Canadian Science Publishing

TUTORIAL

Derivation of general relativistic gravitational potential energy using principle of equivalence and gravitational time dilation

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Abstract: Einstein's theory of general relativity, which has been experimentally proved to be a true theory of gravity does not need gravitational potential energy to predict the trajectory of particles in space. This is because general relativity is a purely geometric theory. Objects move along the geodesics in the curved space–time. The energy–momentum tensor that warps space–time as per Einstein's field equations takes into account only the energy–momentum of matter and radiation. Thus, gravitational potential energy does not come into the picture in Einstein's theory of gravity and its role is taken over by the curvature of space–time. However, a general relativistically correct expression for gravitational potential energy is required for energy conservation and some energy-based approaches in physics. Conventionally, the correct form of gravitational potential energy is derived by using the full mathematical formality of general relativity. In this paper, we derive the same general relativistic expression for gravitational potential energy simply by using the principle of equivalence and gravitational time dilation.

Key words: gravitational potential energy, principle of equivalence, general relativity, general relativistic potential energy, special theory of relativity.

Résumé : La théorie de la relativité générale d'Einstein, qui a été démontrée être une vraie théorie de la gravité, n'a pas besoin d'une énergie potentielle gravitationnelle pour prédire les trajectoires des particules dans l'espace. Ceci est dû au fait que la relativité générale est une théorie purement géométrique. Les objets se déplacent selon les géodésiques dans l'espace-temps courbe. Le tenseur d'énergie- impulsion qui déforme l'espace-temps selon les équations de champ d'Einstein ne prend en compte que l'impulsion/énergie de la matière et de la radiation. Ainsi, l'énergie potentielle gravitationnelle n'entre pas en jeu dans la théorie de la gravité d'Einstein et son rôle est repris par la courbure de l'espace-temps. Cependant, une expression correcte en relativité générale est requise pour étudier la conservation de l'énergie et pour certaines approches physiques basées sur l'énergie. Conventionnellement, une forme correcte pour l'énergie potentielle gravitationnelle est dérivée en utilisant la formulation mathématique complète de la relativité générale. Dans la présente, nous décrivons un événement par lequel nous dérivons la même expression de relativité générale pour l'énergie potentielle gravitationnelle, simplement en utilisant le principe d'équivalence et la dilatation du temps. [Traduit par la Rédaction]

Mots-clés : énergie potentielle gravitationnelle, principe d'équivalence, théorie générale de la relativité, énergie potentielle en relativité générale, théorie de la relativité spéciale.

1. Introduction

As per Newton's law, gravitational force between two material bodies is calculated by taking the derivative of the gravitational potential energy, which is given by $\phi = -GMm/r$. The gravitational force finally decides the motion of a particle in space populated with an arbitrary distribution of mass. However, Einstein's theory of general relativity is a geometric theory of gravity in which the dynamics of the objects are governed by curvature of space–time created by distribution of energy–momentum in the universe. Every object follows a geodesic path in curved space–time and we do not need to calculate gravitational potential energy to predict the trajectory. General relativity has established itself as the true theory of gravity by numerous experimental confirmations, such as deflection of light rays coming from a distant star by the sun, precession of the perihelion of mercury, and gravitational time dilation [1–2]. The metric of a space–time in general relativity is found by solving the Einstein field equation given by [1]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is Ricci tensor, R is Ricci scalar, $g_{\mu\nu}$ is metric tensor, and $T_{\mu\nu}$ is the energy–momentum tensor. It is important to note that the energy–momentum tensor on the right-hand side of eq. (1) contains only matter or radiation terms, not the gravitational potential energy. This does not mean that gravitational potential energy does not exist, it just means that, in Einstein's theory of gravity, we deal with the curvature of space–time to calculate the dynamics of material bodies. The role of gravitational potential energy has been taken over by the curvature of space–time. That is why it is stated that *covariant* divergence (not the "ordinary divergence") of the energy–momentum tensor is zero. Covariant divergence includes a term related to curvature. So, we can say that gravitational potential energy exists in the form of the curvature of space–time.

In some energy-based branches of physics, like Hamiltonian approaches of quantum mechanics, we need the exact form of

Received 22 September 2020. Accepted 18 July 2021.

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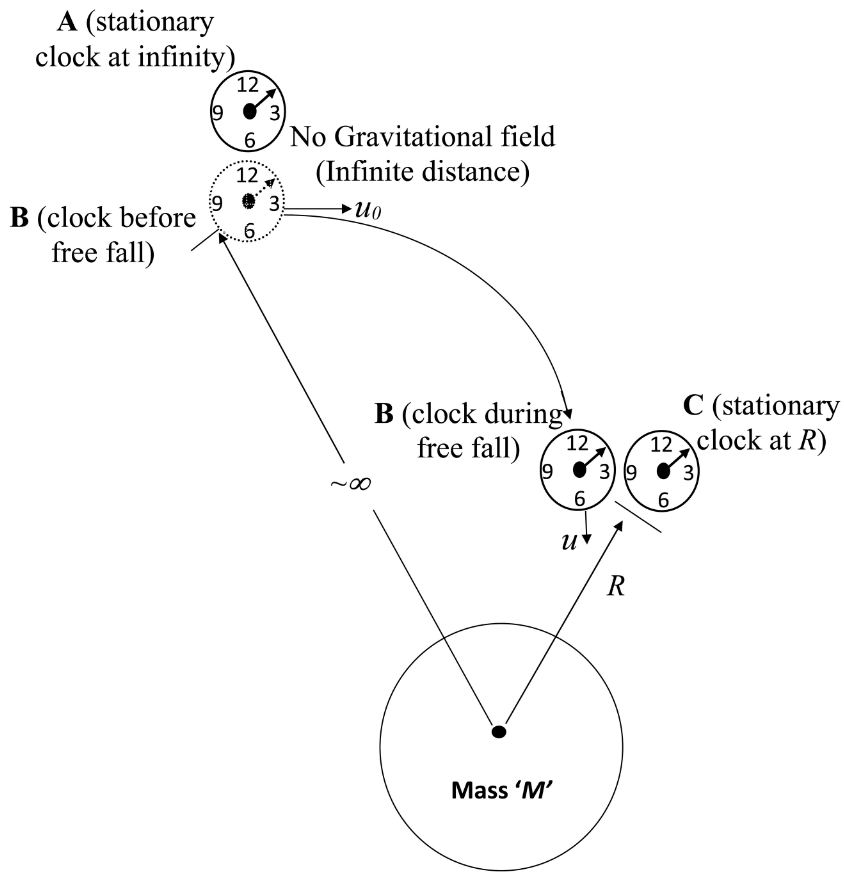
Fig. 1. Positions of stationary clocks A and C and freely falling particle with attached clock B used for derivation of general relativistic gravitational potential energy.



gravitational potential energy. Newton's law gives only an approximate value of this potential energy, just like it predicts the approximate trajectory or dynamics of material objects. We have to apply the general theory of relativity to exactly calculate the gravitational potential energy between two masses, one large mass M and other infinitesimal mass m at a distance R from it. This has been done previously [3] by using the full formality of general relativity and the potential energy (U) is given by,

$$U = mc^2\left[\left(1 - \frac{2GM}{c^2R}\right)^{1/2} - 1\right] \tag{2}$$

We can observe that if Schwarzschild radius ($2GM/c^2$) is too small as compared to distance R, eq. (2) reduces to

$$U = mc^2\left[\left(1 - \frac{1}{2}\frac{2GM}{c^2R}\right) - 1\right] = -\frac{GMm}{R}$$

which is Newton's law.

In this paper, we will derive eq. (2) using a simpler method by means of a thought experiment without using the full formality of general relativity. We will however use the principle of equivalence, which is the foundation on which the whole of general relativity stands and expressions of gravitational time dilation and special relativity. The advantage of our method is that the expression of general relativistic gravitational potential energy is derived by physical arguments and interesting consequence of the principle of equivalence becomes evident.

2. Derivation of gravitational potential energy from the principle of equivalence and gravitational time dilation

Let us consider a spherical mass M (a planet or star), as shown in Fig. 1. There are two stationary clocks, A and C. Clock A is at infinite distance from M and stationary clock C is at a distance R from M. Let a particle with initial velocity u_0 and small rest mass m_0 with clock B attached to it fall freely from infinity. It crosses past clock C with velocity u.

As the particle falls freely in the gravitational field, by the principle of equivalence [1, 4, 5], space–time looks flat to a tiny observer standing on this particle. In other words, it is in a *local inertial frame* of reference and it does not experience any gravitational field in its local region of space during its free motion. Hence, the tiny observer on the particle is entitled to apply the special theory of relativity to events happening in its *local region* at any instant of time. Let dT_A be the time interval (proper time) between two consecutive ticks of clock A. When the clock B crosses past clock A at infinity, the same time interval as measured by clock B will be given by (applying special theory of relativity),

$$(dT_\mathrm{A})_{\mathrm{B}\infty} = \frac{dT_\mathrm{A}}{\sqrt{1 - (u_0^2/c^2)}} \tag{3}$$

Due to gravitational time dilation [1], time interval in clock A as measured by clock C is given by,

$$(dT_A)_C = dT_A\sqrt{1-\frac{2GM}{c^2R}} \tag{4}$$

Note that we can take the expression for gravitational time dilation from general relativity or from the Schwarzschild solution as the aim of this paper is just to demonstrate how a modification of time coordinate leads to a change in gravitational potential energy.

When clock B crosses past clock C at radial distance R from M, since it is in a local inertial frame due to free fall, it can apply special relativity to translate the time $(dT_A)_C$ in terms of its own time by relation,

$$(dT_A)_{BR} = \frac{(dT_A)_C}{\sqrt{1-(u^2/c^2)}}$$

Putting eq. (4) in the above expression,

$$(dT_A)_{BR} = dT_A\sqrt{1-\frac{2GM}{c^2R}}\frac{1}{\sqrt{1-(u^2/c^2)}} \tag{5}$$

Since clock B is always in a locally inertial frame of reference due to free fall, its rate of ticking remains the same and its measured value for the same parameter will also be the same irrespective of where it is in space. So,

$$(dT_A)_{B\infty} = (dT_A)_{BR}$$

Using eq. (3) and (5) in above,

$$\frac{dT_A}{\sqrt{1-(u_0^2/c^2)}} = dT_A\sqrt{1-\frac{2GM}{c^2R}}\frac{1}{\sqrt{1-(u^2/c^2)}}$$

$$\Rightarrow \sqrt{1-\frac{u^2}{c^2}}\frac{1}{\sqrt{1-(u_0^2/c^2)}} = \sqrt{1-\frac{2GM}{c^2R}} \tag{6}$$

Taking the gravitational potential energy as zero at infinity and applying the law of conservation of energy to the particle with rest mass m_0 described at the beginning of this section,

$$(\text{Sum of rest and kinetic energy})_\infty = (\text{Sum of rest and kinetic energy})_R + U_R$$

$$\Rightarrow U_R = \frac{m_0c^2}{\sqrt{1-(u_0^2/c^2)}} - \frac{m_0c^2}{\sqrt{1-(u^2/c^2)}} \Rightarrow U_R = \frac{m_0c^2}{\sqrt{1-(u^2/c^2)}}\left[\frac{\sqrt{1-(u^2/c^2)}}{\sqrt{1-(u_0^2/c^2)}}-1\right] \quad \text{or} \quad U_R = mc^2\left[\frac{\sqrt{1-(u^2/c^2)}}{\sqrt{1-(u_0^2/c^2)}}-1\right] \tag{7}$$

where $m = m_0/\sqrt{1-(u^2/c^2)}$ is the relativistic mass of the particle.

Using eq. (6) in eq. (7), we get,

$$U_R = mc^2\left[\left(1-\frac{2GM}{c^2R}\right)^{1/2}-1\right] \tag{8}$$

Thus, we have derived the exact expression for the relativistic gravitation potential energy, which is the same as eq. (2) using the principle of equivalence, gravitational time dilation, and special theory of relativity.

3. Conclusion

The general theory of relativity is an exact theory of gravity that has seen no violation of its predictions in any of the experimental observations conducted so far. In this theory, dynamics of the material bodies are determined by curvature of space–time given by Einstein's field equation, which takes into account only the distribution of energy–momentum due to matter and radiation, excluding gravitational potential energy. However, as covariant divergence of energy–momentum tensor for the law of energy conservation includes a term related to curvature in space–time, this curvature can be interpreted as representing gravitational potential energy. In some branches of physics, such as the Hamiltonian approach, the exact (general relativistically correct) form of this gravitational potential energy is required. In this paper, we have presented a sequence of events through which we could derive the general relativistic expression for gravitational potential energy simply by using the principle of equivalence, gravitational time dilation, and expressions of the special theory of relativity. Our expression exactly matches the expression conventionally derived using full formality of general relativity. In the special case, when the distance is much more than Schwarzschild radius, the general relativistic expression for gravitational potential energy reduces to the Newtonian expression of gravitational potential energy.

Acknowledgement

The author is thankful to Archana Sharma, Associate Director of BTD Group, Bhabha Atomic Research Centre, for encouragement and support during this research.

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