ, this gives $\frac{d}{d\tau}\left(r^2\dot{\varphi}+0\right)=0\rightarrow L=r^2\dot{\varphi}$ with L the angular momentum per unit mass.

For x=r, we make use of the four-velocity $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=c^2$ and this gives

$$\left(1-\frac{r_s}{r}\right)c^2\dot{t}^2-h(r)\dot{r}^2-r^2\dot{\varphi}^2=c^2\rightarrow\dot{r}^2=\frac{1}{h(r)}\left\{\left(1-\frac{r_s}{r}\right)c^2\frac{E^2}{c^4\left(1-\frac{r_s}{r}\right)^2}-\frac{r^2L^2}{r^4}-c^2\right\}$$

after inserting the value for $\dot{t}^2$ as a function of E.

Simplification of the latter expression leads us to

$$\rightarrow\dot{r}^2=\left\{\frac{E^2}{c^2\left(1-\frac{r_s}{r}\right)h(r)}-\frac{L^2}{h(r)r^2}-\frac{c^2}{h(r)}\right\}$$

We now limit ourselves to radial geodesics, $\dot{\varphi}=0\rightarrow L=0$, we then get the parameter equations for the geodesic

$$\dot{t}=\frac{E}{c^2\left(1-\frac{r_s}{r}\right)}\quad\text{(a)}$$

$$\dot{r}^2=\frac{E^2}{c^2h(r)\left(1-\frac{r_s}{r}\right)}-\frac{c^2}{h(r)}\quad\text{(b)}$$

We dropped the apple from a standstill from a certain point from the center of the black hole. Say that this height corresponds to a surface coordinate r_0 in space-time We then have $\dot{r}\big|_{r=r_0}=0$ (only valid for $r_0>r_s$) and (b) becomes

$$0=\frac{E^2}{c^2h(r_0)(1-\frac{r_s}{r_0})}-\frac{c^2}{h(r_0)}\rightarrow E^2=c^4\left(1-\frac{r_s}{r_0}\right)$$

So we can then write (a) and (b) as

$$\dot{t}=\frac{\sqrt{1-\frac{r_s}{r_0}}}{1-\frac{r_s}{r}}\quad\text{(a')}$$

$$\dot{r}^2=\frac{c^2}{h(r)}\left(\frac{\frac{r_s}{r}-\frac{r_s}{r_0}}{1-\frac{r_s}{r}}\right)\quad\text{(b') and}$$

$$\dot{r}=\frac{dr}{d\tau}=\frac{\pm c}{\sqrt{h(r)}}\sqrt{\left(\frac{\frac{r_s}{r}-\frac{r_s}{r_0}}{1-\frac{r_s}{r}}\right)}=\frac{\pm c\sqrt{\frac{r_s}{r}-\frac{r_s}{r_0}}}{\sqrt{h(r)(1-\frac{r_s}{r})}}$$

We retain the minus-sign because $dr < 0$ and $d\tau > 0$
From this expression we can now calculate the acceleration along the geodesic. It must be equal to the acceleration on Earth, which gives rise to tidal forces.

We first put $\dot{r} = u \cdot \frac{1}{v}$ with $u = -c\left(\frac{r_s}{r} - \frac{r_s}{r_0}\right)^{\frac{1}{2}}$ and $v = \sqrt{h(r)\left(1 - \frac{r_s}{r}\right)}$

so that $\ddot{r} = \frac{d^2 r}{d\tau^2} = \frac{1}{v}\frac{du}{d\tau} - \frac{u}{v^2}\frac{dv}{d\tau}$ with $\frac{d\cdot}{d\tau} = \frac{d\cdot}{dr}\frac{dr}{d\tau} = \frac{d\cdot}{dr}\dot{r}$

Calculation of both terms gives

$$a = \frac{d\dot{r}}{d\tau} = \frac{\frac{1}{2}c^2(\frac{r_s}{r^2})}{h(r)\left(1 - \frac{r_s}{r}\right)} + \frac{c^2}{2}\left(\frac{r_s}{r} - \frac{r_s}{r_0}\right)\frac{g(r)}{h(r)^2\left(1 - \frac{r_s}{r}\right)^2} .$$

The first term can also be written as $\frac{\frac{1}{2}c^2 \frac{2GM_a}{c^2}\frac{1}{r^2}}{h(r)\left(1 - \frac{r_s}{r}\right)} = \frac{GM_a}{r^2}\frac{1}{h(r)\left(1 - \frac{r_s}{r}\right)}$

For $r = r_0$ the approximate acceleration becomes ($d\tau \approx dt$ for $r > r_0$)

$$a\,\Big|_{r=r_0} = \frac{d\dot{r}}{d\tau}\Big|_{r=r_0} = \frac{GM_a}{r_0^2}\frac{1}{h(r_0)\left(1 - \frac{r_s}{r_0}\right)} + 0$$

Only a short-cut expression of the calculation of the second term is shown above with

$$g(r) = h'(r)\left(1 - \frac{r_s}{r}\right) + h(r)\frac{r_s}{r^2}.$$

Given the approximate expression of the universal gravitational potential equal to $-\frac{GM_a}{r}$ and its derived acceleration $\frac{GM_a}{r^2}$ we may conclude that $g(r)$ must be identical zero or $h'(r)r(r - r_s) + h(r)r_s \equiv 0$ for $r > r_0$. The integration of this equation is given in section 3 above, where we found that

$$h(r) = \frac{1}{1 - \frac{r_s}{r}}$$

The condition C1 is satisfied: for $r \to \infty$, $h(\infty) = 1$ and also C2 is satisfied:
This value for $h(r)$ reduces the first term to the required approximate form

$$a = \frac{GM_a}{r^2}$$

5. Determination of the as yet unknown function $h(r)$ by direct comparison in the Newtonian limit

We start from the weak field metric and apply the correction using the function $h(r)$

$$\overline{ds}^2 = (1 - \tfrac{r_s}{r})\overline{ct}^2 - h(r)\overline{dr}^2 - r^2\overline{d\theta}^2 - r^2\sin^2(\theta)\overline{d\varphi}^2$$

The Mathematica notebook 'curvature' gives us the Christoffel symbols for this metric

Γ[1, 2, 1]	$\frac{Rs}{2\,r^2-2\,r\,Rs}$
Γ[2, 1, 1]	$\frac{Rs}{2\,r^2\,h[r]}$
Γ[2, 2, 2]	$\frac{h'[r]}{2\,h[r]}$
Γ[2, 3, 3]	$-\frac{r}{h[r]}$
Γ[2, 4, 4]	$-\frac{r\,Sin[\theta]^2}{h[r]}$

where $\Gamma[1,2,1]$ stands for Γ^0_{10}

From our previous discussion in section 1 we know that Γ^i_{00} are the only relevant components for the geodesic equation of the weak field which is given in terms of the Christoffel symbols by

$$\frac{d^2x^i}{d\tau^2}+\Gamma^i_{00}(u^0)^2=0$$

The spatial components of the four velocity for a particle initially at rest are all zero and $U^2=g_{\mu\nu}u^\mu u^\nu=g_{00}(u^0)^2=c^2 \Rightarrow (u^0)^2=\frac{c^2}{g_{00}}$

The radial geodesics of such a particle in spherical coordinates now reads after inserting the value for $\Gamma[2,1,1]$ $\frac{d^2r}{d\tau^2}=-\frac{c^2\Gamma^1_{00}}{g_{00}}=\frac{-c^2r_s}{2g_{00}r^2h(r)}=-\frac{GM}{r^2}\cdot\frac{1}{\left(1-\frac{r_s}{r}\right)h(r)}$

For large r the Scharzschild metric becomes essentially flat and the radial surface coordinate becomes the Newtonian radial coordinate and the proper time τ becomes the Newtonian time t. We are now able to compare the previous expression with the Newtonian law of gravity $\frac{d^2r}{dt^2}=-\frac{GM}{r^2}$

In order for this comparison to hold, $\left(1-\frac{r_s}{r}\right)h(r)$ should be set equal to 1 or

$$h(r)=\frac{1}{1-\frac{r_s}{r}}$$

Conclusion: The function $h(r)$ can be determined from the weak-field approach without calculating and referring to the components of the Ricci-tensor.